Optimal Consumption and Investment with Asymmetric Long-term/Short-term Capital Gains Taxes*

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Journal of Economic Literature Classification Numbers: G11, H24, K34, D91.
Keywords: Capital Gains Tax Law, Portfolio Selection, Consumption, Asymmetric Tax Rates

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Abstract

We propose an optimal consumption and investment model with asymmetric long-term/short-term tax rates for a small investor. We also consider both the case where an investor can get full tax rebate for capital losses and the case where the investor can only carry over capital losses. The full rebate case is a better model for low income investors while the carry over case is more suited for wealthy investors. The optimal trading strategy is characterized by a time-varying no-transaction region outside which it is optimal to realize capital gain or loss to achieve the optimal fraction of after-tax wealth invested in stock. We show that the optimal consumption and investment policy is qualitatively different from what is found in the standard literature and the optimal policy for low income investors is qualitatively different from that for wealthy investors. More specifically, for low income investors, (1) it can be optimal to defer capital loss realization beyond one year even in the absence of transaction costs and wash sale restriction; (2) it can be optimal to defer long-term capital gains even when the rebate rate for short-term losses is high; (3) raising the short-term tax rate can significantly increase both consumption and stock investment; (4) higher tax rates (such those for wealthy investors) can make them significantly better off.

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1. Introduction

As shown by the existing literature (e.g., Koo and Dybvig (1996), Constantinides (1983, 1984), Dammon, Spatt, and Zhang (2001)), capital gain tax can significantly affect the optimal trading strategies of investors. Two important features of the current tax code are that the tax rate for long term investment can be much lower than the short term tax rate for most investors and the capital loss rebate is limited ($3,000 per year) with the rest carried over to the future indefinitely. However, the research on how these important features affect the optimal consumption and investment is limited.

Optimal investment in the presence of capital gain tax is in general extremely difficult because of the path dependency of capital gain tax basis, as pointed out by Koo and Dybvig (1996). Most existing literature approximate the exact tax basis using the average tax basis to reduce the dimensionality of the optimization problem (e.g., Dammon, Spatt, and Zhang (2001, 2004), Gallmeyer, Kaniel and Tompaidis (2006)). Unfortunately, this approximation method is not suitable for studying the impact of differential long-term/short-term tax rates, because with average tax basis, it may be optimal to hold shares that are purchased at different times and the optimal investment again becomes path dependent due to the need to keep track of the purchase dates of each purchase. Using exact tax basis, DeMiguel and Uppal (2005) show that "...the investor typically holds shares with only a single tax basis, and even when shares with a second tax basis are held the proportion is less than 4%." As explained by DeMiguel and Uppal (2005), the main reason for this finding is that "when the stock price goes up, a risk-averse investor would rarely purchase any additional shares of stock because of diversification reasons. Consequently, the tax basis of the shares held after an increase in the stock price is the same as in the previous time period. On the other hand, when the stock price goes down, to get a tax rebate the investor sells all those shares whose tax basis is above the current stock price, and then buys shares at the current stock price." Based on this finding, we develop a continuous-time optimal consumption and investment model with a single tax basis to study the impact of asymmetric long-term and short-term tax rates and the capital loss carry over provision.[1]

More specifically, we consider the optimal consumption and investment problem of a small, constant relative risk aversion (CRRA) investor who can continuously trade a risk free asset and a risky stock to maximize his expected utility from intertemporal consumption and bequest. There is no transaction cost, but the investor must pay the capital gains tax with asymmetric long-term/short-term rates. Based on the finding of DeMiguel and Uppal (2005) and as in Dammon and Spatt (1996), we assume that the investor always maintains a single tax basis. We consider both the case where an investor can get full tax rebate for any capital loss and the case where the investor can only carry over capital loss. The full rebate case is a better model for low income investors whose capital loss is likely below \$3,000 a year, while the carry over case is more suited for wealthy investors whose potential capital loss can far and frequently exceed \$3,000 a year. In addition, we assume capital gains and losses can be realized immediately, there is no wash sale restriction and shorting against the box is prohibited. The optimal trading strategy is characterized by a time-varying no-transaction region outside which it is optimal to realize at least some

[1] To compare the impact of the single-basis assumption and the commonly adopted average-basis assumption, we also solve a corresponding average-basis model with full capital loss rebate and full carry over when the long term and short term tax rates are the same. The optimal trading strategies are quite similar, which is consistent with the finding of DeMiguel and Uppal (2005) that most of the time investors only carry a single tax basis.

capital gains or losses to achieve the optimal fraction of after-tax wealth invested in stock. We implement an iterative procedure to numerically compute these boundaries and conduct an extensive analysis of the effect of asymmetric tax rates.

We show that the optimal consumption and investment policy is qualitatively different from what is found in the standard literature and the optimal policy for low income investors is qualitatively different from that for wealthy investors. In contrast to the case with equal tax rates across holding durations as considered by the existing literature, it may be optimal for a low income investor to defer the realization of capital loss beyond one year if the long-term tax rate is lower than the short-term tax rate. This is because immediately realizing capital loss when it occurs resets the holding period to zero, which makes it take longer to qualify for the lower long-term tax rate. When the short-term tax rate is significantly higher than the long-term rate, it is optimal to realize long-term capital gain immediately to re-establish the short-term tax status for the benefit of higher future capital loss tax credits. In addition, the option of realizing capital loss short-term and realizing capital gain long-term can make a low income investor prefer taxable securities to tax-exempt ones even when these securities have exactly the same price processes (e.g., same before-tax expected returns and volatilities). This suggests that a low income investor can be worse off with a tax-exempt status. In addition, as tax rates increase, a low income investor typically increases the optimal target stock holdings because capital gain tax effectively reduces the return volatility in the full tax rebate case. In contrast, a wealthy investor whom the carry-over case fits better is always worse off with capital gain tax and prefers tax exempt securities *ceteris paribus*. In addition, as tax rates increases, she decreases her optimal target stock holdings. She generally invests more than the no-tax case when she has tax losses because of the carried-over tax losses.

We show that lower income investors are willing to pay a significant fraction of their initial wealth to gain the same capital gains tax treatment as rich investors have. For example, suppose the interest rate is zero, the expected stock return is 4%, the volatility is 20%, the dividend yield is 2%, and investors have an expected remaining life time of 80 years and a relative risk aversion coefficient of 3. Consider a relatively poor investor with a marginal ordinary income tax rate of 10%, and a long-term capital gains tax rate of 0%. This investor would be willing to pay as much as 33.4% (about 33.4%/80 = 0.42% per year) of her initial wealth to gain the same capital gains tax treatment as that of a rich investor who has a marginal ordinary income tax rate of 35% and a long-term capital gains tax rate of 15%. Even when an investor is forced to realize short-term gains more often, e.g., due to liquidity shocks, higher rates would still make her significantly better off. For example, if a large liquidity shock that requires the liquidation of the entire stock position occurs once a year on average and thus the expected investment horizon is only one year, the low income investor would still be willing to pay 2.8% of the initial wealth to gain the same capital gains tax treatment as that of the rich investor. The higher tax rates on capital losses and the ability to defer short-term capital gains effectively make stock investment much less risky. As a result, a low income investor with higher tax rates would also invest and consume significantly more (as a fraction of their wealth). As Wilson and Liddell (2010) reported, in 2007, tax returns with an adjusted gross income of $100,000 or less have short-term net losses on average. These returns, totaling about 6 million, account for more than half of the total returns that have short-term gains or losses. This implies that many lower income investors can benefit from higher short-term capital gains tax rates. Our finding implies that raising capital gains tax rates for lower income investors to the levels of high income investors can decrease the after-tax riskiness of stock investment for lower income investors and significantly increase their stock market participation.

Because tax is only paid when it is realized, an investor has the option to defer capital gains tax and realize capital losses early. When long-term/short-term tax rates are the same, the value of this deferring comes only from earning the interest on the capital gains tax by not paying it sooner. When long-term rates are lower than short-term rates, the value of this deferring also comes from the benefit of realizing gains at a lower (long-term) rate in the future. We show that the value of deferring from realizing gains at a lower rate in the future is much greater than that from saving interest for low income investors. For example, for a low income investor with a short-term rate of 20% and a long-term rate of 15%, the value of deferring from realizing gain at the long-term rate (instead of the short-term rate) can be as high as 10% of the initial wealth, in contrast to a mere 1.2% for the value from interest saving. Therefore, by ignoring the difference between the long-term and short term rates, most of the existing literature significantly underestimates the value of deferring capital gains tax and largely overestimates the effective tax rates for lower income investors. In contrast, for wealthy investors, the value of deferring mainly comes from the interest saving, because a higher tax rebate does not affect how much loss they carry over.

As far as we know, among the few studies that discuss asymmetric long-term/short-term tax rates (e.g., Constantinides (1984), Dammon and Spatt (1996)), they all assume that the tax rate for long-term capital losses is the same as that for long-term capital gains, instead of the same as the marginal ordinary income tax rate as the law stipulates. We show that while this assumption does not significantly change the optimal trading strategy for a low income investor with a short-term status or a wealthy investor, it qualitatively changes the optimal trading strategy for a low income investor with a long-term status. More specifically, under this assumption, it is always optimal to realize all short-term capital losses before they turn into long-term because of the higher short-term rate. In addition, even when the short-term tax rate is only slightly higher than the long-term rate, it is also optimal to realize any long-term capital gains immediately to re-establish the short-term tax status for the benefit of the higher rate for capital losses. In contrast, when the rate for the long-term losses is equal to the marginal ordinary income tax rate as the law stipulates, we show that it can be optimal to defer short-term losses beyond the one year threshold and it is always optimal to defer small long-term capital gains and small long-term capital losses, no matter how high the short-term tax rate is relative to the long-term rate. This is because the long-term status strictly dominates the short-term status under the current law in the sense that compared to an investor with a short-term position, an investor with a long-term position pays the same tax rate on losses and a lower tax rate on gains.

While we have several simplifying assumptions, their impact on our main results is likely limited. The impact of immediate rebate assumption is probably small, especially when interest rate is low. This is because if interest rate is zero, then the investor can borrow against the tax rebate to be received later without interest cost. Accordingly, to address the concern, we have set the interest rate to be zero in all the relevant analysis. The assumption of no wash sale restriction and no transaction costs also unlikely changes our main results because investors can purchase similar stocks for 30 days without much loss and transaction cost rates are typically very small, especially for liquid stocks and in recent years.

As far as we know, this is the first paper to examine the optimal consumption and investment problem with asymmetric long-term/short-term tax rates and limited use of capital loss and the first to show that low income investors can be better off with higher tax rates. There are two important exceptions: Constantinides (1984) and Dammon and Spatt (1996), who consider the impact of asymmetric long-term/short-term tax rates on the capital gains/losses realization timing of one share of stock in a discrete-time setting where an investor sells and repurchases the stock only for tax reasons (e.g., not for consumption or portfolio rebalancing). Both of these studies assume the tax rate for long-term losses is the same as that for long-term gains. In contrast, we examine the impact of asymmetric long-term/short-term tax rates on optimal consumption and optimal risk exposure in a continuous time setting, assuming the tax rate for long-term losses is the same as the marginal ordinary income tax rate as the law stipulates. Therefore, investors realize capital gains/losses not only for tax reasons, but also for consumption and portfolio rebalancing purposes. While Constantinides (1984) and Dammon and Spatt (1996) provide important theoretical insights into the cost and benefit of realizing gains and losses later or sooner, they do not offer explicit guidance to the actual trading policy for an investor who maximizes the expected utility from intertemporal consumption. For example, the realization boundaries in Constantinides (1984) and Dammon and Spatt (1996) only depend on the basis to price ratio. In contrast, when a CRRA investor maximizes the expected utility from intertemporal consumption, the realization boundaries depend not only on the basis to price ratio, but also on the current fraction of wealth invested in the stock, in addition to other factors such as risk aversion, patience and life expectancy. This implies that the optimal tax realization strategy can also be substantially different. Indeed, as discussed above, in contrast to Constantinides (1984) and Dammon and Spatt (1996), it can be optimal to defer short-term losses beyond the one year threshold and it is always optimal to defer small long-term capital gains and small long-term capital losses. Also different from Constantinides (1984) and Dammon and Spatt (1996), we show that it can be optimal to realize short-term gains even when the long-term rate is much lower than the short-term rate, because of the benefit from achieving a better risk exposure.

