Dynamic Asset Allocation, Heterogeneity of Preferences, Cross Section of Volatility, Skewness, and Expected Returns: Theory and Evidence

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Abstract

This article studies equilibrium asset pricing when agents have heterogeneous preferences in a multi-period economy. It is shown that the economy's stochastic discount factor (hereafter SDF) is a function of the market return, the squared market return, the cubic market return, the market aggregate volatility risk, the market aggregate skewness risk, the market aggregate correlation risk, the market-mimicking aggregate volatility product, and the market-mimicking squared market product. The SDF has a structural interpretation in terms of the distribution of investor preferences. This leads to an asset pricing model in which the expected return on risky assets depends explicitly on the asset's coskewness, cokurtosis, and covariance with the aggregate volatility risk, aggregate skewness risk, aggregate correlation risk, and other higher-order related moments. Empirical results indicate that the risk factors appearing in the SDF have a significant impact on assets' risk premia. Their influence depends on the cross-sectional variance of investor risk aversions, the cross-sectional variance of investor skewness preferences, and the cross-sectional covariance of investor risk aversions with skewness preferences.

JEL Classification: G0, G1, C2.

Keywords: asset pricing, coskewness, cokurtosis, aggregate volatility risk, aggregate skewness risk, correlation risk, risk aversions, skewness preferences.

1 Introduction

As Wang (2003) notes, asset pricing models particularly the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965) and its multifactor extensions have been a key element of finance. The multifactor extensions of the CAPM, to name a few, include the Harvey and Siddique (2000) market coskewness model, Dittmar's (2002) market cokurtosis model, Chen's (2000) intertemporal CAPM model, and more recently, the Ang et al. (2006) aggregate market volatility risk model.[1] The usual approach is to assume the existence of a representative agent, then use the Taylor expansion series of the representative agent marginal utility, and then drop all the higher terms that the researchers believe are unimportant for explaining the expected excess return of the risky assets. Harvey and Siddique use the third order expansion of the representative investor marginal utility to derive an asset pricing model in which assets' risk premia depend on the CAPM beta and an additional component refereed to as coskewness (the beta with respect to the squared market return) risk premium. Dittmar (2002) goes beyond the third order expansion of investor marginal utility and derives an asset pricing model in which both coskewness and cokurtosis are priced. Ang et al. and Boehme et al. (2007) investigate whether the aggregate volatility risk factor is priced. They empirically show that the aggregate volatility risk is priced and its price is negative. Ang et al. use Chen's results to support the theoretical justification of their empirical model. However, Chen considers an economy in which the SDF is derived from the Epstein and Zin (1989, hereafter EZ) asset pricing model.[2] He then uses a log-linear approximation of the representative investor aggregate budget constraint combined with the EZ SDF to derive an asset pricing model in which the asset's expected excess return depends on the change in the exponentially weighted forecasts of future market variance. While the EZ SDF is consistent with an underlying equilibrium model, Chen's approximation could not be. Approximating the SDF of the representative investor's marginal utility leads to a SDF that is not necessarily consistent with the optimization behavior of investors. Boehme et al. construct a model where returns are multivariate normal but investors are uncertain about the first and second moments of the joint distribution of returns and signals. They assume heterogeneous investors with

[1] Boehme et al. (2007) also derive a conditional CAPM in which the aggregate volatility risk is priced.

[2] Note that the Epstein and Zin SDF is a nonlinear function of consumption growth and market return and does not depend on the market aggregate volatility.

exponential utility function and arrive endogenously at a conditional CAPM where the market risk-premium, the market volatility, and the systematic risk of individual assets are all sensitive to information that affects the market's conditional covariance of expected return.

The SDFs derived in these papers have no structural interpretations in terms of investor preferences. More specifically, these studies leave unanswered the question of whether or how the distribution of investor preferences affects the SDF in equilibrium. Moreover, there is no a unified framework that delivers a SDF in equilibrium which can be viewed as an extension of both Harvey and Siddique, Dittmar, and Ang et al. asset pricing models. In addition, multifactor extensions of the CAPM model do not derive the assets' risk premia in an equilibrium model. They leave unanswered the question of whether their findings are consistent with an underlying equilibrium model. For example, Fama (1991) and more recently Cochrane (2001) criticize the use of the intertemporal CAPM model as a "fishing license" to allow a variety of factors into a SDF without verifying whether such factors are consistent with the optimization behavior of investors. The purpose of this paper is to close these gaps.

We go beyond the representative agent utility models by allowing heterogeneity of preferences among agents. In particular, without making any assumption about the functional form of investor utility functions and the distribution of asset returns, we provide a general framework that maps heterogeneity of preferences into the SDF. Hence, the first contribution of our paper is to provide a structural interpretation to the SDF involving higher moments. Our SDF is a function of deep structural parameters namely the average value of investor risk-tolerances (inverse of the Arrow-Pratt measure of investor risk aversions), skew-tolerances and kurtosis-tolerances, the cross-sectional variances of investor risk- and skew-tolerances, and finally the cross-sectional covariance of investor risk-tolerances with skew-tolerances. We extend the frameworks of Samuelson (1970), Judd and Guu (2001), and Chabi-Yo, Leisen and Renault (2007) to a dynamic market model in which each investor maximizes his or her expected utility. Our intertemporal portfolio choice is a dynamic problem in which each investor chooses his or her consumption level and asset allocation conditional on her current wealth. To make each decision, the investor takes into account the fact that at any future date, the portfolio weight will be optimally revised conditional on the available wealth. Our model setup can be viewed as a dynamic extension of Harvey and Siddique (2000), Kraus and Litzenberger (1976), Rubinstein (1973, 1974) and is also related to Brandt et al. (2005). Our approach allows us to derive the optimal shares of wealth invested in each security, the assets' risk premia, and the SDF in equilibrium while other approaches do not. We show that the optimal portfolio demand for stocks includes a skewness hedging component, an intertemporal volatility hedging component and other higher-order related intertemporal hedging components. The hedging components depend on the correlation between the aggregate volatility risk and risky assets, the correlation between the aggregate skewness risk and risky assets, and the distribution of investor preferences.

The paper's second contribution is that our theoretical analysis reveals that the SDFs studied by the aforementioned authors omit components pertaining to the cross-sectional variance of investor risk-tolerances, the cross-sectional variance of investor skew-tolerances, and the cross-sectional covariance of investor risk-tolerances with skew-tolerances. More specifically, the SDF derived in this paper involves the market return, the squared market return, the cubic market return, the market aggregate volatility risk, the market aggregate skewness risk, the market-mimicking aggregate volatility product, market-mimicking squared market return product, and finally the aggregate correlation risk. Hence, stocks with different sensitivities to these risk factors have different expected returns.

We derive a structural interpretation of the market prices of the risk factors appearing in the SDF in terms of the average value of investor risk-, skew-, and kurtosis-tolerances, the cross-sectional variance of investor risk-tolerances, the cross-sectional variance of investor skew-tolerances, and finally the cross-sectional covariance of investor risk-tolerances with skew-tolerances. The fact that heterogeneity of preferences gives rise to additional pricing factors is related to the general theory of pricing heterogeneity (see Rubinstein (1974), Constantinides and Duffie (1996), Heaton and Lucas (1996), and more recently Balduzzi and Yao (2007)). The former focuses on incomplete consumption insurance while we focus instead on incompleteness with respect to nonlinear risks in a dynamic model. Constantinides and Duffie (1996) find that the cross-sectional variance of consumption growth across agent gives rise to an additional priced risk factor, but we find that heterogeneity of investor preferences gives rise to several risk factors.

A third contribution of the paper is that our closed-form formulas for SDFs can be easily applied to real data to recover the distribution of investor risk-, skew-, and kurtosis-tolerances without knowledge of the functional form of the investors utility function and risky assets joint distribution. Thus, this article sheds some light on the ability of the squared market return, the cubic market return, the aggregate volatility risk, the aggregate skewness risk, the aggregate correlation risk and other higher-order related moments to explain risky asset returns. None of the existing literature covers this case.

The remainder of this article is organized as follows. Section 2 describes the basic model and derives the optimal shares of wealth invested in each security, the assets' risk premia, and the SDF in equilibrium. Section 3 develops the empirical implications of the model. Section 4 specifies the test design and presents the results of the empirical tests. Section 5 summarizes and concludes.

2 The model

2.1 Returns

We consider a complete market economy with a large number of investors with heterogeneous preferences and endowments. In this economy, investors are indexed by $i = 1, ..., I$ and trade in n risky assets and a safe asset at times $\tau = t,\ t+1, ...,T-1$. We denote respectively by $R_{k\tau+1}$, $k = 1, ...n$, the return from investing \$1 at time τ in each of security $k = 1, ...n$. All assets are traded in competitive markets without transaction costs and taxes. We follow closely Samuelson (1970) and Judd and Guu (2001), assuming that the distribution of the returns belongs to a family $\mathbb{P}$ of "compact" or "small-risk" distributions defined such as some specified parameter σ goes to zero, all the distributions converge to a sure outcome. The $\mathbb{P}$ family has the following property:

$$\begin{aligned} E_\tau R_{k,\tau+1} - R_f &= \sigma^2 a_{k,\tau}(\sigma) \text{ for } k = 1, ..., n, \\ E_\tau (R_{k,\tau+1} - E_\tau R_{k,\tau+1})^r &= \sigma^r E_\tau Y^r_{k,\tau+1} \text{ for } k = 1, ..., n,\ r \geq 2, \end{aligned}$$

where r is a finite number, and $Y_{k,\tau+1}$ is a random variable with zero expectation. Given the scale risk parameter σ, there exists a unique random variable $Y_{k,\tau+1}$ such as $R_{k,\tau+1} - E_\tau R_{k,\tau+1} = \sigma Y_{k,\tau+1}$. Thus, the random variable $Y_{k,\tau+1}$ is well defined. An explanation may be needed to illustrate why the asset's expected return is a function of σ. Assume that $E_\tau R_{k\tau+1} = \mu_{\tau k}$. If the $\mu_{\tau k}$s were not all equal, then the risky asset with the largest

$\mu_{\tau k}$ would dominate the rest as σ approaches zero, and arbitrage would be possible.[3] As σ converges to zero, it becomes certain that the distribution of $(R_{k\tau+1})_{k=1,\ldots n}$ converges to a sure outcome $(R_f)_{k=1,\ldots n}$. Any random vector $R_{k\tau+1}$ that belongs to $\mathbb{P}$ can be decomposed as follows:

$$R_{k\tau+1}(\sigma) = R_f + \sigma^2 a_{k\tau}(\sigma) + \sigma Y_{k\tau+1}. \tag{1}$$

Here, the coefficient $a_{kt}(\sigma)$ is a function of σ. The σ parameter, which characterizes the scale of risk, is important for the analysis. Specification (1) refers to the small noise expansion and it provides a convenient framework to analyze portfolio holdings and resulting equilibrium allocations for a given random vector $Y_{\tau+1} = (Y_{k\tau+1})_{1\leq k\leq n}$, where $E_\tau[Y_{\tau+1}] = 0$ and $Var_\tau(Y_{\tau+1}) = \Sigma_\tau$ is a symmetric and positive definite matrix. Thinking in terms of Brownian motion, σ may be thought as the square root of time, while the drift and diffusion terms are given by $\left(R_f + \sigma^2 a_{k\tau}(\sigma)\right)$ and $(\sigma Y_{k\tau+1})$ respectively.[4]

Note that all asset returns are not correlated through σ since the correlation between two assets k and j, $Cov_\tau(Y_{k\tau+1}, Y_{j\tau+1}) / \sqrt{Var_\tau(Y_{k\tau+1})\, Var_\tau(Y_{j\tau+1})}$, is independent of σ. In Equation (1), the term

$$\sigma^2 a_{k\tau}(\sigma) = E_\tau R_{k\tau+1} - R_f$$

has the interpretation of time-varying risk premium. Note that the return specification in Equation (1) is similar to Samuelson's return decomposition, except that Samuelson restricts the function $a_{k\tau}(\sigma)$ to constant. In addition, Samuelson assumes that the risk premium function $a_{k\tau}(\sigma)$ is not time-varying and is independent of σ. Under this assumption, risk premia are proportional to the squared scale of risk. Within a representative agent framework, Samuelson shows that the optimal portfolio coincides with the mean-variance optimal portfolio when σ approaches zero. Contrary to Samuelson, we assume that $a_{k\tau}(\sigma)$ is a function of σ. An intuition behind this assumption is that anyone familiar with a continuous time model will identify $\left(R_f + \sigma^2 a_{k\tau}(\sigma)\right)$ and $(\sigma Y_{k\tau+1})$ with the drift and the diffusion terms of a diffusion process, where the drift term is a nonlinear function of σ. This assumption would allow the following to be derived: the optimal portfolio weights and the market price of aggregate volatility risk, aggregate skewness risk, aggregate correlation risk, and kurtosis risk in equilibrium.

[3] Samuelson (1970) and Judd and Guu (2001) provide an explanation on why the asset's expected return is a function of σ.

[4] Note that there are no restrictions on the distribution of $Y_{k\tau+1}$. Hence, the distribution of $R_{k\tau+1}$ is general and is not restricted to a specific distribution.

2.2 Investor Preferences and Portfolio Optimization

We consider the portfolio choice at time t of investor i who maximizes the expected utility of wealth at some terminal date T by trading in n risky assets and a safe asset at times $\tau = t, ..., T-1$. Without loss of generality, we assume that $t = 0$ and $T = 2$. The investor's optimization problem is to choose each period $[\tau, \tau + 1]$, the level of consumption $C_\tau^{(i)}$ and the asset allocation $\omega_\tau^{(i)}$ for the wealth that is not consumed. And since the risky returns are functions of the scale risk parameter σ, then the level of consumption $C_\tau^{(i)}(\sigma)$ and the asset allocation $\omega_\tau^{(i)}(\sigma)$ are also functions of σ. We assume the functions $C_\tau^{(i)}(\sigma)$, $\omega_\tau^{(i)}(\sigma)$, and $a_{k\tau}(\sigma)$ are at least r times differentiable in the neighborhood of zero.