This paper is also closely related to Dammon, Spatt, and Zhang (2001, 2004) and Gallmeyer, Kaniel and Tompaidis (2006). In a discrete time setting, Dammon, Spatt, and Zhang (2001) consider the optimal consumption and portfolio decisions with capital gains tax and short-sale constraints, assuming a binomial stock price process and average tax basis approximation. They show that contrary to standard financial advice, optimal equity holding can increase until late in lifetime because of the forgiveness of capital gains tax at death. While their main analysis focuses on the symmetric long-term and short-term tax rates case, they do consider the impact of asymmetric tax rates on optimal life cycle equity holding in Subsection 3.2. In contrast to our model, however, they assume an investor can only trade once a year and Constantinides (1984)'s condition for realizing long-term gains each year is always satisfied, therefore, it is always optimal to realize all short-term losses and all long-term gains in their model, even with asymmetric tax rates. In addition, as Constantinides (1984) and Dammon and Spatt (1996), Dammon, Spatt, and Zhang (2001) also assume that the tax rate for long-term losses is the same as that for long-term gains. Dammon, Spatt, and Zhang (2004) examine optimal asset allocation and location decisions for investors making taxable and tax-deferred investments. They show that it is significantly advantageous to hold bonds in the tax-deferred account and equity in the taxable account. Gallmeyer, Kaniel and Tompaidis (2006) consider the optimal consumption-portfolio problem with symmetric capital gains tax rates and multiple stocks to understand how short selling influences portfolio choice when shorting against the box is prohibited. They find that shorting one stock even when no stock has embedded gain may be optimal. In addition, when short selling is costly, the benefit of trading separately in multiple stocks is not economically significant.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 provides some analytical results, a verification theorem, and the numerical solution procedure. In Section 4, we conduct numerical analysis on the optimal consumption and trading strategies, the bias against lower income investors and the value of deferring realization. Section 5 concludes and all the proofs are relegated to the Appendix.

2. The Model

Throughout this paper we are assuming a probability space $(\Omega, \mathcal{F}, P)$. Uncertainty and the filtration $\{\mathcal{F}_t\}$ in the model are generated by a standard one dimensional Brownian motion w and a Poisson process defined below. We will assume that all stochastic processes are adapted.

There are two assets an investor can trade without any transaction costs. The first asset is a money market account growing at a continuously compounded, constant rate r. The second asset ("the stock") is a risky investment. The ex-dividend stock price S_t follows the process

$$dS_t = \mu S_t dt + \sigma S_t dw_t, \tag{1}$$

where μ and σ are constants with $\mu > r$.

The investor is subject to capital gains tax. We assume that the tax on dividend and interest is due when they are paid. Optimal investment in the presence of capital gains tax is in general extremely difficult because of the path dependency of tax basis, as pointed out by Dybvig and Koo (1996). Most existing literature approximate the exact tax basis using the average tax basis to reduce the dimensionality of the optimization problem (e.g., Dammon, Spatt, and Zhang (2001, 2004), Gallmeyer, Kaniel and Tompaidis (2006)). Unfortunately, this approximation method is not suitable for studying the impact of asymmetric long-term/short-term tax rates, because with average tax basis, it may be optimal to hold shares that are purchased at different times and the optimal investment problem again becomes path dependent due to the need to keep track of the time of each purchase. Using exact tax basis, DeMiguel and Uppal (2005) find that the investor almost always holds shares with a single tax basis. As explained by DeMiguel and Uppal (2005), the main reason for this finding is that "when the stock price goes up, a risk-averse investor would rarely purchase any additional shares of stock because of diversification reasons. Consequently, the tax basis of the shares held after an increase in the stock price is the same as in the previous time period. On the other hand, when the stock price goes down, to get a tax rebate the investor sells all those shares whose tax basis is above the current stock price, and then buys shares at the current stock price." Based on this finding, we assume that an investor always keeps a single tax basis.[2] As a consequence, the investor should liquidate all stock holdings before purchases. In addition, we assume that (1) capital gains and capital losses can be realized immediately after sale; (2) no wash sale restriction; (3) shorting against the box is prohibited. The investor is

[2] To compare the impact of the single-basis assumption and the commonly adopted average-basis assumption, we also solve a corresponding average-basis model when the long-term and short-term tax rates are the same. The optimal trading strategies are quite similar, which is consistent with the finding of DeMiguel and Uppal (2005) that most of the time investors only carry a single tax basis.

endowed with x_{0-} dollars in the risk free account and y_{0-} dollars in the stock account. This initial endowment includes the present value of all future after-tax ordinary income (i.e., all except capital gains/losses).[3] Let x_t denote the dollar amount invested in the riskless asset, y_t denote the current value of the stock holding, B_t be the tax basis at time t, and s_t be the length of stock holding period since last purchase.

Between purchases, we then have

$$\begin{aligned} dx_t &= (r(1-\tau_i)x_{t-} + (1-\tau_d)\delta y_{t-} - c_t)dt + \left[y_{t-} - \tau(s_t)\left(y_{t-} - B_{t-}\right)\right. \\ &\quad \left. + (1-\omega)\,\kappa\left(\tau_S - \tau(s_t)\right)\left(B_{t-} - y_{t-}\right)^+ - \omega\tau\left(s_t\right)\left(B_{t-} - y_{t-}\right)^+\right] dM_t, \end{aligned} \tag{2}$$

$$dy_t = \mu y_{t-}dt + \sigma y_{t-}dw_t - y_{t-}dM_t, \tag{3}$$

$$dB_t = \left[-B_{t-} + \omega\left(B_{t-} - y_{t-}\right)^+\right] dM_t, \tag{4}$$

$$ds_t = dt, \tag{5}$$

where r and δ are before-tax interest rate and dividend yield with tax rates τ_i and τ_d respectively, dM_t represents the fraction of the current stock position that is sold at t, $\omega = 0$ or 1 corresponds to the full-rebate case and the full carry-over case, respectively, $\kappa = 0$ corresponds to the case of applying the long-term tax rate to long-term losses and $\kappa = 1$ to the case of applying the short-term tax rate to all losses, and τ is the tax rate function for capital gains with

$$\tau(s) = \begin{cases} \tau_S & if \quad s < H \\ \tau_L & if \quad s \geq H, \end{cases} \tag{6}$$

where H is the shortest time for qualifying a long term tax status. The bracketed term in (2) denotes the after-tax dollar amount if the entire stock position is sold at

[3] More specifically, suppose the investor has a constant after-tax income rate of L until death. Then with a complete market, the present value of the after-tax income is equal to $L/(r+\lambda)$, which can be simply added to her initial wealth to arrive at the initial endowment we consider, and all our analysis would remain valid. We do not explicitly model the ordinary income tax payment to simplify the exposition and focus on the main analysis.

t. Because only a fraction dM_t is sold, the after-tax dollar amount and the basis are both proportionally reduced. Because $\tau_S \geq \tau(s)$, compared to the case with $\kappa = 0$, an investor with a long-term status in the $\kappa = 1$ case has an additional put option represented by the term $(B - y)^+$, which enables the investor to realize long-term losses at the (higher) short-term rate.

At a purchase time η (a stopping time) with a holding period of s since last purchase, we have

$$\begin{aligned} x_\eta &= x_{\eta-} + y_{\eta-} - \tau(s_{\eta-})\,(y_{\eta-} - B_{\eta-}) + (1-\omega)\,\kappa\,(\tau_S - \tau(s_{\eta-}))\,(B_{\eta-} - y_{\eta-})^+ \\ &\quad -\omega\tau(s_{\eta-})(B_{\eta-} - y_{\eta-})^+ - I_\eta, \end{aligned} \tag{7}$$

$$y_\eta = I_\eta, \tag{8}$$

$$B_\eta = I_\eta + \omega(B_{\eta-} - y_{\eta-})^+, \tag{9}$$

$$s_\eta = 0, \tag{10}$$

where I is the dollar amount of the stock bought immediately after a sale.

The investor maximizes expected utility from intertemporal consumption and the final after-tax wealth at the first jump time $\mathcal{T}$ of an independent Poisson process with intensity λ. This Poisson process can represent the time of a liquidity shock upon which one must liquidate the entire portfolio or the death time of the investor or the performance evaluation time of a fund.[4] If it represents a death time, then capital gains tax may be forgiven (e.g., in USA) or may be not (e.g., in Canada). Let $V(x_0, y_0, B_0, s_0)$ be the time 0 value function, which is equal to

$$\sup_{\{M_t,\eta,I_\eta,c_t\}} E\left[\alpha\int_0^{\mathcal{T}} e^{-\beta t}u(c_t)dt + (1-\alpha)e^{-\beta\mathcal{T}}u\left((1-\iota)(x_{\mathcal{T}} + y_{\mathcal{T}}) + \iota f(x_{\mathcal{T}}, y_{\mathcal{T}}, B_{\mathcal{T}}, s_{\mathcal{T}})\right)\right], \tag{11}$$

[4]As shown by Carr (1998) and Liu and Loewenstein (2002), one can use a series of random times to approximate a fixed time (e.g., of performance evaluation).

subject to (2)-(10) and the solvency constraint

$$f(x_t, y_t, B_t, s_t) \geq 0,\ (1-\iota)(x_t + y_t) \geq 0,\ \forall t \geq 0, \tag{12}$$

where $\beta > 0$ is the subjective discount rate, $\alpha \in [0,1]$ is the weight on intertemporal consumption, $\iota \in \{0,1\}$ indicating if tax is due or not at $\mathcal{T}$,

$$\begin{aligned} f(x,y,B,s) \ \equiv\ & x + y - \tau(s)(y-B) + (1-\omega)\kappa(\tau_S - \tau(s))(B-y)^+ \\ & + \omega\tau(s)(B-y)^+ \end{aligned}$$

is the after-tax wealth, and

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

with the relative risk aversion coefficient γ. If $\alpha = 0$, the problem can be also interpreted as an investment problem of a fund whose manager's compensation is proportional to the before-tax ($\iota = 0$) or after-tax ($\iota = 1$) asset under management.

Using the dynamic programming principle and integrating out the Poisson jump, we can rewrite the investor's problem in a recursive form as

$$\begin{aligned} & V(x_0, y_0, B_0, s_0) \\ = \ & \sup_{\{M_t, \eta, I_\eta, c_t\}} E\Big[\int_0^\eta e^{-(\beta+\lambda)t}\left(\alpha u(c_t) + (1-\alpha)\lambda u((1-\iota)(x_t+y_t) + \iota f(x_t, y_t, B_t, s_t)\right) dt \\ & + e^{-(\beta+\lambda)\eta} V\left(f(x_{\eta-}, y_{\eta-}, B_{\eta-}, s) - I_\eta, I_\eta, I_\eta, 0\right)\Big], \end{aligned} \tag{13}$$

subject to (2)-(10) and the solvency constraint (12).