In this economy, investors have additive time-separable preferences and the value function of each investor is:

$$V_0^{(i)} = \max_{\left\{\omega_\tau^{(i)}(\sigma), C_\tau^{(i)}(\sigma)\right\}_{\tau=0}^{\tau=T-1}} E_0\left[\sum_{\tau=0}^{T} \beta^\tau u_i\left(C_\tau^{(i)}(\sigma)\right)\right] \qquad i = 1, ..., I \tag{2}$$

subject to the sequence of budget constraints

$$W_{\tau+1}^{(i)}(\sigma) = \left(W_\tau^{(i)}(\sigma) - C_\tau^{(i)}(\sigma)\right)\left(\omega_\tau^{(i)\intercal}(\sigma)\,\mathbf{R}_{\tau+1}^e(\sigma) + R_f\right) \quad \tau = 0, 1$$

and the terminal condition $W_T^{(i)}(\sigma) = C_T^{(i)}(\sigma)$. $\mathbf{R}_{\tau+1}^e(\sigma)$ is the vector of excess returns on the n risky assets from time τ to $\tau+1$, and β is a subjective discount factor.[5] The function $u_i(.)$ measures the investor's utility of consumption. Without loss of generality, we assume that the investor's initial wealth is equal to 1, and the individual asset allocation shares fulfill the market clearing conditions

$$\sum_{i=1}^{I}\left(W_\tau^{(i)}(\sigma) - C_\tau^{(i)}(\sigma)\right)\omega_\tau^{(i)}(\sigma) = \overline{\theta}_\tau \text{ for } \tau = t, ..., T-1.$$

We next assume that all agents have utility functions that exhibit non-satiation ($u_i^{'} > 0$), risk aversion ($u_i^{''} < 0$), a preference for positive skewness ($u_i^{'''} > 0$), and kurtosis preference ($u_i^{''''} < 0$). The intertemporal portfolio choice in Equation (2) is a dynamic problem. At time τ, each investor chooses his or her consumption level $C_\tau^{(i)}(\sigma)$ and asset allocation $\omega_\tau^{(i)}(\sigma)$ conditional on having wealth $W_\tau^{(i)}(\sigma)$. To make his or her decision at time τ,

[5]The upper symbol $\intercal$ refers to the transpose.

investor i takes into account the fact that at any future date τ, the portfolio weight will be optimally revised conditional on the wealth $W_{\tau}^{(i)}(\sigma)$. If $\sigma = 0$, there are no risky assets and $C_{\tau}^{(i)}(0)$ represents the optimal consumption for a deterministic problem in which the investor's wealth grows at the risk-free rate for the remaining $T - t$ periods.

We solve the dynamic portfolio choice in Equation (2) by expressing the multiperiod problem (2) as single-period problems:

$$V_0^{(i)} = \max_{\left\{\omega_0^{(i)}(\sigma), C_0^{(i)}(\sigma)\right\}} \left\{ \max_{\left\{\omega_1^{(i)}(\sigma), C_1^{(i)}(\sigma)\right\}} E_1 \left(\sum_{\tau=0}^{T} \beta^{\tau} u_i \left(C_{\tau}^{(i)}(\sigma) \right) \right) \right\}. \tag{3}$$

Note that all investors have homogeneous expectations. The source of heterogeneity in this economy comes from investor preferences, consumptions, and endowments.[6] The feature of the model presented in this section resembles the dynamic extension of Cass and Stiglitz (1970) with heterogeneous investors. Note that Cass and Stiglitz (1970) is a static single-period model in which there is a representative investor. The feature of our model is also related to that of Rubinstein (1974) and Brandt et al. (2005). However, there are many differences between our model and those of Rubinstein and Brandt et al. First, in our model investors have heterogeneous preferences, endowments and consumptions, while Brandt et al. consider an economy in which there is a representative investor. Second, Brandt et al. assume that the representative investor has a CRRA utility function and use a Taylor expansion series to derive a recursive relationship that allows them to derive the optimal portfolio weights. In our model, we assume the risky assets can be specified as in Equation (1), which resembles the single-factor model conventionally used in finance theory. As stated above, specification (1) is general, and risky assets can be decomposed as in Equation (1) without knowledge of the assets' return distribution. As will be seen in the succeeding sections, instead of using a Taylor expansion series, we use a small noise expansion approach proposed by Judd and Guu (2001). The advantage of this approach is that it is less restrictive than the Taylor expansion series and it allows us to derive the closed-form solution of investor optimal portfolio weights at the beginning of each period. Moreover, it allows us to derive the closed-form expression of the asset's risk premium and the aggregate SDF expression in equilibrium.[7] Second, Rubinstein's framework is a

[6] The case in which investors have heterogeneous expectations is beyond the scope of this paper, and will be explored in future research.

[7] It is useful to point out that Constantinides (1982), Ross (1973), and Wilson (1968) have constructed

one-period model, while we are interested in the investor's dynamic problem. Moreover, to derive the aggregate SDF, Rubinstein assumes that in equilibrium, all individuals have partially similar economic characteristics such as identical present consumption and identical future wealth. As will be seen in the succeeding sections, the aggregation results derived in this paper are obtained without making any assumptions about the investor's present consumption and future wealth.

2.3 Notations

The investor's preferences are characterized by the parameters:

$$\begin{array}{llllll} \tau_i = & -\frac{u_i'\left(W_2^{(i)}(0)\right)}{u_i''\left(W_2^{(i)}(0)\right)}, & \rho_i = & \frac{\tau_i^2}{2}\frac{u_i'''\left(W_2^{(i)}(0)\right)}{u_i'\left(W_2^{(i)}(0)\right)}, & \kappa_i = & -\frac{\tau_i^3}{3}\frac{u_i''''\left(W_2^{(i)}(0)\right)}{u_i'\left(W_2^{(i)}(0)\right)}, \\ \tau_{ic} = & -\frac{u_i'\left(C_1^{(i)}(0)\right)}{u_i''\left(C_1^{(i)}(0)\right)}, & \rho_{ic} = & \frac{\tau_{ic}^2}{2}\frac{u_i'''\left(C_1^{(i)}(0)\right)}{u_i'\left(C_1^{(i)}(0)\right)}, & \kappa_{ic} = & -\frac{\tau_{ic}^3}{3}\frac{u_i''''\left(C_1^{(i)}(0)\right)}{u_i'\left(C_1^{(i)}(0)\right)}, \end{array}$$

where τ_i represents the risk-tolerance parameter at $W_2^{(i)}(0)$ ($1/\tau_i$ is the Arrow-Pratt absolute measure of risk aversion), and ρ_i and κ_i represent the skew-tolerance and kurtosis-tolerance at $W_2^{(i)}(0)$. $W_2^{(i)}(0)$ represents the optimal wealth for a deterministic problem in which the investor's wealth grows at the risk-free rate for the remaining $T-t$ periods. The parameters τ_{ic}, ρ_{ic} and κ_{ic} represent the risk-tolerance, skew-tolerance and kurtosis-tolerance at $C_1^{(i)}(0)$. In addition, we define the following quantity:

$$\tau_{\nu_i} = \frac{\tau_i}{1-\nu_i} \quad \text{with} \quad \nu_i = \frac{\beta R_f^2}{\frac{u_i''\left(C_1^{(i)}(0)\right)}{u_i''\left(W_2^{(i)}(0)\right)}+\beta R_f^2},$$

and consider the average values, variances and covariance:

$$\begin{array}{llllll} \overline{\tau} = & \frac{1}{I}\sum_{i=1}^{I}\tau_i, & \overline{\rho} = & \frac{\sum_{i=1}^{I}\rho_i\tau_i}{\sum_{i=1}^{I}\tau_i}, & \overline{\kappa} = & \frac{\sum_{i=1}^{I}\kappa_i\tau_i}{\sum_{i=1}^{I}\tau_i}, \\ \overline{\tau^2}-\overline{\tau}^2 = & \frac{1}{I}\sum_{i=1}^{I}\left(\tau_i-\overline{\tau}\right)^2, & \overline{\rho\tau}-\overline{\rho}.\overline{\tau} = & \frac{\sum_{i=1}^{I}(\rho_i-\overline{\rho})(\tau_i-\overline{\tau})\tau_i}{\sum_{i=1}^{I}\tau_i}, & \overline{\rho^2}-\overline{\rho}^2 = & \frac{\sum_{i=1}^{I}(\rho_i-\overline{\rho})^2\tau_i}{\sum_{i=1}^{I}\tau_i}, \end{array}$$

an aggregate pricing kernel with weights for individual utilities. Under the assumption of complete markets, the aggregate pricing kernel is a solution to a social planner problem in maximizing a weighted average of individual marginal utility functions.