The associated HJB equation is

$$\begin{aligned} \max\Big\{ & V_s + \mathcal{L}_0 V, \sup_I V\left(f(x,y,B,s) - I, I, I + \omega(B-y)^+, 0\right) - V(x,y,B,s), \\ & f(0,y,B,s) V_x - y V_y - \left(B - \omega(B-y)^+\right) V_B \Big\} = 0 \end{aligned} \tag{14}$$

in $s > 0$, $B > 0$, $y > 0$, $f(x, y, B, s) > 0$, and $x + y > 0$ (if $\iota = 0$), where

$$\begin{aligned}\mathcal{L}_0 V &= \frac{1}{2}\sigma^2 y^2 V_{yy} + \mu y V_y + \left((1-\tau_i) r x + (1-\tau_d)\delta y\right) V_x - (\beta+\lambda) V \\ &\quad + \alpha^{1/\gamma}\frac{\gamma}{1-\gamma}\left(V_x\right)^{-\frac{1-\gamma}{\gamma}} + \frac{(1-\alpha)\lambda}{1-\gamma}\left((1-\iota)(x+y) + \iota f(x,y,B,s)\right)^{1-\gamma}.\end{aligned}$$

Using the homogeneity property of the value function, we can reduce the dimensionality of the problem by the following transformation:

$$\begin{aligned} z &= \frac{x}{y}, \\ b &= \frac{B}{y}, \\ V(x,y,B,s) &= y^{1-\gamma}\Phi(z,b,s), \end{aligned}$$

for some functions Φ, where b is equal to the basis per share divided by the stock price, and so will be simply referred to as the basis to price ratio. Then (14) can be reduced to

$$\max\left\{\Phi_s + \mathcal{L}_1\Phi, G(z,b,s;\Phi) - \Phi(z,b,s),\ -(1-\gamma)\Phi + f(z,1,b,s)\Phi_z + \omega(b-1)^+\Phi_b\right\} = 0 \tag{15}$$

in $s > 0$, $b > 0$, $f(z, 1, b, s) > 0$, and $z + 1 > 0$ (if $\iota = 0$), where

$$\begin{aligned}\mathcal{L}_1\Phi &= \frac{1}{2}\sigma^2 z^2\Phi_{zz} + \frac{1}{2}\sigma^2 b^2\Phi_{bb} + \sigma^2 z b\Phi_{zb} - \left(\mu - \gamma\sigma^2\right) b\Phi_b \\ &\quad - \left[(\mu - (1-\tau_i) r - \gamma\sigma^2) z - (1-\tau_d)\delta\right]\Phi_z + \left[(1-\gamma)(\mu - \frac{1}{2}\gamma\sigma^2) - \beta - \lambda\right]\Phi \\ &\quad + \frac{\gamma\alpha^{1/\gamma}}{1-\gamma}\left(\Phi_z\right)^{-\frac{1-\gamma}{\gamma}} + \frac{(1-\alpha)\lambda}{1-\gamma}\left((1-\iota)(z+1) + \iota f(z,1,b,s)\right)^{1-\gamma}\end{aligned}$$

and

$$G(z,b,s;\Phi) = f(z,1,b,s)^{1-\gamma}\sup_{k\in(-1,+\infty)}(k+1)^{\gamma-1}\Phi\left(k, 1 + \omega(k+1)\frac{(b-1)^+}{f(z,1,b,s)}, 0\right).$$

In terms of the fraction of wealth $\pi \equiv y/f(x, y, B, s) = 1/f(z, 1, b, s)$, the optimal trading strategy of the investor can be characterized by a sell boundary $\bar{\pi}(b, s)$ and a buy boundary $\underline{\pi}(b,s)$ in the π-b plane: transact to the sell boundary $(\bar{\pi}(b,s),b)$ if $\pi(z,b,s) > \bar{\pi}(b,s)$; no transaction if $\underline{\pi}(b,s) < \pi(z,b,s) < \bar{\pi}(b,s)$; otherwise liquidate the entire stock position and transact to π^*, where $\pi^* = \bar{\pi}(1,0) = \underline{\pi}(1,0)$. In other words, if the current fraction π is above the sell boundary then vertically transact to the sell boundary; if the current fraction π is between the sell boundary and the buy boundary, then no transaction; otherwise sell all the stock and rebalance to π^*.

3. Theoretical Analysis and Numerical Procedure

Before presenting numerical procedure to solve the HJB equation (15), we conduct some theoretical analysis.

The following proposition provides a sub-optimal Merton-like strategy that keeps a constant fraction of wealth in stock. Moreover, if the risk free rate is 0 and tax rates are constant, then such a strategy is optimal. Define

$$\rho \equiv \beta + \lambda - (1-\gamma)\left(\frac{\left((1-\tau_S)\,\mu + (1-\tau_d)\,\delta - (1-\tau_i)\,r\right)^2}{2\,(1-\tau_S)^2\,\gamma\sigma^2} + (1-\tau_i)\,r\right).$$

Proposition 1 *Assume $\iota = 1$ and $\rho > 0$.*

1. *Within the class of strategies with a constant fraction of wealth in stock, the optimal fraction is*

$$\begin{aligned}
\frac{y_t}{x_t + y_t - \tau_S\,(y_t - B_t)} &= \frac{(1-\tau)\,\mu + (1-\tau_d)\,\delta - (1-\tau_i)\,r}{(1-\tau_S)^2\,\gamma\sigma^2}, \\
\frac{c_t^*}{x_t + y_t - \tau_S\,(y_t - B_t)} &= \alpha^{1/\gamma}\nu^{-(1-\gamma)/\gamma},
\end{aligned}$$

and the associated value function is

$$\frac{\left[\nu\,(x + y - \tau_S\,(y - B))\right]^{1-\gamma}}{1-\gamma},$$

where ν is the unique positive root of

$$-\rho\nu^{1-\gamma} + \gamma\alpha^{1/\gamma}\upsilon^{-(1-\gamma)^2/\gamma} + (1-\alpha)\,\lambda = 0. \tag{16}$$

2. *if $r = 0$, $\omega = 0$, and $\tau_S = \tau_L$, then the above strategy is indeed optimal within the class of all feasible strategies.*

Proof: see Appendix.

Proposition 1 implies that if $r = 0$, all tax rates are equal, and an investor starts with all cash and gets the full tax rebate, then the investor can achieve the same expected utility as in the case without tax. However, the investor invests a greater fraction of the after-tax wealth in the stock. Intuitively, with zero interest rate and same tax rates, there is no benefit of deferring the realization of capital gains or capital losses. Thus the investor trades the stock continuously and the after-tax expected return becomes $(1-\tau_S)\mu + (1-\tau_d)\,\delta$, while the after-tax volatility becomes $(1-\tau_S)\sigma$. Therefore, if $\tau_d = \tau_S$, the stock investment increases because it is proportional to the ratio of the after-tax expected return to the after-tax variance. However, the expected utility does not change because it is determined by the Sharpe ratio, which remains the same when tax rates are all the same.

In the following we focus on the full rebate case $\omega = 0$. Let us present the verification theorem.

Proposition 2 *(verification theorem). Assume $r > 0$ or $r = 0$ with $\tau_S > \tau_L$, and $\omega = 0$. Let $\Phi(z, b, s)$ be a solution to the HJB equation (15) satisfying certain regularity conditions. Define the no-trading region NT, the buy region BR, and the sell region*

SR as follows:

$$\begin{aligned} NT &= \left\{ (z,b,s): \begin{array}{l} \Phi(z,b,s) > G(z,b,s;\Phi), \\ -(1-\gamma)\,\Phi + f\,(z,1,b,s)\,\Phi_z < 0 \end{array} \right\}, \\ BR &= \{(z,b,s): \Phi(z,b,s) = G(z,b,s;\Phi)\}, \\ SR &= \{(z,b,s): -(1-\gamma)\,\Phi + f\,(z,1,b,s)\,\Phi_z = 0\}. \end{aligned}$$

Assume ∂NT is sufficiently smooth and

$$\partial NT \cap \{b = 1, s = 0\} = (z^*, 1, 0)$$

for some $z^* > 0$. Denote $\partial B = \overline{NT} \cap BR$ and $\partial S = \overline{NT} \cap SR$. Define

$$V\,(x,y,B,s) = y^{1-\gamma}\Phi\left(\frac{x}{y}, \frac{B}{y}, s\right).$$

Then $V\,(x,y,B,s)$ is the value function, and the optimal control is given as follows:
i) optimal consumption: $c^*(x_t, y_t, B_t, t) = y_t \left(\Phi_z(\frac{x_t}{y_t}, \frac{B_t}{y_t}, t)\right)^{-1/\gamma}$;
ii) sell strategy: at ∂S,

$$M_t = \int_0^t 1_{\left\{\left(\frac{x_\xi}{y_\xi}, \frac{B_\xi}{y_\xi}, s\right) \in \partial S\right\}} dM_\xi;$$

iii) buy strategy $(\eta_1^*, \eta_2^*, ...; I_1^*, I_2^*, ...)$: put $\eta_0^* = 0$ and inductively

$$\begin{aligned} \eta_{n+1}^* &= \inf\left\{t > \eta_n^*; \left(\frac{X^{(n)}(t)}{Y^{(n)}(t)}, \frac{B^{(n)}(t)}{Y^{(n)}(t)}, t - \eta_n^*\right) \in BR\right\}, \\ I_{n+1}^* &= \frac{1}{1+z^*} f\Big(X^{(n)}(\eta_{n+1}^{*-}), Y^{(n)}(\eta_{n+1}^{*-}), B^{(n)}(\eta_{n+1}^{*-}), \eta_{n+1}^{*-} - \eta_n^*\Big), \end{aligned}$$

where $\left(X^{(n)}(t), Y^{(n)}(t), B^{(n)}(t)\right)$ is the induced process with the combined control $(c_t^*, M_t, (\eta_1^*, \eta_2^*, ...; I_1^*, I_2^*, ...))$.

Proof: The proof is similar to that in Davis and Norman (1990) and Øksendal and Sulem (2002).

Because $G(z,b,H;\Phi)$ in (15) depends on Φ, we need to provide an iterative procedure to find the solution satisfying the requirements of Proposition 2. Noting that the tax rates remain constant after the holding period exceeds H, we must have $\Phi_s = 0$ and $f(x,y,B,s) = f(x,y,B,H)$ for any $s > H$. We denote

$$\varphi(z,b) \equiv \Phi(z,b,s) \text{ for } s \geq H.$$

Then we have

$$\begin{aligned} \max\Big\{ & \mathcal{L}_1\varphi,\ G(z,b,H;\varphi) - \varphi(z,b), \\ & -(1-\gamma)\varphi + f(z,1,b,H)\varphi_z = 0\Big\}, \quad s > H. \end{aligned} \tag{17}$$

Hence, we can instead solve the system: (17) in $s > H$, (15) in $s < H$ with the terminal condition at $s = H : \Phi(z,b,H) \equiv \varphi(z,b)$. This motivates us to propose the following algorithm.

The algorithm of finding the solution numerically ($\omega = 0$):

1. Set

$$M_0 = \left(\frac{\gamma}{\beta - (1-\gamma)r}\right)^{\gamma/(1-\gamma)}.$$

2. For given M_i, use a penalty method with a finite difference scheme (e.g., Dai and Zhong (2010)) to solve

$$\max\Big\{\mathcal{L}_1\varphi,\ \frac{(M_i f(z,1,b,H))^{1-\gamma}}{1-\gamma} - \varphi(z,b), \tag{18}$$

$$-(1-\gamma)\varphi + f(z,1,b,H)\varphi_z = 0\Big\}, \quad s > H, \tag{19}$$

$$\begin{aligned} \max\Big\{ & \Phi_s + \mathcal{L}_1\Phi,\ \frac{(M_i f(z,1,b,s))^{1-\gamma}}{1-\gamma} - \Phi(z,b,s), \\ & -(1-\gamma)\Phi + f(z,1,b,s)\Phi_z\Big\} = 0, \quad s < H, \end{aligned} \tag{20}$$

with the terminal condition at $s = H$:

$$\Phi(z,b,H) = \varphi(z,b); \tag{21}$$

3. Set

$$M_{i+1} = \left((1-\gamma) \sup_{k \in (-1,+\infty)} (k+1)^{\gamma-1} \Phi(k,1,0) \right)^{1/(1-\gamma)};$$

4. If $|M_{i+1} - M_i| <$tolerance, then stop, otherwise set $M_i = M_{i+1}$ and go to step 2.

The reason why we choose the initial guess M_0 is that $\frac{(M_0 f(x,y,B,0))^{1-\gamma}}{1-\gamma}$ is the value function associated with the trivial suboptimal strategy of investing only in the risk free asset. It can be shown that the above iterative procedure yields a monotonically increasing sequence $\{M_i\}_{i=1,2,...}$. Hence the algorithm must be convergent. A proof is in Appendix.

Remark: When $\iota = 1$, we can also choose $M_0 = \nu$, as given in (16), which corresponds to another suboptimal strategy of always liquidating all stock holdings to keep a constant fraction of wealth in stock.

The carry over case $\omega = 1$ is rather complicated and is addressed in Appendix.

4. Numerical Analysis

In this section, we provide some numerical analysis on the solution of the investor's problem. We set the default parameter values as follows: $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\delta = 0.02$, $\sigma = 0.2$, $\alpha = 0.9$, $\tau_i = \tau_d = \tau_S$, $\tau_S = 0.15$, $\tau_L = 0.15$, and $\iota = 1$.

4.1 Optimal trading boundaries

Figure 1 plots the optimal trading boundaries against the basis-price ratio b for three tax rates when the short term and long term rates are the same, for both the full rebate and the full carry over cases. When tax rate is zero, we have the standard

Merton solution where the investor invests a constant fraction 50% of wealth in the stock. When the interest rate is zero, it is also optimal to keep a constant fraction of wealth in stock, as shown in Proposition 1. However, because the capital gains tax and capital losses credits effectively reduce the variance of the stock return more than the expected return, the fractions become higher for higher tax rates, e.g., 58.8% if tax rates are 15% and 76.9% if tax rates are 35%, as shown by the horizontal lines.

With positive tax rates and a positive interest rate, Figure 1 shows that it is optimal to have a no-transaction region when there are capital gains (i.e., $b < 1$) in both the full rebate and the full carry over cases. More specifically, if the fraction of wealth in the stock is (vertically) above the sell boundary, then the investor sells a minimum amount (and thus realizes some capital gains) to stay at or below the sell boundary. The trading direction is vertically downward in the figure because the basis-price ratio b does not change as the investor sells. If the fraction of wealth in the stock is (vertically) below the buy boundary, then the investor liquidates the entire stock position and rebalances to the corresponding dotted position at $b = 1$. Because the no transaction region is bounded above and below, Figure 1 shows that it can be optimal to realize capital gains even when the interest rate is positive. Intuitively, the no transaction region is a reflection of the tradeoff between the benefit of deferring tax payment from saving interest and the cost of suboptimal risk exposure. When the fraction of wealth in stock is too low or too high relative to the optimal level for the case with zero interest rate and the same before-tax risk premium $\mu + \delta - r$, the cost of suboptimal risk exposure is greater than the benefit of deferring capital gains tax, therefore, the investor sells the stock and pays the tax. As tax rate increases, the no transaction region shifts up and the investor on average invests more in the stock as in the zero interest rate case. Since the long term and short term rates are the same, the optimal trading boundaries are independent of holding duration.

As predicted by the standard literature, Figure 1(a) shows that in the full rebate case, the entire region with capital losses (i.e., $b > 1$) belongs to the transaction region, which implies that it is always optimal to immediately realize any capital loss (by trading to the corresponding dotted positions at $b = 1$). Intuitively, immediately realizing losses can not only earn interest on the tax rebate earlier but also reduce the duration of a sub-optimal position. As shown next, always realizing losses immediately is no longer true when the long-term and short-term tax rates differ.

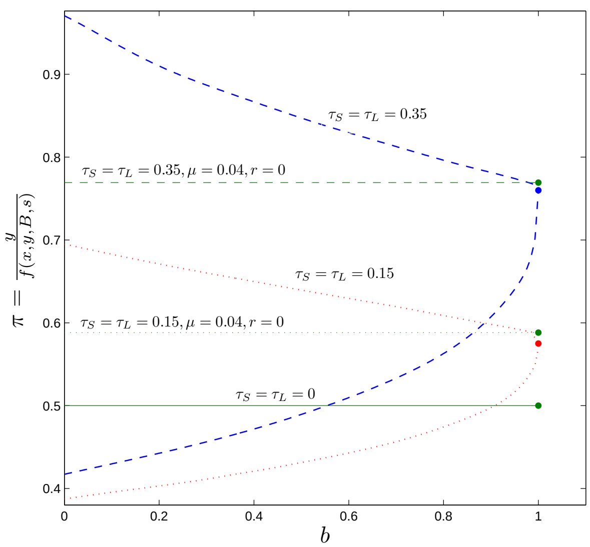
Even though there is no tax rebate for capital losses, the investor still prefers to realize capital losses immediately, because of the benefit of achieving the optimal risk exposure sooner. Indeed, Figure 1(b) shows if there is a capital loss (i.e., $b > 1$), then it is optimal to continuously realize losses to stay at the corresponding lines for different tax rates. In contrast to the full rebate case, the distance between optimal fraction at $b = 1$ and the dotted line for $b > 1$ suggests that the optimal fraction of wealth invested in the stock is discontinuous at $b = 1$. This is due to the difference in the tax treatment of capital gain and capital loss, because the investor needs to pay tax for a capital gain but can only carry over capital losses. Because of this discrepancy, the investor tends to invest less than the Merton fraction when he does not carry any capital losses and invest more than the Merton fraction when he is carrying some capital losses that can offset some potential capital gains.

Figure 1(b) shows that different from the full rebate case, it is also optimal to defer very small capital gains, i.e., even if b is very close to 1 and the fraction of wealth in stock is far from the optimal target (the dots), the investor still prefers to defer the realization. Intuitively, the benefit of achieving the optimal target sooner is smaller than the cost from the asymmetric treatment of gains and losses. In addition, as the tax rates increases, the optimal fraction at $b = 1$ (represented by the red or blue dot) always decreases. This is because for the carry over case, the after-tax stock return is always smaller than the no-tax return and tax does not reduce negative return fluctuation.

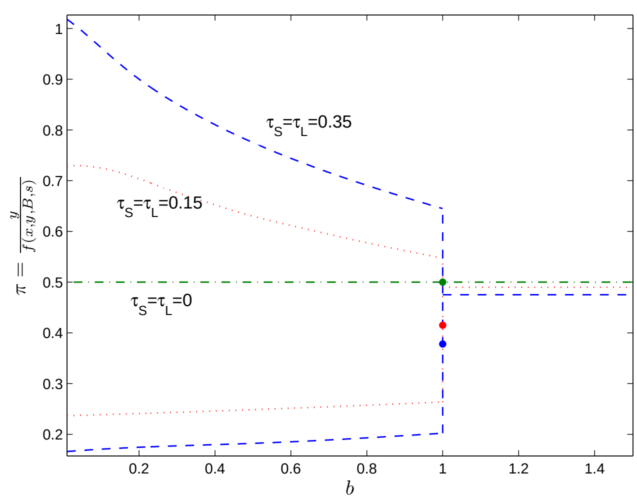
Figure 2 plots the optimal trading boundaries against the basis-price ratio b for four different holding times when the short-term and long-term rates differ, for $\kappa = 0$ and $\kappa = 1$ and both the full rebate and the full carry over cases. Compared to the same rate case with $\tau_L = \tau_S = 0.35$, the optimal target level at $b = 1$ and $s = 0$ is significantly higher and the no transaction region is much wider, a reflection of the greater benefit of deferring capital gains tax because the long-term tax rate is lower. Still, in contrast to Constantinides (1984) and Dammon and Spatt (1996), Figure 2 implies that it can be optimal to realize short-term gains even when the long-term rate is much lower than the short-term rate, as long as the fraction of wealth in the stock becomes too high or too low relative to the optimal risk exposure. Different from the symmetric rate case, Figure 2(a) shows that with full rebate, when the investor has already held the stock for some time (e.g., $s > 0.5$), there can be a no transaction region even when there is a capital loss (i.e., $b > 1$). Therefore, in contrast to the prediction by standard models and as pointed out by Dammon and Spatt (1996), it may be optimal to also defer tax losses realization even when there is no transaction cost or wash sale restriction. This is because there is a benefit of paying a lower (long-term) tax rate in case of an eventual gain if the investor does not realize the small tax loss and keep the long-term status.

Most of the existing literature on optimal consumption and investment assume that a long-term loss is taxed at the long-term capital gains tax rate (i.e., $\kappa = 0$) instead of the marginal ordinary income rate (i.e., $\kappa = 1$). Next we examine how

Figure 1: Optimal trading boundaries against basis-price ratio b.



(a) Full Rebate



(b) Full Carry Over

Parameter default values: $\omega = 0$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 1$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = \tau_d = \tau_S$, and $\kappa = 1$.

the optimal trading strategies differ across these two models. Figure 2(a) shows that when the holding duration is short, the boundaries for these two models are virtually indistinguishable (e.g., $s = 0$, $s = 0.5$). However, as the holding period approaches or exceeds the short-term threshold of 1 year, the optimal trading strategies become significantly different. If $\kappa = 0$, because the option value of realizing capital losses later at the higher short-term rate decreases, the total benefit of not immediately realizing capital losses declines. Therefore the capital loss threshold beyond which it is optimal to realize the loss immediately becomes smaller when the holding time gets close to the short-term threshold of 1 year. The boundary at $s = 1^-$ shows that it is always optimal to realize all losses at the end of the year. Right after the status becomes long-term (i.e., $s = 1^+$), the entire region becomes the transaction region if $\kappa = 0$, i.e., it is optimal for the investor to sell all the stock and rebalance to the dotted position for $s = 0$ and thus realize all capital gains or capital losses immediately. These findings confirm the predictions of the existing literature (e.g., Constantinides (1984)). Intuitively, immediately realizing all the capital gains or losses converts the status to short-term, which entitles the investor to realize future loss at the higher short-term rate. Because $\kappa = 0$, the long-term tax status has costs and benefits, compared to the short-term status. The benefit is the capability of realizing capital gains at a lower rate. The cost is that the tax (rebate) rate for capital losses is also lower. When the short-term tax rate is sufficiently higher than the long-term rate (as it is in the figure), the benefit of deferring tax is smaller than the option value of realizing losses at the higher short-term rate, therefore, it is optimal for the investor to realize all the long-term gains and losses.[5]

In contrast to the predictions of the existing literature, the boundaries for $s = 1^-$ and $s \geq 1$ with $\kappa = 1$ imply that it can be optimal to defer some short-term losses

[5]Our additional numerical results unreported in the paper show that for $\tau_S = 0.35$, as long as τ_L is less than 0.34, it is optimal for the investor to realize all long-term gains and losses.

beyond the end of a year and to defer small long-term gains and losses if $\kappa = 1$.[6] This is because if $\kappa = 1$, then the long-term status strictly dominates the short-term status since the investor can realize losses at the higher rate and gains at the lower rate when she has the long-term status.