where $\overline{\tau}$ is the average of investor risk-tolerances, $\overline{\rho}$ and $\overline{\kappa}$ represent the weighted average of investor skew-tolerances and kurtosis-tolerances respectively, $\overline{\tau^2} - \overline{\tau}^2$ captures the cross-sectional variance of investor risk-tolerances, $\overline{\rho\tau} - \overline{\rho}.\overline{\tau}$ captures the cross-sectional covariance of investor risk-tolerances with skew-tolerances, and $\overline{\rho^2} - \overline{\rho}^2$ represents the cross-sectional variance of investor skew-tolerances. We also define the quantities

$$\overline{\tau}_\nu = \frac{1}{I}\sum_{i=1}^{I} \tau_{\nu_i} \text{ and } \overline{\rho}_c = \sum_{i=1}^{I} \left(\rho_{ic}\nu_i^2.\tau_{\nu_i}^2/\tau_{ic}\right) / \sum_{i=1}^{I} \tau_{\nu_i}.$$

2.4 Asset Allocations, Volatility Risk and Skewness Preferences

Note that the portfolio weight function $\omega_\tau^{(i)}(\sigma)$ is r times differentiable in the neighborhood of zero. As σ approaches 0, the portfolio weight can be well approximated by $\omega_\tau^{(i)}(\sigma) = \sum_{j=0}^{r} \frac{\sigma^j}{j!} \omega_\tau^{(i)^{[j]}}(0)$ where $\omega_\tau^{(i)^{[j]}}(0)$ represents the jth derivative of the portfolio weight with respect to σ evaluated at 0. Similarly, the function $a_{k\tau}(\sigma)$ can be well approximated by $a_{k,\tau}(\sigma) = \sum_{j=0}^{r} \frac{\sigma^j}{j!} a_{k,\tau}^{[j]}(0)$ where $a_{k,\tau}^{[j]}(0)$ represents the jth derivative of $a_{k,\tau}(\sigma)$ with respect to σ evaluated at 0. In this section, we assume that $r = 1$, and then use the first-order conditions in Equation (3) to derive the optimal asset allocations, assets' risk premia and the SDF in equilibrium.[8]

Proposition 1 *In a two-period investor's problem with heterogeneous preferences and homogeneous expectation, the optimal shares of wealth invested in risky securities are characterized*

- *at date 1 by:*

$$\omega_1^{(i)}(\sigma) = \left(\phi_{1i}^\intercal \varkappa_1\right)\overline{\omega}_1 + \psi_{1i}\overline{\varsigma}_1$$

with

$$\overline{\omega}_1 = \left(Var_1(\mathbf{R}_2)\right)^{-1} Cov_1(\mathbf{R}_2, r_{M2}), \qquad \overline{\varsigma}_1 = \left(Var_1(\mathbf{R}_2)\right)^{-1} Cov_1\left(\mathbf{R}_2, r_{M2}^2\right)$$

and

$$\varkappa_1^\intercal = [1, r_{M1}] \ , \ \phi_{1i}^\intercal = \frac{R_f}{W_2^{(i)}(0)}\left[\frac{\tau_i}{\overline{\tau}}, \frac{2R_f\tau_i(\rho_i - \overline{\rho})}{\overline{\tau}_\nu\overline{\tau}} - \frac{R_f}{W_2^{(i)}(0)}\frac{\tau_i^2}{\overline{\tau}_\nu\overline{\tau}}\right] \quad \psi_{1i} = \frac{R_f}{W_2^{(i)}(0)}\frac{(\rho_i - \overline{\rho})\tau_i}{\overline{\tau}^2} \ ,$$

[8]The case where $r = 0$, that is $\omega_\tau^{(i)}(\sigma) = \omega_\tau^{(i)}(0)$ and $a_{k\tau}(\sigma) = a_{k\tau}(0)$ produces an equilibrium model in which the market return is the only risk factor (the CAPM model).

here, $r_{M1} = R_{M1} - E_0 R_{M1}$ *and* $r_{M2} = R_{M2} - E_1 R_{M2}$*.*

- *at date 0 by:*

$$\omega_0^{(i)}(\sigma) = \quad \phi_{0i}\overline{\omega}_0 + \psi_{0i}\overline{\varsigma}_0 + \varphi_{0i}\overline{\varsigma}_{0,\vartheta} \quad , \tag{4}$$

where:

$$\begin{aligned}
\overline{\omega}_0 = & \quad (Var_0(\mathbf{R}_1))^{-1} Cov_0(\mathbf{R}_1, r_{M1}), \\
\overline{\varsigma}_0 = & \quad (Var_0(\mathbf{R}_1))^{-1} Cov_0\left(\mathbf{R}_1, r_{M1}^2\right), \\
\overline{\varsigma}_{0,\vartheta} = & \quad (Var_0(\mathbf{R}_1))^{-1} Cov_0(\mathbf{R}_1, Var_1(r_{M2})),
\end{aligned}$$

with:

$$\begin{aligned}
\phi_{0i} = & \quad \frac{R_f}{W_1^{(i)}(0)} \frac{\tau_{\nu_i}}{\overline{\tau}_\nu}, \\
\psi_{0i} = & \quad \frac{R_f}{W_1^{(i)}(0)} \left(\left(\frac{\rho_i \tau_i}{\overline{\tau}_\nu^2} - \frac{\tau_{\nu_i}\overline{\tau}.\overline{\rho}}{\overline{\tau}_\nu^3} \right) R_f + \left(\frac{\rho_{ic}\nu_i^2 \tau_{\nu_i}^2}{\tau_{ic}\overline{\tau}_\nu^2} - \frac{\tau_{\nu_i}\overline{\rho}_c}{\overline{\tau}_\nu^2} \right) \right), \\
\varphi_{0i} = & \quad \frac{1}{W_1^{(i)}(0)} \left(\frac{\tau_{\nu_i}(1-\overline{\rho})}{\overline{\tau}_\nu \overline{\tau}} - \frac{\tau_i(1-\rho_i)}{\overline{\tau}^2} \right).
\end{aligned}$$

Here, $\mathbf{R}_\tau = (R_{k\tau})_{k=1,\ldots n}$ *for* $\tau = 1, 2$*.*

Proof. See the Appendix. ■

Proposition 1 shows that at date 1, investors hold two portfolios: the market portfolio $\overline{\omega}_1$; and the mimicking skewness portfolio $\overline{\varsigma}_1$. Note that the return, $\overline{\varsigma}_1^\intercal \mathbf{R}_2$, on the mimicking skewness portfolio is the payoff that mimics the squared excess market return r_{M2}^2. The share invested in the market portfolio is a linear function of the excess market return r_{M1}. The economic intuition that could explain this linear relationship is that the investor observes at time 1 a signal equals to the excess market return. The investor's response to the signal is to adjust his or her share as a linear function of the excess market return at time 1. The share invested in the mimicking skewness portfolio is time-independent. When the joint distribution of asset returns is normal, the weight $\overline{\varsigma}_1$ of the mimicking skewness portfolio is equal to zero, and investors hold only the market portfolio.