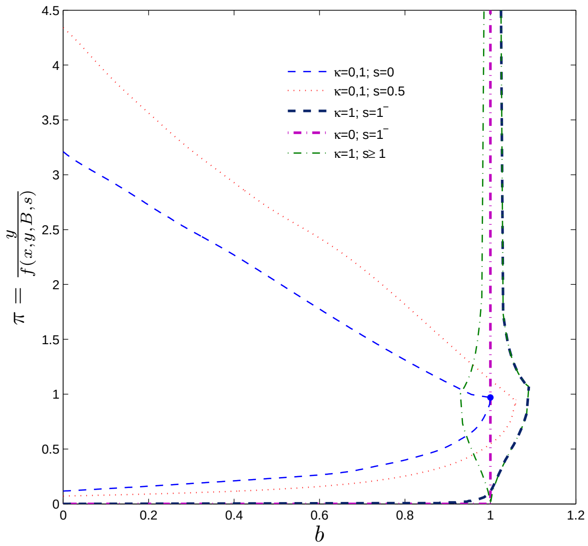
Figure 2(a) also suggests that if $\kappa = 1$, the optimal trading strategies for short-term status and long-term status are qualitatively different. An investor tends to defer the realization of large short-term capital gains, as reflected by the wider no transaction region when b is small and $s < 1$. In contrast, it is always optimal to immediately realize large long-term capital gains, as reflected by the fact all positions with a large capital gain are in the transaction region if $s \geq 1$. The main intuition for always immediately realizing a large long-term capital gain is that if it is not realized and stock price goes down, then effectively the investor realizes the incremental loss at the lower long-term rate because the investor still has a cumulative capital gain, whereas if it is realized and stock price goes down, then the investor can realize the incremental loss at the higher short-term rate.

In contrast to the full rebate case with asymmetric tax rates, Figure 2(b) shows that with full carry over, it is still optimal to immediately realize any losses as in the symmetric rate case. This is because the change of tax rates does not affect the benefit of carrying over the tax loss and adjusting the risk exposure.

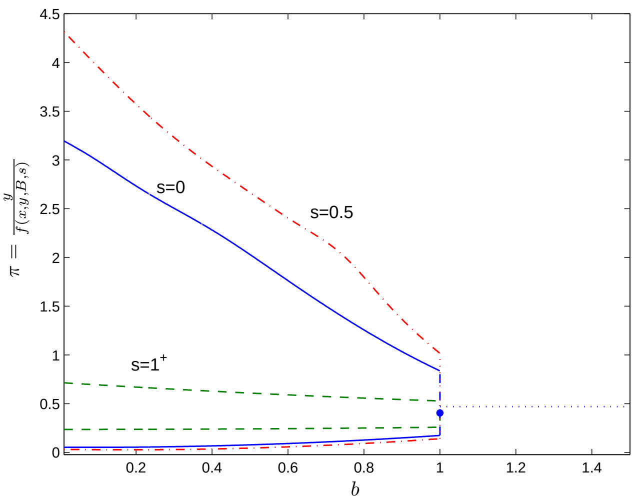
Also different from the full rebate case with asymmetric tax rates, it is not optimal to always realize large long-term capital gains ($s = 1^+$), although the no transaction region shrinks significantly for $s > 1$ because of the lower long-term capital gain tax rate. As discussed earlier under Figure 2(a), the benefit of realizing large long-term

[6] Since the tax rates no longer change for holding period beyond 1 year, the trading boundaries are the same for all $s \geq 1$.

Figure 2: Optimal trading boundaries against basis-price ratio b with asymmetric long-term/short-term rates.



(a) Full Rebate



(b) Full Carry Over

Parameter default values: $\omega = 0$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 1$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = \tau_d = 0.35$, $\tau_L = 0.15$, and $\tau_S = 0.35$.

capital gains is that in case the stock price declines, the investor can realize the loss at a higher short-term rate in the full rebate case. In the full carry over case, however, this benefit is absent and thus the investor prefers to defer capital gains, long-term or short-term, although the deferring region is smaller for long-term gains because of the lower tax rate.

4.2 Would higher tax rates make low income investors worse off?

The short-term tax rate is set to be equal to the marginal ordinary income tax rate applicable to the investor, while the long-term rate is independent of the income level as long as the ordinary tax rate is higher than the long-term rate. This implies that short-term rates applied to lower income investors are lower than those for higher income investors. Because investors can choose to realize gains at the long-term rate and losses at the higher short-term rate, this implies that low income investors may be better off with higher tax rates such as those for high income investors. We next examine whether higher tax rates can indeed make low income investors better off. To this end, we compute the equivalent wealth loss of a lower income investor for facing the implied lower short-term rate. More specifically, let $V_L(x, y, B, s)$ and $V_H(x, y, B, s)$ denote the value functions for a lower income investor with lower tax rates and a lower income investor with higher tax rates. Let Δ be the equivalent wealth loss (EWL, in terms of the fraction of the initial wealth) at time 0 of the lower income investor from the lower tax rates, with the initial wealth of W_0 all in the riskless asset, i.e.,

$$V_L(W_0, 0, 0, 0) = V_H((1 - \Delta)W_0, 0, 0, 0).$$

Table 1: Default Tax Rates for Figure 3 and Table 2

Tax Rate Levels	Before 2012				After 2012			
	τ_i	τ_d	τ_S	τ_L	τ_i	τ_d	τ_S	τ_L
Low	0.1	0	0.1	0	0.15	0.15	0.15	0.1
Midium	0.25	0.15	0.25	0.15	0.28	0.28	0.28	0.2
High	0.35	0.35	0.35	0.15	0.396	0.396	0.396	0.2

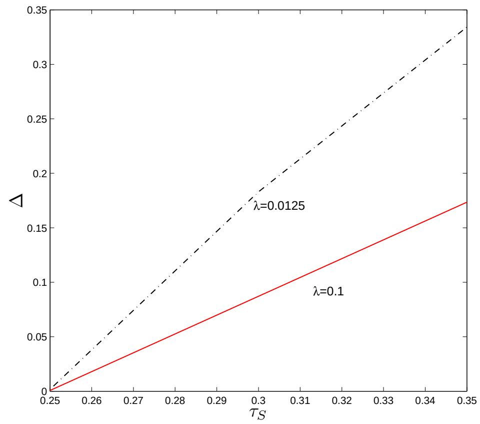
Because of the homogeneity of the value functions, Δ is independent of W_0 which can thus be set to 1.

The current tax schedules are set to expire at the end of 2012 and new schedules will start in 2013. Accordingly, we consider several cases for this comparison: three different sets of tax rates for before and after 2012 as described in Table 1. For convenience of reference, we call the rates for low income investors, "Low Rates," for medium income investors, "Medium Rates," for high income investors, "High Rates" and use the tax rates in Table 1 for the three levels. Because investors have CRRA preferences, the initial wealth is normalized to \$1 for *all* investors and therefore none of the results reported below comes from the difference in initial wealth levels and only comes from the difference in the tax rates they face.

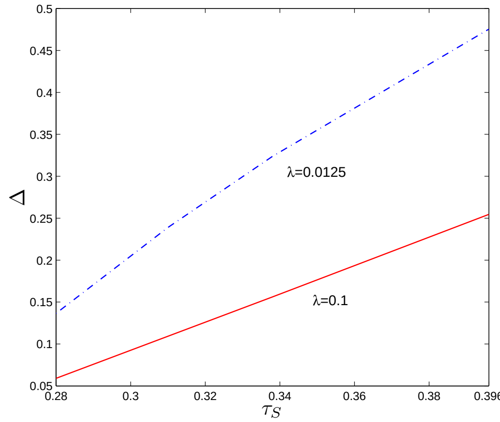
In Figure 3, we plot the EWLs of a low income investor with low and medium rates relative to with high rates against the short-term tax rate τ_S. Figure 3 shows that not only a low income investor can be better off from facing a higher short-term tax rate, but also the benefit a higher short-term tax rate can be very significant. For example, Figures 3 (a) and (b) show that an investor with low tax rates is willing to pay as much as 33% (before 2012) or 34% (after 2012) of his initial wealth to have the higher short-term tax rate of 35%. Figures 3 (c) and (d) show that an investor with medium tax rates is willing to pay as much as 28% (before 2012) or 24% (after 2012) of his initial wealth to have the higher short-term tax rate of 35%. As the ordinary income tax rate increases, the benefit from facing a higher short-term capital gains tax rate increases dramatically. Figure 3 implies that indeed raising capital gains tax rates can make low income investors significantly better off. Furthermore, Figure 3 also shows that even elder low income investors can also be significantly better off with higher tax rates. For example, when the expected time to death is only 10 years ($\lambda = 0.1$), the equivalent wealth loss of investors from the low tax rates relative to a high rate of 35% is as high as 17% of their initial wealth before 2012 and 16% after 2012.

To help evaluate the robustness of the results shown in Figure 3, we report in Table 2 how optimal initial target fractions of wealth invested in stock, initial consumption wealth ratios and EWLs of the investor with low, medium, and high rates (as defined in Table 1) change as we change a set of parameter values. Table 2 suggests that the significant welfare gain from higher tax rates is robust to these parameter value changes. For example, under the current tax code (before 2012), even when an elder investor with low rates (medium rates, respectively) has only an expected 5-year remaining lifetime ($\lambda = 0.2$), he is still willing to pay as much as 11.2% (column P-R) (9.5% (column M-R), respectively) of his current wealth to qualify for the higher 35% short-term rate. As the stock market volatility increases, the gain becomes even greater. For example, when the volatility increases to 30%, the EWL of an investor from low rates relative relative to high rates increases to 36.9% before 2012 and 49.8% after 2012. As the expected return μ of the stock increases, the EWLs decrease because the chance of realizing losses decreases. However, the decrease in EWLs is small. For example, a 50% increase in the expected return (from 0.04 to 0.06) only results in a 9.5% reduction in the EWL of an investor from low rates relative

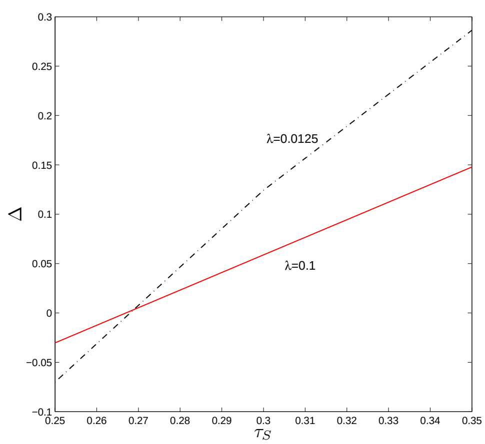
Figure 3: EWLs before and after 2012.



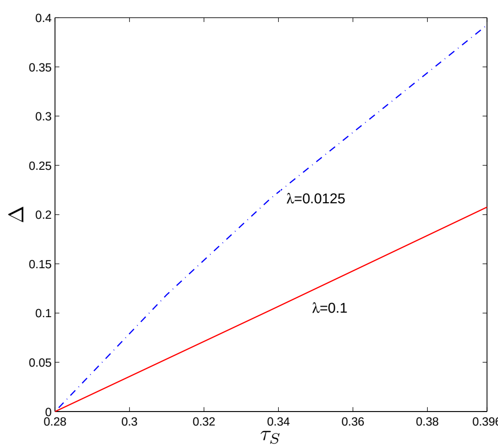
(a) Poor before 2012



(b) Poor after 2012



(c) Middle Class before 2012



(d) Middle Class after 2012

Parameter default values: $\omega = 0$, $\gamma = 3$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, and $\kappa = 1$; for the poor: before 2012, $\tau_i = 0.10$, $\tau_d = 0$, $\tau_L = 0$, $\tau_S = 0.10$, after 2012, $\tau_i = 0.15$, $\tau_d = 0.15$, $\tau_L = 0.1$, $\tau_S = 0.15$; for the middle class, before 2012, $\tau_i = 0.25$, $\tau_d = 0.15$, $\tau_L = 0.15$, $\tau_S = 0.25$, after 2012, $\tau_i = 0.28$, $\tau_d = 0.28$, $\tau_L = 0.20$, $\tau_S = 0.28$; for the higher income investor, $\tau_i = \tau_S$, $\tau_d = \tau_S$, before 2012, $\tau_L = 0.15$, after 2012, $\tau_L = 0.2$.

to high rates (column P-R) before 2012. As the dividend yield decreases, the EWLs increase because the disadvantage of higher income investors from paying higher tax rates on dividend payment decreases. This suggests that lower income investors prefer stocks with high dividend yield and higher income investors prefer stocks with low dividend yield. As risk aversion increases from 3 to 5, the optimal stock investment decreases, the optimal consumption increases, and the EWLs also decrease, but only slightly. With an expected time to death of 80 years, the tax forgiveness at death has almost no impact on the EWLs and optimal consumption and investment, as shown by the row with $\iota = 1$. So our main results also apply to investors in Canada where capital gains tax is not forgiven at death.

An investor with a higher short-term rate pays more tax when she realizes a short-term gain. This is why as shown in Figures 1 and 2, the no-realization region for an investor with a short-term gain and a higher short-term rate is wider than an investor with a lower short-term rate. However, investors may be forced to liquidate due to liquidity reasons. To examine the impact of this forced liquidation on the EWLs of an investor from low rates relative to high rates, we report the results when an investor is forced to liquidate the entire stock position due to a large liquidity shock that occurs at the intensity of $\lambda = 0.5, 1$, i.e., $\iota = 1$. Indeed the EWLs are much smaller because the investor has to realize short-term gains more often and the expected investment horizon is much shorter. For example, when the large liquidity shock occurs once a year on average ($\lambda = 1$), the EWL of an investor from low rates relative to high rates (column P-R) before 2012 decreases to 2.8% of the initial wealth. On the other hand, since the expected investment horizon is now only 1 year, the 2.8% loss is still quite economically significant. In addition, to bias against us, we assume that the large liquidity shock occurs once a year and when the liquidity shock occurs, the investor must liquidate the entire position and thus likely realize more gains than necessary for most liquidity needs in practice. These findings suggest that liquidity induced capital gain realization can unlikely eliminate or reverse the bias.

Table 2 also shows that an investor with higher tax rates invests more and consumes more.

To summarize, we show that low income investors can be significantly better off with higher tax rates like those faced by high income investors and raising their tax rates would make them invest more and consume more.

4.3 The value of deferring capital gains realization

Because tax is only paid when it is realized, an investor has the option to defer capital gains tax and immediately realize capital losses. When long-term/short-term tax rates are the same, the value of this deferring comes only from earning the interest on the capital gains tax, which is why as we show in Proposition 1, it would be optimal not to defer capital gains tax if the interest rate were zero. When long-term rates are lower than short-term rates, the value of this deferring can also come from the benefit of realizing gains at a lower long-term rate. We next decompose the value of deferring into these two sources to compare their relative magnitude. More specifically, let $\underline{V}(x, y, B, s; \tau_S, \tau_L)$ be the value function when the investor cannot defer capital gains realization, i.e., is forced to realize both gains and losses continuously (and thus short-term). Let $V(x, y, B, s; \tau_S, \tau_S)$ be the value function when the investor can defer capital gains realization, but long-term rates are equal to the short-term rates, and $V(x, y, B, s; \tau_S, \tau_L)$ be the value function when the investor can defer capital gains realization and long-term rates are lower than the short-term rates. We use the time 0 EWLs (again in terms of the fraction of the initial wealth) Δ_0 and Δ_1 to measure the values of deferring from these two sources respectively, assuming all the initial wealth is in the risk free asset, i.e.,

$$\underline{V}(1, 0, 0, 0; \tau_S, \tau_L) = V(1 - \Delta_0, 0, 0, 0; \tau_S, \tau_S)$$

and

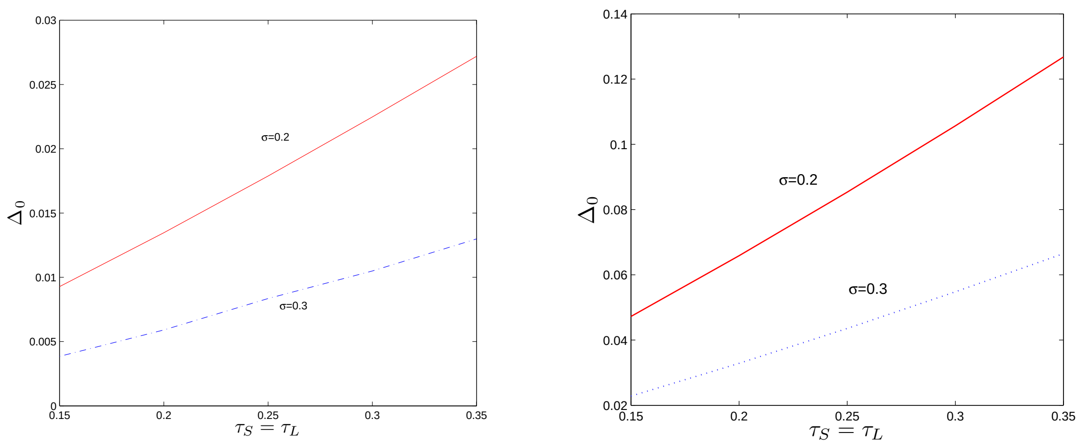
$$V(1, 0, 0, 0; \tau_S, \tau_S) = V(1 - \Delta_1, 0, 0, 0; \tau_S, \tau_L).$$

Figure 4 plots the equivalent wealth loss Δ_0 for two volatility levels $\sigma = 0.2$ and $\sigma = 0.3$ for both the full rebate and the full carry over cases. As expected, this figure shows that as the single tax rate increases, the value of deferring from saving the interest on tax increases, because of the increase in the capital gains tax. The EWL magnitude varies from 0.81% to 2.6% of the initial wealth for the full rebate case and from 2.1% to 12.7% of the initial wealth for the full carry over case. As the stock volatility increases, this value from the interest saving decreases. This is because with a higher volatility, the investor invests less in the stock and thus the dollar amount of the capital gains tax deferred decreases.

Figure 5 plots the equivalent wealth loss Δ_1 for two volatility levels $\sigma = 0.2$ and $\sigma = 0.3$. This figure shows that as the short-term tax rate increases, the value of deferring from realizing gains at a lower rate increases significantly, because of the increase in the difference between the long-term rate and the short-term rate. The value of deferring from realizing gains at a lower rate is much greater than that from saving interest. For the full rebate case, the EWL magnitude can be as high as 48% of the initial wealth, in contrast to 2.6% for the benefit of interest saving. Therefore, by ignoring the difference between the long-term and short term rates, most of the existing literature significantly underestimates the value of deferring capital gains tax and largely overestimates the effective tax rates. Similar to Figure 4, the value of deferring from realizing gains at a lower rate also decreases with volatility because of the reduced investment in the stock.

In contrast, for the full carry over case, the EWL Δ_1 is much smaller because realizing losses short-term does not provide any additional benefit compared to realizing them long-term even when short-term rates are much higher. Also different from the full carry over case, an increase in the volatility increases the EWL.

Figure 4: The value of deferring capital gains tax from saving the interest.



(a) Full Rebate

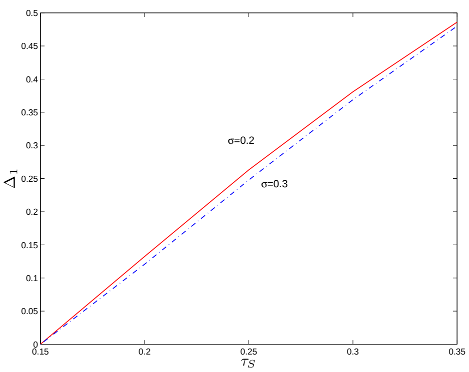
(b) Full Carry Over

Parameter default values: $\omega = 0$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 1$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = \tau_d = \tau_S$, $\tau_L = \tau_S$, and $\kappa = 1$.

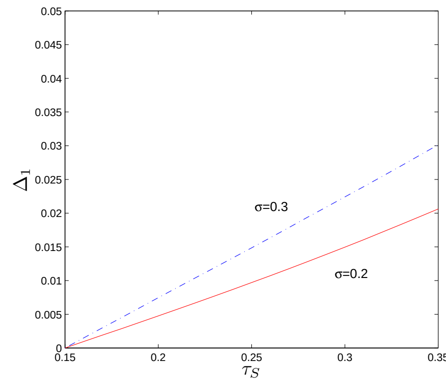
4.4 The initial portfolio fraction target and consumption

Figure 6 plots the optimal initial portfolio fraction target π^* at $b = 1$ against short term tax rate for two different long-term tax rates: $\tau_L = 0.15$ and $\tau_L = 0.2$. For the full rebate case, as the short-term tax rate increases, the optimal target significantly increases. Recall that without any tax, it is optimal to keep 50% in the stock. Figure 6 shows that the presence of a higher short-term tax rate can significantly increase stock investment (e.g., at $\tau_S = 0.35$). This is because the after-tax risk (variance) of

Figure 5: The value of deferring capital gains tax from realizing gains at the lower long-term rate.



(a) Full Rebate



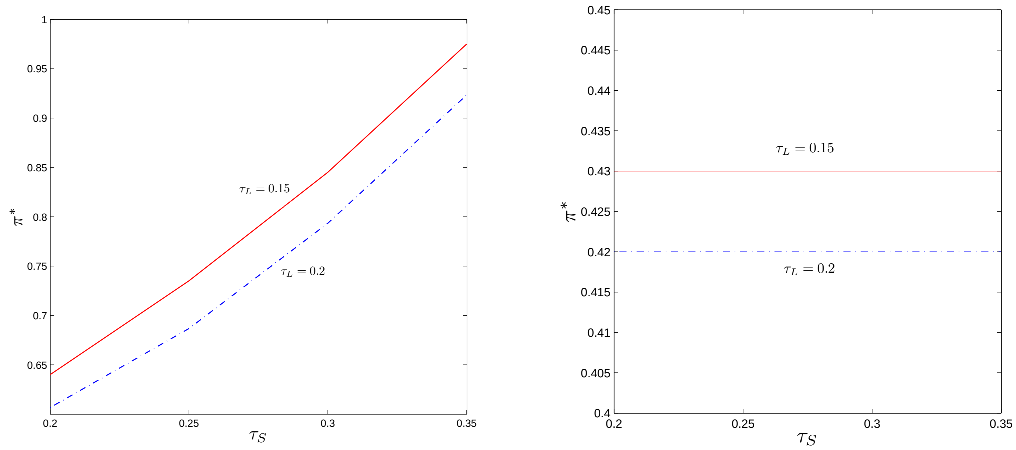
(b) Full Carry Over

Parameter default values: $\omega = 0$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 1$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = \tau_d = \tau_S$, $\tau_L = \tau_S$, and $\kappa = 1$.

investing in the stock is reduced more than the after-tax expected return. With a higher short-term tax rate, the effective loss from stock investment is further reduced and thus the stock becomes more attractive. After 2012, the long-term tax rate will be raised to 20%. Figure 6 shows that increasing the long-term rate can significantly reduce stock investment. For example, at $\tau_S = 0.35$, the 5% increase in the long-term tax rate results in more than 10% reduction in the stock investment. Similar intuition suggests that the consumption also increases with the short term tax rate, as shown in Figure 7.

In contrast, for the full carry over case, both the initial target and the initial consumption are insensitive to the changes in the short-term tax rate. This is because the investor rarely realizes short-term capital gains and with full carry over, short-term rates have no direct impact on the realization of capital losses.