On the other hand, at date 0, investors hold the market portfolio $\overline{\omega}_0$, the mimicking skewness portfolio $\overline{\varsigma}_0$, and a third portfolio $\overline{\varsigma}_{0,\vartheta}$ defined by the covariance between the aggregate volatility risk and the basis asset returns. The first two terms in Equation (4) are called myopic demands because they represent vectors of portfolio weights for an investor who has only a single-period objective for a short-period investment problem. The first term represents the myopic mean-variance optimal demand. The second term represents the skewness hedging demand. The sign of the skewness hedging demand depends on the correlation between the squared excess market return and stock returns via $\overline{\varsigma}_0$, and the coefficient ψ_{0i}. The third term in Equation (4) represents the intertemporal volatility hedging demand, which is determined by $\overline{\varsigma}_{0,\vartheta}$ and the constant $\varphi_{0i} = \frac{1}{W_1^{(i)}(0)}\left(\frac{\tau_{\nu_i}(1-\overline{\rho})}{\overline{\tau}\tau_\nu} - \frac{\tau_i(1-\rho_i)}{\overline{\tau}^2}\right)$. The sign of the volatility hedging demand depends on the correlation between volatility shocks and returns via $\overline{\varsigma}_{0,\vartheta}$, and the coefficient φ_{0i}. For instance, when φ_{0i} is negative, positive (negative) hedging demand against volatility risk arises if and only if volatility shocks and stock returns are negatively (positively) correlated. Note that if there is no intermediate consumption $\tau_{\nu_i} = \tau_i$, then the share invested in the volatility hedging portfolio reduces to $\frac{1}{W_1^{(i)}(0)}\frac{\tau_i(\rho_i-\overline{\rho})}{\overline{\tau}^2}$. This implies that investors hold a positive share of the intertemporal volatility hedging portfolio if the investor's skew-tolerance is higher than the average $\rho_i > \overline{\rho}$.[9] Note also that $\overline{\varsigma}_{0,\vartheta}^{\intercal}\mathbf{R}_1$ represents the return on the mimicking aggregate volatility risk $Var_1(r_{M2})$. When asset returns are jointly normally distributed, it is straightforward to show that $\overline{\varsigma}_0 = 0$ and the hedging demand for skewness risk is equal to zero.[10] However, the intertemporal volatility hedging demand is not null. In the next proposition, we derive the risk premium on the risky assets.

Proposition 2 *In a two-period investor's problem with heterogeneous preferences and homogeneous expectation, the risk premium on asset k is given*

- *at date 1 by:*

$$E_1 R_{k,2} - R_f = \overline{A}_1 Cov_1(r_{M2}, R_{k,2}) + \overline{B}_1 Cov_1\left(r_{M2}^2, R_{k,2}\right),$$

with:

$$\overline{A}_1 = \frac{1}{\overline{\tau}} + \frac{R_f(1-2\overline{\rho})}{\overline{\tau}_\nu.\overline{\tau}} r_{M1} \qquad \overline{B}_1 = -\frac{\overline{\rho}}{\overline{\tau}^2}$$

[9] In incomplete markets, Chacko and Viceira (2005) examine the optimal consumption and portfolio choice problem of long-horizon investors who have access to a risk-free asset and a risky asset. They find that the optimal demand for the risky assets includes an intertemporal hedging component that is negative when investors have coefficients of relative risk aversions larger than one, and the instantaneous correlation between volatility and stock return is negative.

[10] To see this, note that $Cov_0\left(\mathbf{R}_1, r_{M1}^2\right)$ can be written as $\sigma^3 E_0\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)^2\mathbf{Y}_1$ which can be simplified to $\sigma^3 E_0\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)\mathbf{Y}_1$. If asset returns are jointly normally distributed, $E_0\left(\mathbf{Y}_1 Y_{j,1} Y_{k,1}\right) = 0$. Hence $E_0\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)\mathbf{Y}_1 = 0$.

- *at date 0 by:*

$$E_0 R_{k,1} - R_f \quad = \quad \overline{A}_0 Cov_0 \left(r_{M1}, R_{k,1}\right) + \overline{B}_0 Cov_0 \left(r_{M1}^2, R_{k,1}\right) + \overline{D}_0 Cov_0 \left(Var_1 \left(r_{M2}\right), R_{k,1}\right) \tag{5}$$

with

$$\overline{A}_0 = \frac{1}{\overline{\tau}_\nu} R_f, \quad \overline{B}_0 = -\left(\frac{\overline{\tau}.\overline{\rho}}{\overline{\tau}_\nu^3} R_f^2 + \frac{\overline{\rho}_c}{\overline{\tau}_\nu^2} R_f\right), \quad \overline{D}_0 = \frac{(1-\overline{\rho})}{\overline{\tau}.\overline{\tau}_\nu}.$$

Here, $r_{M1} = R_{M1} - E_0 R_{M1}$ *and* $r_{M2} = R_{M2} - E_1 R_{M2}$.

Proof. See the Appendix. ■

At date 1, the asset's risk premium can be decomposed into two components: the risk-premium due to the asset's covariance risk, namely, $\overline{A}_1 Cov_1 \left(r_{M2}, R_{k,2}\right)$ and the risk premium due to the asset's coskewness risk, namely, $\overline{B}_1 Cov_1 \left(r_{M2}^2, R_{k,2}\right)$. Moreover, the asset's covariance risk premium is time-varying and conditional on the market excess return r_{M1}. Since $\overline{B}_1 < 0$, all else equal, assets with negative covariance with the square of the market return have higher expected returns than assets with positive or zero covariance with the squared market return. This result is consistent with Harvey and Siddique (2000). When $\overline{\rho} = 0$, the asset's risk premium equals to the conditional version of the CAPM with a time-varying parameter $\overline{A}_1$. All else equal, assets with positive covariance with the market return have higher expected returns than assets with negative covariance with the market return if $\overline{\rho} < \frac{1}{2}$ and $r_{M1} > 0$. The conditional risk premium is derived without making any assumption about the time-varying nature of the beta coefficients in the asset's risk premium decomposition. A number of studies that investigate the performance of the conditional CAPM model assume that the beta implied by the CAPM model is a linear function of one or more pre-specified conditional variables (for e.g. Harvey (1996), Ghysels (1998) and, Brandt (1999)). As Wang (2003) states: "while this trade-off between time-varying risk and expected returns makes such models intuitively appealing, it is empirically challenging, since there is no theoretical guidance on how betas and risk premia vary with variables that represent conditioning information" Proposition 1 theoretically shows, in a context of an equilibrium model, that the beta coefficient $\overline{A}_1$ is a linear function of the excess market return. In the case where investors are homogeneous with respect to their preferences, the parameter $\overline{A}_1$ is still time-varying. It is useful to point out that the expected value of $\overline{A}_1$ is $\frac{1}{\overline{\tau}}$ since $E_0 r_{1M} = 0$. Therefore, on average, all else equal, assets with positive covariance with the market return have higher expected returns than assets with negative covariance with the market return. This is consistent with the unconditional version of the CAPM. On the other hand, at date 0, the asset's risk premium is a sum of three components: the risk premium due to the asset's covariance risk, namely, $\overline{A}_0 Cov_0 (r_{M1}, R_{k,1})$; and the risk premium due to the asset's coskewness risk, namely, $\overline{B}_0 Cov_0 \left(r_{M1}^2, R_{k,1}\right)$; and the risk premium due to the aggregate volatility risk premium, namely, $\overline{D}_0 Cov_0 \left(Var_1 (r_{M2}), R_{k,1}\right)$. The coefficient $\overline{B}_0$ is negative, implying that, all else equal, assets with a negative covariance with the square of the market return have higher expected returns than assets with zero or positive covariance with the square of the market return. Furthermore, our model predicts that, all else equal, assets with a negative covariance with the market volatility have higher expected returns than assets with positive or zero covariance with the market volatility if the average value of the investors' skew-tolerances $\overline{\rho}$ is higher than 1. When all investors have a Constant Relative Risk Aversion (CRRA) type of utility function with identical or different risk aversion coefficients greater than 1, it can be shown that $\overline{\rho} > 1$. As a result, $\overline{D}_0 < 0$ and our model predicts that, all else equal, assets with a negative covariance with the aggregate volatility risk have higher expected returns than assets with positive or zero covariance with the aggregate volatility risk. Equation (5) generalizes the empirical model used by Ang, Hodrick, Xing and Zhang (2006, hereafter AHXZ) to examine the pricing of aggregate volatility risk in the cross-section of stock returns. The AHXZ empirical model assumes that the return data generating process is a linear function of the market and the aggregate volatility risk factors (see Equation 3, Page 266). In this paper, we provide a structural interpretation of the market price of coskewness and aggregate volatility risk in terms of the average value of investor risk- and skew-tolerances. Our results indicate that the sign of the market price of aggregate volatility risk depends on the average values of investor preference parameters. When asset returns are jointly normally distributed, the risk premium due to the asset's coskewness risk, $\overline{B}_0 Cov_0 \left(r_{M1}^2, R_{k,1}\right)$, vanishes in Equation (5), and Equation (5) reduces to the AHXZ empirical multi-beta pricing model.[11]

We next use Proposition 2 to recover the functional form of the SDF.

Proposition 3 *In a two-period investor's problem with heterogeneous preferences and ho-*

[11] To see this, note that $Cov_0 \left(r_{M1}^2, R_{k,1}\right)$ can be written as $\sigma^3 E_0 \left(\overline{\omega}_0^\intercal Y_1\right)^2 Y_{k,1}$ which can be simplified to $\sigma^3 \overline{\omega}_0^\intercal E_0 \left(Y_1 Y_1^\intercal Y_{k,1}\right) \overline{\omega}_0$. If asset returns are jointly normally distributed, $E_0 \left(Y_1 Y_1^\intercal Y_{k,1}\right) = 0$.

mogeneous expectation, the aggregate SDF is given by:

$$m_{0,2} = m_{0,1} m_{1,2}$$

with

$$m_{1,2} = \frac{1}{R_f} - \frac{\overline{A}_1}{R_f} r_{M2} - \frac{\overline{B}_1}{R_f} \left[r_{M2}^2 - E_1 r_{M2}^2 \right]$$

$$m_{0,1} = \frac{1}{R_f} - \frac{\overline{A}_0}{R_f} r_{M1} - \frac{\overline{B}_0}{R_f} \left[r_{M1}^2 - E_0 r_{M1}^2 \right] - \frac{\overline{D}_0}{R_f} \left[Var_1 \left(r_{M2} \right) - E_0 Var_1 \left(r_{M2} \right) \right] \qquad (6)$$

where $\overline{A}_1$, $\overline{B}_1$, $\overline{A}_0$, $\overline{B}_0$, and $\overline{D}_0$ are defined in Proposition 2, $m_{1,2}$ and $m_{1,2}$ represent the SDF for the periods $[0,1]$ and $[1,2]$ respectively, and $r_{M1} = R_{M1} - E_0 R_{M1}$ and $r_{M2} = R_{M2} - E_1 R_{M2}$.

Proof. See the Appendix. ■

Proposition 3 shows that the aggregate SDF for the time period $[1,2]$ is a quadratic function of the excess market return. This SDF involves r_{M2} and r_{M2}^2; hence, covariance and coskewness remain important elements for asset pricing. More interestingly, the coefficient $\overline{A}_1$ of the risk factor r_{M2} is a linear function of the excess market return r_{M1}. This implies that the time-varying market price of covariance risk is also important for asset pricing. The functional form of the SDF $m_{1,2}$ clearly illustrates how this article differs from existing literature. For example, conditional CAPM models allow the CAPM betas to be a linear function of one or more conditioning variables. This specification makes these models intuitively appealing. However, there is no theoretical guidance on how betas and the risk premia change with variables that represent conditioning information. This paper theoretically derives, in a context of an equilibrium model, the closed-form expression for the higher-order moment SDF with time-varying coefficients.

On the other hand, the SDF for the period $[0,1]$ is equal to the quadratic function of the market return augmented with the aggregate volatility risk factor $Var_1 \left(r_{M2} \right)$. The aggregate volatility risk factor that appears in the SDF is due to the intertemporal volatility hedging demand. If all investors are myopic, it is straightforward to show that the market volatility factor vanishes in the SDF specification and the SDF $m_{0,1}$ reduces to the SDF, as implied by the Harvey and Siddique (2000) market coskewness model. Therefore, the SDF $m_{0,1}$ appears as a natural extension of the Harvey and Siddique (2000) market coskewness model in a context of a dynamic equilibrium model.

2.5 Aggregate Volatility Risk and Aggregate Skewness Risk

In this section, we consider the approximations $\omega_{\tau}^{(i)}(\sigma) = \sum_{j=0}^{r} \frac{\sigma^j}{j!} \omega_{\tau}^{(i)^{[j]}}(0)$ and $a_{k,\tau}(\sigma) = \sum_{j=0}^{r} \frac{\sigma^j}{j!} a_{k,\tau}^{[j]}(0)$ with $r = 2$, and then use the first-order conditions in Equation (3) to derive the assets' risk premia and the SDF in equilibrium.[12] These approximations involve large amounts of algebraic manipulation. To simplify this manipulation, we assume there is no intermediate consumption.[13]