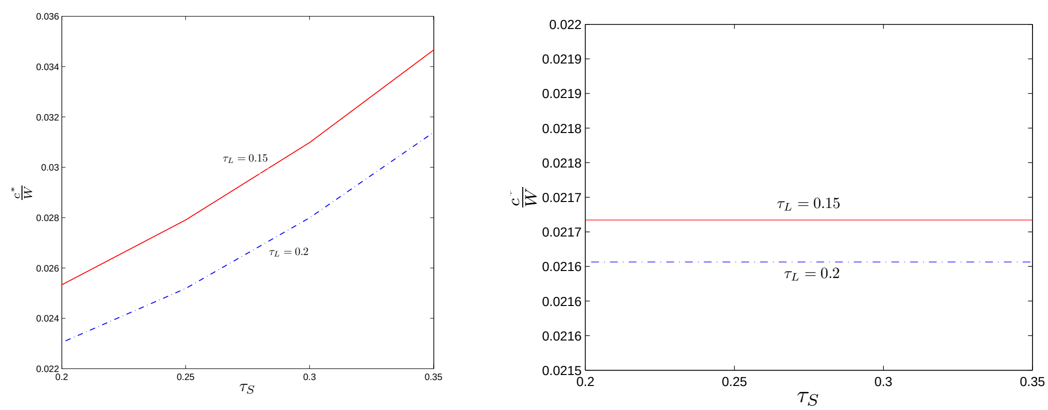
Figure 6: Optimal initial fraction of wealth in stock against the short-term tax rate.



(a) Full Rebate (b) Full Carry Over

Parameter default values: $\omega = 0$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = 0.35$, $\tau_d = 0.15$, and $\kappa = 1$.

Figure 7: Optimal initial consumption to wealth ratio against the short-term tax rate.



(a) Full Rebate (b) Full Carry Over

Parameter default values: $\omega = 0$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = 0.35$, $\tau_d = 0.15$, and $\kappa = 1$.

4.5 $\kappa = 0$ versus $\kappa = 1$

In the limited literature that considers asymmetric long-term/short-term tax rates, it is assumed that the long-term tax rate for capital losses is the same as the long-term rate for capital gains, corresponding to $\kappa = 0$ in our model. However, under the current tax code the long-term tax rate for losses is the same as the short-term tax rate, corresponding to $\kappa = 1$ in our model. Since the short-term rate is higher than the long-term rate, the existing literature underestimates the value of deferring in the full rebate case.[7] For the full rebate case, Figure 8 plots the equivalent wealth loss Δ_2 from $\kappa = 1$ to $\kappa = 0$ against the short-term tax-rate for two risk aversion levels: $\gamma = 2$ and $\gamma = 3$, where Δ_2 solves

$$V(1, 0, 0, 0; \kappa = 0) = V(1 - \Delta_2, 0, 0, 0; \kappa = 1).$$

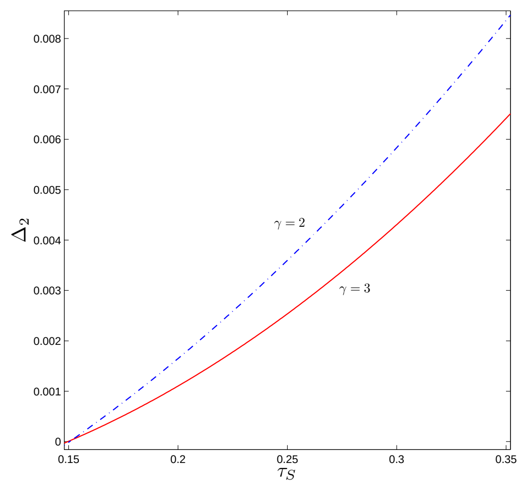
This figure shows that the EWL is much smaller than the EWL from the asymmetric rates case to the single tax rate as shown in Figure 4, but can still be significant. For example, if the short-term rate is equal to 0.35, the EWL is about 0.66%. For an investor with a relative risk aversion coefficient of 2, the EWL can be as high as 0.90%.

4.6 Simulated sample paths

To keep a single tax basis, we assume that if an investor needs to buy more stock, she must first liquidate the entire stock position before buying. This assumption is not as restrictive as it may appear. Intuitively, if there is a capital loss, then it is optimal to sell the entire stock position to realize the capital loss earlier. The only case this assumption might be restrictive is when there is a capital gain and the investor wants to buy more of the stock. This happens when the fraction of wealth invested in the stock moves downward and reaches the buy boundary. However, since the stock has a higher expected return than the risk free asset and intertemporal consumption

[7] Clearly, different rates for long-term losses have no impact for the full carry over case.

Figure 8: The equivalent wealth loss from $\kappa = 1$ to $\kappa = 0$ against tax-rate τ_S.



Parameter default values: $\omega = 0$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = 0.35$, $\tau_d = 0.15$, and $\tau_L = 0.15$.

withdrawn from the risk free asset account also tends to move the fraction of wealth invested in the stock upward, the likelihood of buying when there is a capital gain is low.

To verify this intuition, we simulated 1000 sample paths of the π and b for a 20 year horizon (2500 trading dates per path) and report the average (across sample paths) number of transactions in these 20 sample paths in Tables 3 and 4. We use nbuy and nsell to denote respectively the average number of transactions involving first liquidating the entire position and then rebalancing to the optimal position and those involving selling a fraction of the current position. We also note if a transaction is a realization of a loss ($b \geq 1$) or a gain ($b < 1$). For these 1000 sample paths and the full rebate case, Table 3 implies that there are 137.43 average number of transactions for a 20 year horizon, 107.11 of which involve first liquidating the entire position and then rebalancing occur to realize *capital losses* (short-term 106.03, long-term 1.08), which is likely optimal even without the assumption of a single tax basis. Out of 2500 possible trading dates for each path, the investor liquidates her entire position to realize capital gains only on an average of 10.11 dates. More importantly, these capital gains are all long-term. Realizing all large long-term capital gains to reestablish the short-term status is also likely optimal even without the assumption of a single tax basis, because of the benefit of higher short-term rate for incremental losses, as explained below Figure 2. Therefore the impact of the assumption of a single tax basis is small and unlikely affects our results significantly. For the full carry over case, Table 4 shows similar patterns.

Table 3: The average number of transactions for 1000 sample paths, Full Rebate

holding period	nbuy ($b \geq 1$)	nbuy ($b < 1$)	nsell ($b \geq 1$)	nsell ($b < 1$)
$s < H$	106.03	0	19.17	1.04
$s \geq H$	1.08	10.11	0	0

Parameter default values: $\omega = 0$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.3$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = 0.35$, $\tau_d = 0.35$, $\tau_S = 0.35$, $\tau_L = 0.15$ and $\kappa = 1$.

Table 4: The average number of transactions for 1000 sample paths, Full Carry Over

holding period	nbuy ($b \geq 1$)	nbuy ($b < 1$)	nsell ($b \geq 1$)	nsell ($b < 1$)
$s < H$	462.95	0	447.10	3.52
$s \geq H$	1.05	0.07	0.17	41.39

Parameter default values: $\omega = 0$, $\gamma = 3$, $\lambda = 0.0125$, $\beta = 0.01$, $\iota = 0$, $H = 1$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.3$, $\alpha = 0.9$, $\delta = 0.02$, $\tau_i = 0.35$, $\tau_d = 0.35$, $\tau_S = 0.35$, $\tau_L = 0.15$ and $\kappa = 1$.

5. Conclusions

The optimal trading strategy with asymmetric long-term/short-term tax rates can be significantly different from that with a single tax rate. In addition, the impact of capital gain tax on a less wealthy investor can be qualitatively different from on a wealthy investor. For example, in contrast to the standard literature, we show that for a less wealthy investor it can be optimal to defer capital losses beyond one year even in the absence of transaction costs and wash sale restriction. In addition, low income investors can be significantly better off with higher tax rates such as those for wealthy investors. Moreover, raising the short-term tax rate can increase both consumption and stock investment for a low income investor.

The existing literature assumes that the tax rate for long-term capital losses is the same as that for long-term capital gains, instead of the same as the marginal ordinary income tax rate as the law stipulates. Under this assumption, it is always optimal to realize all short-term capital losses before they turn into long-term. In contrast, for low income investors, we show that it can be optimal to defer long-term capital losses, no matter how high the short-term tax rate is compared to the long-term rate.

In contrast, for an investor whose majority of capital losses can only be carried over (e.g., wealthy investors), capital gain tax always makes him worse off and he is willing to hold a tax-exempt security with a much lower expected return. In addition, an increase in tax rates tends to decrease the optimal target stock holdings of a wealthy investor and it can be optimal for him to defer even large long-term capital gains.

To conclude, our paper shows that the impact of capital gain tax on the optimal trading strategy critically depends on the difference between long-term and short-term tax rates and whether capital loss rebate or carry-over is more representative of the capital loss treatment.

APPENDIX

In this Appendix, we provide the proof for Proposition 1.

A.1 Proof of Proposition 1

Let

$$W_t = x_t + y_t - \tau_S (y_t - B_t).$$

It is easy to verify

$$dW_t = [(1-\tau_i) r x_t - c_t + ((1-\tau_S)\mu + (1-\tau_d)\delta) y_t] dt + (1-\tau_S)\sigma y_t dw_t. \quad \text{(A-1)}$$

Because the investor always liquidates the entire stock holdings to maintain a constant fraction of wealth in stock, it follows $y_t = B_t$, $x_t = W_t - y_t$. So, problem (11) reduces to a classical Merton's consumption-investment problem with interest rate $(1-\tau_i) r$ and wealth process following (A-1) and stock prices following

$$\frac{dP_t}{P_t} = [(1-\tau_S)\mu + (1-\tau_d)\delta] dt + (1-\tau_S)\sigma dw_t.$$

It is well-known that Merton problem's optimal consumption and investment strategy is

$$\begin{cases} \frac{c_t^*}{W_t} = \alpha^{1/\gamma} v^{-(1-\gamma)/\gamma}, \\ \frac{y_t^*}{W_t} = \frac{(1-\tau_S)\mu + (1-\tau_d)\delta - (1-\tau_i) r}{\gamma (1-\tau_S)^2 \sigma^2}, \end{cases}$$

which gives a value function $\frac{(vW)^{1-\gamma}}{1-\gamma}$. Notice that Merton's optimal strategy is admissible to problem (11), then these two value functions coincide. It is easy to show that (16) has a unique positive root.

If $r = 0$ and $\tau_S = \tau_L = \tau$, then for any allowable strategy, we always have

$$dW_t = -c_t dt + y_t [((1-\tau)\mu + (1-\tau_d)\delta) dt + (1-\tau)\sigma dw_t], \quad \text{(A-2)}$$

which is independent of the tax basis B_t. Since the dynamics of y_t in (3) is independent of B_t, we again obtain a Merton-like problem, and the resulting Merton's strategy must be optimal. The proof is complete.

A.2 Convergence of the algorithm

By step 3 of the algorithm,

$$\Phi(k,1,0) \leq \frac{(M_{i+1}(k+1))^{1-\gamma}}{1-\gamma} \text{ for all } k \in (-1,+\infty). \tag{A-3}$$

From (20), we infer

$$\Phi(k,1,0) \geq \frac{(M_i f(k,1,b,0))^{1-\gamma}}{1-\gamma} = \frac{(M_i(k+1))^{1-\gamma}}{1-\gamma}, \tag{A-4}$$

for all $k \in (-1,+\infty)$. Combination of (A-3) and (A-4) gives

$$M_i \leq M_{i+1}.$$

Hence, $\{M_i\}_{i=1,2,\ldots}$ is a monotonically increasing sequence. It remains to find an upper bound of the sequence.