Proposition 4 *In a two-period investor's problem with heterogeneous preferences and homogeneous expectation, the risk premium on asset k over the first period is:*

$$\begin{aligned} E_0 R_{k,1} - R_f \quad = \quad & \overline{\alpha}_0 Cov_0 \left(r_{M1}, R_{k,1} \right) + \overline{\alpha}_1 Cov_0 \left(r_{M1}^2, R_{k,1} \right) + \overline{\alpha}_2 Cov_0 \left(r_{M1}^3, R_{k,1} \right) \\ & + \overline{\alpha}_3 Cov_0 \left(Var_1 \left(r_{M2} \right), R_{k,1} \right) + \overline{\alpha}_4 Cov_0 \left(Skew_1 \left(r_{M2} \right), R_{k,1} \right) \\ & + \overline{\alpha}_5 Cov_0 \left(r_{M1} Var_1 \left(r_{M2} \right), R_{k,1} \right) + \overline{\alpha}_6 Cov_0 \left(r_{M1}^{(\rho)}, R_{k,1} \right) \\ & + \overline{\alpha}_7 Cov_0 \left(r_{M1}^{\overline{\varsigma}_0} . r_{M1}, R_{k,1} \right) + \overline{\alpha}_8 Cov_0 \left(r_{M1}^{\overline{\varsigma}_{0,\vartheta}} . r_{M1}, R_{k,1} \right) \end{aligned} \tag{7}$$

with:

$$\begin{aligned} r_{M1}^{(\rho)} = \quad & Cov_1 \left(r_{M2}, r_{M2}^{\overline{\varsigma}_1} \right), \quad r_{M1}^{\overline{\varsigma}_0} = \quad \overline{\varsigma}_0^{\intercal} \left(\mathbf{R}_1 - E_0 \mathbf{R}_1 \right), \quad Skew_1 \left(r_{M2} \right) = \quad E_1 r_{M2}^3, \\ r_{M2}^{\overline{\varsigma}_1} = \quad & \overline{\varsigma}_1^{\intercal} \left(\mathbf{R}_2 - E_1 \mathbf{R}_2 \right), \quad r_{M1}^{\overline{\varsigma}_{0,\vartheta}} = \quad \overline{\varsigma}_{0,\vartheta}^{\intercal} \left(\mathbf{R}_1 - E_0 \mathbf{R}_1 \right) \end{aligned}$$

and:

$$\begin{aligned} \overline{\varsigma}_1 = \quad & \left(Var_1 \left(\mathbf{R}_2 \right) \right)^{-1} Cov_1 \left(\mathbf{R}_2, r_{M2}^2 \right), \\ \overline{\varsigma}_0 = \quad & \left(Var_0 \left(\mathbf{R}_1 \right) \right)^{-1} Cov_0 \left(\mathbf{R}_1, r_{M1}^2 \right), \\ \overline{\varsigma}_{0,\vartheta} = \quad & \left(Var_0 \left(\mathbf{R}_1 \right) \right)^{-1} Cov_0 \left(\mathbf{R}_1, Var_1 \left(r_{M2} \right) \right), \end{aligned}$$

where the parameters $\overline{\alpha}_j$'s are functions of the distribution of investor preference parameters

[12] Due to space consideration, we report only the assets' risk premia and the SDF in equilibrium. The closed-form solution for the optimal asset allocations is available on request. It includes a skewness hedging component, an intertemporal volatility hedging component and other higher-order related intertemporal hedging components. The hedging components depend on the correlation between the aggregate volatility risk with risky assets, the correlation between the aggregate skewness risk with risky asset, and the distribution of investor preferences (see the proof of Proposition 4 in the Appendix).

[13] The results with intermediate consumption is similar and is available on request. With intermediate consumption, the risk premium has the same functional form as in Equation (7), except that the expression of the market prices of covariance risk $\overline{\alpha}_i s$ are complicated functions of investor preference parameters.

that is characterized by $(\overline{\tau}, \overline{\tau^2} - \overline{\tau}^2, \overline{\rho}, \overline{\rho^2} - \overline{\rho}^2, \overline{\rho\tau} - \overline{\rho}.\overline{\tau}, \overline{\kappa})$*:*

$$
\begin{aligned}
\overline{\alpha}_0 &= \tfrac{1}{\overline{\tau}} R_f \\
&\quad + \left(\tfrac{2}{\overline{\tau}^3} R_f^2 - 2\left(1 + 3R_f^2\right) \tfrac{\overline{\rho}}{\overline{\tau}^3} R_f \right) Var_0\left(r_{M1}\right), \\
\overline{\alpha}_1 &= -\tfrac{\overline{\rho}}{\overline{\tau}^2} R_f^2, \\
\overline{\alpha}_2 &= \tfrac{\overline{\kappa}}{\overline{\tau}^3} R_f^3, \\
\overline{\alpha}_3 &= \tfrac{(3-\overline{\rho})}{\overline{\tau}^2} - \tfrac{2}{R_f^2} \tfrac{1}{\overline{\tau}^2} - \tfrac{2}{R_f^2} \tfrac{\left(\overline{\tau^2} - \overline{\tau}^2\right)}{\overline{\tau}^4}, \\
\overline{\alpha}_4 &= -\tfrac{2}{R_f^2} \tfrac{\overline{\rho}}{\overline{\tau}^2} + \tfrac{\overline{\kappa}}{\overline{\tau}^3} - \tfrac{2}{R_f^2} \tfrac{\overline{\rho\tau} - \overline{\rho}.\overline{\tau}}{\overline{\tau}^3}, \\
\overline{\alpha}_5 &= \tfrac{3\overline{\kappa}}{\overline{\tau}^3} R_f - 4R_f \tfrac{\overline{\rho}}{\overline{\tau}^3} - \tfrac{8R_f\left(\overline{\rho^2} - \overline{\rho}^2\right)}{\overline{\tau}^3} \\
&\quad + \tfrac{2}{R_f} \tfrac{(1-2\overline{\rho})\left(\overline{\tau^2} - \overline{\tau}^2\right)}{\overline{\tau}^4} + \tfrac{2}{R_f} \tfrac{(1-2\overline{\rho})}{\overline{\tau}^2}, \\
\overline{\alpha}_6 &= -4 \tfrac{\left(\overline{\rho^2} - \overline{\rho}^2\right)}{\overline{\tau}^3} + \tfrac{2}{R_f^2} \tfrac{\left(\overline{\rho\tau} - \overline{\rho}.\overline{\tau}\right)}{\overline{\tau}^3} - \tfrac{2}{R_f^2} \tfrac{\overline{\rho}\left(\overline{\tau^2} - \overline{\tau}^2\right)}{\overline{\tau}^4}, \\
\overline{\alpha}_8 &= -4R_f \tfrac{\left(\overline{\rho^2} - \overline{\rho}^2\right)}{\overline{\tau}^3}, \\
\overline{\alpha}_7 &= -4 \tfrac{\left(\overline{\rho^2} - \overline{\rho}^2\right)}{\overline{\tau}^3} R_f^3.
\end{aligned}
$$

Here, $\mathbf{R}_\tau = \left(R_{k\tau}\right)_{k=1,\ldots n}$ *for* $\tau = 1, 2$.