Let $\Phi_0(z,b,s)$ be the solution to the HJB equation (15), as given in Proposition 2. We will prove

$$M_i \leq \left((1-\gamma)\sup_{k\in(-1,+\infty)}(k+1)^{\gamma-1}\Phi_0(k,1,0)\right)^{1/(1-\gamma)}$$

or equivalently,

$$\frac{M_i^{1-\gamma}}{1-\gamma} \leq \sup_{k\in(-1,+\infty)}(k+1)^{\gamma-1}\Phi_0(k,1,0) \tag{A-5}$$

for all i. Clearly it is true for $i=0$ because Φ_0 corresponds to the value function and M_0 is only associated with a suboptimal strategy. Suppose (A-5) is true for i. Then

$$\frac{(M_i f(z,1,b,s))^{1-\gamma}}{1-\gamma} \leq f(z,1,b,s)^{1-\gamma}\sup_{k\in(-1,+\infty)}(k+1)^{\gamma-1}\Phi_0(k,1,0).$$

Applying the maximum principle (cf. [15]) to the problem (18)-(21), we have

$$\Phi\left(z,b,s\right) \leq \Phi_0(z,b,s) \text{ for all } z,b,s.$$

Therefore,

$$\begin{aligned} \sup_{k\in(-1,+\infty)} (k+1)^{\gamma-1}\Phi_0\left(k,1,0\right) &\geq \sup_{k\in(-1,+\infty)} (k+1)^{\gamma-1}\Phi\left(k,1,0\right) \\ &= \frac{M_{i+1}^{1-\gamma}}{1-\gamma}, \end{aligned}$$

which is desired.

A.3 The carry-over case

Consider the carry-over case $\omega = 0$. First we claim that incessant trading is necessary for $B > y$ and $s = 0$. Indeed, for $B > y$ and $s = 0$

$$\begin{aligned} V\left(x,y,B,0\right) &\geq V\left(f\left(x,y,B,0\right) - I, I, I + (B-y)^+, 0\right) \\ &= V\left(x+y-I, I, I+B-y, 0\right). \end{aligned}$$

Denote

$$I = y + \delta,\ \ \delta > 0.$$

Then we have

$$V\left(x,y,B,0\right) \geq V\left(x-\delta, y+\delta, B+\delta, 0\right),$$

from which we deduce

$$-V_x + V_y + V_B \leq 0,\ \text{ for } B > y.$$

On the other hand, we always have

$$yV_x - yV_y - yV_B \leq 0,\ \text{ for } B > y.$$

So,

$$V_x - V_y - yV_B = 0 \text{ for } B > y,$$

which implies incessant trading for $B > y$.

Now let us derive an equivalent partial different equation by which we can numerically solve for $V(x, y, B, 0)$, $B > y$.

Define

$$\begin{aligned} x_t + B_t &= Z_t \\ x_t + y_t &= W_t. \end{aligned}$$

We will restrict attention to $Z_t > W_t$ which is equivalent to $B_t > y_t$. It is easy to verify

$$\begin{aligned} dW_t &= [(1-\tau_i)rW_t - c_t + (\mu + (1-\tau_d)\delta - (1-\tau_i)r)\, y_t]\, dt + \sigma y_t dw_t, \\ dZ_t &= [(1-\tau_i)rx + (1-\tau_d)\delta y - c_t]\, dt \\ &= [(1-\tau_i)rW_t + ((1-\tau_d)\delta - (1-\tau_i)r)y_t - c_t]\, dt, \text{ for } y_t < B_t. \end{aligned}$$

Consider the value function

$$J(W, Z) = \max_{c,y} E\left[\int_0^\infty e^{-\beta t} u(c_t) dt | W_0 = W, Z_0 = Z\right]$$

in $B_t > y_t$, which is governed by

$$\begin{aligned} &\max_{c,y}\left\{u(c) - \beta J + \frac{1}{2}\sigma^2 y^2 J_{WW}\right. \\ &+ [(1-\tau_i)rW - c + (\mu + (1-\tau_d)\delta - (1-\tau_i)r)\, y]\, J_W \\ &\left. + [(1-\tau_i)rW + ((1-\tau_d)\delta - (1-\tau_i)r)y - c]\, J_Z\right\} = 0 \end{aligned}$$

in $0 < W < Z$.

By using the transformation

$$J(W,Z)=W^{1-\gamma}J\left(1,\frac{Z}{W}\right)=W^{1-\gamma}\zeta\left(\eta\right),\ \eta=\frac{W}{Z},$$

we obtain

$$L\zeta=0 \quad \text{in} \quad 0<\eta<1, \tag{A-6}$$

where

$$\begin{aligned}
& L\zeta \\
= \;& U(c^*)+\frac{1}{2}\sigma^2\pi^{*2}\eta^2\zeta_{\eta\eta} \\
+ \;& \left\{\left[(1-\tau_i)\,r\,(1-\pi^*)+(1-\tau_d)\,\delta\pi^*-c^*\right](1-\eta)+\mu\pi^*+\sigma^2\pi^{*2}\,(1-\gamma)\right\}\eta\zeta_\eta \\
+ \;& \left\{\left[(1-\tau_i)\,r\,(1-\pi^*)+(\mu+(1-\tau_d)\,\delta)\,\pi^*-\frac{1}{2}\gamma\sigma^2\pi^{*2}-c^*\right](1-\gamma)-\beta\right\}\zeta,
\end{aligned}$$

$$c^*=\left[(1-\eta)\,\eta\zeta_\eta+(1-\gamma)\,\zeta\right]^{-1/\gamma},$$

and

$$\pi^*=\frac{\left[(1-\tau_d)\,\delta-(1-\tau_i)\,r\right]\eta^2\zeta_\eta-\left[\mu+(1-\tau_d)\,\delta-(1-\tau_i)\,r\right]\left[\eta\zeta_\eta+(1-\gamma)\,\zeta\right]}{\eta^2\zeta_{\eta\eta}+2\,(1-\gamma)\,\eta\zeta_\eta-\gamma\,(1-\gamma)\,\zeta}.$$

One boundary condition is

$$\zeta(0) \quad = \quad \text{Merton solution with scaled wealth.} \tag{A-7}$$

The solution is unique provided that $\zeta(1)$ is given. This motivates us to present the following numerical algorithm.

The algorithm of finding the minimum solution numerically $(\omega=1)$**:**

1. Set

$$M_0=\text{initial guess}.$$

2. For given M_i, solve (A-6) for ζ_i with (A-7) and $\zeta(1)=M_i$. . Then we denote

$$\Phi_i\left(z,b,0\right)=\left(z+1\right)^{1-\gamma}\zeta_i\left(\frac{z+1}{z+b}\right).$$

3. By using $\Phi_i(z,b,0)$, solve

$$\max\left\{\mathcal{L}_1\varphi, G\left(z,b,H;\Phi_i\right)-\varphi(z,b),\right.$$
$$\left.-\left(1-\gamma\right)\varphi+f\left(z,1,b,s\right)\varphi_z+\left(b-1\right)^+\varphi_b=0\right\}, \quad s>H.$$

$$\max\left\{\Phi_s+\mathcal{L}_1\Phi, \quad G\left(z,b,s;\Phi_i\right)-\Phi(z,b,s),\right.$$
$$\left.-\left(1-\gamma\right)\Phi+f\left(z,1,b,s\right)\Phi_z+\left(b-1\right)^+\Phi_b\right\}=0, \quad s<H.$$

with the terminal condition at $s=H$:

$$\Phi(z,b,H)=\varphi(z,b),$$

where

$$G\left(z,b,s;\Phi_i\right)=f\left(z,1,b,s_0\right)^{1-\gamma}\sup_{k>-1}\left(k+1\right)^{\gamma-1}\Phi_i\left(k,1+\left(k+1\right)\frac{\left(b-1\right)^+}{f\left(z,1,b,s\right)},0\right).$$

4. Set
$$M_{i+1}=\left(\left(1-\gamma\right)\sup_{k>-1}\left(k+1\right)^{\gamma-1}\Phi\left(k,1,0\right)\right)^{1/(1-\gamma)};$$

5. If $M_{i+1}=M_i$, then stop, otherwise $M_i=M_{i+1}$ and go to step 2.

References

[1] I. Ben Tahar, H. M. Soner, and N. Touzi, The dynamic programming equation for the problem of optimal investment under capital gains taxes, *SIAM Journal of Control and Optimization* 46 (2007) 1779–1801.

[2] I. Ben Tahar, H. M. Soner, and N. Touzi, Merton problem with taxes: characterization, computation, and approximation, *SIAM Journal of Finanancial Mathematics* 1 (2010) 366–395.

[3] G. M. Constantinides, Capital market equilibrium with personal taxes, *Econometrica* 51 (1983) 611–636.

[4] G. M. Constantinides, Optimal stock trading with personal taxes: Implication for prices and the abnormal January returns, *Journal of Financial Economics* 13 (1984) 65–89.

[5] G. M. Constantinides, Capital market equilibrium with transaction costs, *Journal of Political Economy* 94 (1986) 842–862.

[6] M. Dai, and Y. Zhong, Penalty methods for continuous-time portfolio selection with proportional transaction costs, *Journal of Computational Finance* 13 (2010) 1–31.

[7] Z. Dai, E. Maydew, Douglas A. Shackelford, and H. H. Zhang, Capital gains taxes and asset prices: Capitalization or lock-in? *Journal of Finance* 63 (2008) 709–742.

[8] R. M. Dammon, and C. S. Spatt, The optimal trading and pricing of securities with asymmetric capital gains taxes and transaction costs, *Review of Financial Studies* 9 (1996) 921–952.

[9] R. M. Dammon, C. S. Spatt, and H. H. Zhang, Optimal consumption and investment with capital gains taxes, *Review of Financial Studies* 14 (2001) 583–616.

[10] R. M. Dammon, C. S. Spatt, and H. H. Zhang, Optimal asset location and allocation with taxable and tax-deferred investing, *Journal of Finance* 59 (2004) 999-1037

[11] M.H.A. Davis, and A.R. Norman, Portfolio selection with transaction costs, *Mathematics of Operations Research* 15 (1990) 676–713.

[12] V. DeMiguel, and R. Uppal, Portfolio investment with the exact tax basis via nonlinear programming, *Management Science* 51 (2005) 277–290.

[13] P. Dybvig, and H. Koo, Investment with Taxes, Working paper, Washington University in St. Louis, 1996.

[14] W.H. Fleming, and H.M. Soner, Controlled Markov Processes and Viscosity Solutions, New York, Springer, 1993.

[15] A. Friedman, Variational Principles and Free-boundary Problems, Wiley, New York, 1982.

[16] M. Gallmeyer, R. Kaniel, and S. Tompaidis, Tax management strategies with multiple risky assets, *Journal of Financial Economics* 80 (2006) 243–291.

[17] J. R. Graham, Taxes and corporate finance: a review, *Review of Financial Studies* 16 (2003) 1075–1129.

[18] J. Huang, Taxable and tax-deferred investing: a tax-arbitrage approach, *Review of Financial Studies* 21 (2008) 2173–2207.

[19] B. Jang, H. Koo, H. Liu, and M. Loewenstein, Liquidity premia and transaction costs, *Journal of Finance* 62 (2007) 2329–2366.

[20] H. Liu, Optimal consumption and investment with transaction costs and multiple risky assets, *Journal of Finance* 59 (2004) 289–338.

[21] H. Liu, and M. Loewenstein, Optimal portfolio selection with transaction costs and finite horizons, *Review of Financial Studies* 15 (2002) 805–835.

[22] H. E. Leland, Optimal portfolio management with transaction costs and capital gains taxes, preprint, 1999.

[23] R. C. Merton, Lifetime portfolio selection under uncertainty: The continuous-time model, *Review of Economic Statistics* 51 (1969) 247–257.

[24] R.C. Merton, Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* 3 (1971) 373–413.

[25] B. Øksendal, and A. Sulem, Optimal consumption and portfolio with both fixed and proportional transaction costs, *SIAM Journal of Control and Optimization* 40 (2002) 1765–1790.