Proof. See the Appendix ■

Equation (7) clearly illustrates how the market return, the squared excess market return r_{M2}^2, the cubic of the excess market return r_{M2}^3, the aggregate volatility risk $Var_1\left(r_{M2}\right)$, the aggregate skewness risk $Skew_1\left(r_{M2}\right)$, and the aggregate correlation risk $r_{M1}^{(\rho)}$ influence the asset's expected return. The market-aggregate volatility product, namely, $r_{M1} Var_1\left(r_{M2}\right)$; the market-mimicking squared market product, namely $r_{M1}^{\overline{\varsigma}_0}.r_{M1}$; and the market-mimicking aggregate volatility risk product, namely $r_{M1}^{\overline{\varsigma}_{0,\vartheta}}.r_{M1}$ also influence the asset's expected return. Equation (7) predicts that the market price of coskewness risk $\overline{\alpha}_1$ is negative while the market price of cokurtosis risk $\overline{\alpha}_2$ is positive. It also predicts that the sign of the market price of the aggregate volatility risk factor, $\overline{\alpha}_3$, depends on the average value of investor risk-tolerances $\overline{\tau}$, skew-tolerances $\overline{\rho}$, and the cross-sectional variance of investor risk-tolerances $\overline{\tau^2} - \overline{\tau}^2$. Further, it predicts that the sign of the market price of the aggregate skewness risk factor, $\overline{\alpha}_4$, depends on the average value of investor risk-tolerances $\overline{\tau}$, kurtosis-tolerances $\overline{\kappa}$, and the cross-sectional covariance of investor risk-tolerances with skew-tolerances $\overline{\rho\tau} - \overline{\rho}.\overline{\tau}$. The sign of the market price of the aggregate correlation risk factor, $\overline{\alpha}_6$, depends on the cross-sectional variance of investor risk-tolerances, skew-tolerances $\overline{\rho^2} - \overline{\rho}^2$ and the cross-sectional covariance of investor risk-tolerances with skew-tolerances. Finally, the market prices $\overline{\alpha}_7$ and $\overline{\alpha}_8$ depend on the cross-sectional variance of investor risk-tolerances, and are negative. This implies that, all else equal, assets with a negative covariance with the risk factors $r_{M1}^{\overline{\varsigma}_0}.r_{M1}$ and $r_{M1}^{\overline{\varsigma}_{0,\vartheta}}.r_{M1}$ have higher expected returns than assets with positive or zero covariance with these factors.[14] The SDF implied by specification (7) is given in the next proposition:

Proposition 5 *In a two-period investor's problem with heterogeneous preferences and homogeneous expectation, the aggregate SDF for the first period* $[0,1]$ *is given by:*

$$\begin{aligned} m_{0,1} &= \frac{1}{R_f} + \overline{D}_0 r_{M1} + \overline{D}_1 \left(r_{M1}^2 - E_0 r_{M1}^2\right) + \overline{D}_2 \left(r_{M1}^3 - E_0 r_{M1}^3\right) \\ &\quad + \overline{D}_3 \left(Var_1\left(r_{M2}\right) - E_0 Var_1\left(r_{M2}\right)\right) + \overline{D}_4 \left(Skew_1\left(r_{M2}\right) - E_0 Skew_1\left(r_{M2}\right)\right) \\ &\quad + \overline{D}_5 \left(r_{M1} Var_1\left(r_{M2}\right) - E_0 r_{M1} Var_1\left(r_{M2}\right)\right) + \overline{D}_6 \left(r_{M1}^{(\rho)} - E_0 r_{M1}^{(\rho)}\right) \\ &\quad + \overline{D}_7 \left(r_{M1}^{\overline{\varsigma}_0} . r_{M1} - E_0 r_{M1}^{\overline{\varsigma}_0} . r_{M1}\right) + \overline{D}_8 \left(r_{M1}^{\overline{\varsigma}_{0,\vartheta}} . r_{M1} - E_0 r_{M1}^{\overline{\varsigma}_{0,\vartheta}} . r_{M1}\right) \end{aligned} \tag{8}$$

where $\overline{D}_i = -\frac{\overline{\alpha}_i}{R_f}$ *and* $\overline{\alpha}_i s$ *are defined in Proposition 4.*

Proof. See the Appendix. ■

As shown in Equation (8), the SDF $m_{0,1}$ involves several factors. The risk factors r_{M1}, r_{M1}^2, and r_{M1}^3 are well-known in the literature involving nonlinear SDFs (see for e.g., Harvey and Siddique (2000) and Dittmar (2002)). The additional new risk factors are: the aggregate volatility risk, namely, $Var_1\left(r_{M2}\right)$; the aggregate skewness risk, namely, $Skew_1\left(r_{M2}\right)$; the market-aggregate volatility product, namely, $r_{M1} Var_1\left(r_{M2}\right)$, the correlation risk factor $r_{M1}^{(\rho)}$, the market-mimicking squared market product $r_{M1}^{\overline{\varsigma}_0} . r_{M1}$, and the market-mimicking aggregate volatility risk $r_{M1}^{\overline{\varsigma}_{0,\vartheta}} . r_{M1}$ product.

As shown in Proposition 4, our theory predicts that the cross-sectional variance and covariance of investor preferences explain the assets risk premia via the parameters $\overline{\tau^2} - \overline{\tau}^2$, $\overline{\rho^2} - \overline{\rho}^2$, and $\overline{\rho\tau} - \overline{\rho}.\overline{\tau}$. If the cross-sectional variance and covariance of investor preferences are not important for asset pricing, $\overline{\tau^2} - \overline{\tau}^2 = \overline{\rho^2} - \overline{\rho}^2 = \overline{\rho\tau} - \overline{\rho}.\overline{\tau} = 0$ and the multi-beta pricing relationship in Equation (7) reduces to:

$$\begin{aligned} E_0 R_{k,1} - R_f &= \overline{\alpha}_0 Cov_0\left(r_{M1}, R_{k,1}\right) + \overline{\alpha}_1 Cov_0\left(r_{M1}^2, R_{k,1}\right) + \overline{\alpha}_2 Cov_0\left(r_{M1}^3, R_{k,1}\right) \\ &\quad + \overline{\alpha}_3 Cov_0\left(Var_1\left(r_{M2}\right), R_{k,1}\right) + \overline{\alpha}_4 Cov_0\left(Skew_1\left(r_{M2}\right), R_{k,1}\right) \\ &\quad + \overline{\alpha}_5 Cov_0\left(r_{M1} Var_1\left(r_{M2}\right), R_{k,1}\right) \end{aligned} \tag{9}$$

[14] The case where the portfolio weight and the asset risk premia functions are approximated by $\omega_\tau^{(i)}(\sigma) = \sum_{j=0}^{r} \frac{\sigma^j}{j!} \omega_\tau^{(i)[j]}(0)$ and $a_{k,\tau}(\sigma) = \sum_{j=0}^{r} \frac{\sigma^j}{j!} a_{k,\tau}^{[j]}(0)$ with $r = 3$ produces an asset pricing model in which the aggregate kurtosis risk is priced. Due to space consideration, the result and its proof are available on request. The result indicates that the cross sectional skewness of investor skew-tolerances, is also important for asset pricing.

with:

$$
\begin{aligned}
\overline{\alpha}_0 &= \tfrac{1}{\overline{\tau}}R_f + \left(\tfrac{2}{\overline{\tau}^3}R_f^2 - 2\left(1+3R_f^2\right)\tfrac{\overline{\rho}}{\overline{\tau}^3}R_f\right)Var_0\left(r_{M1}\right), & \overline{\alpha}_1 &= -\tfrac{\overline{\rho}}{\overline{\tau}^2}R_f^2,\\
\overline{\alpha}_2 &= \tfrac{\overline{\kappa}}{\overline{\tau}^3}R_f^3, & \overline{\alpha}_3 &= \tfrac{(3-\overline{\rho})}{\overline{\tau}^2} - \tfrac{2}{R_f^2}\tfrac{1}{\overline{\tau}^2},\\
\overline{\alpha}_4 &= \tfrac{\overline{\kappa}}{\overline{\tau}^3} - \tfrac{2}{R_f^2}\tfrac{\overline{\rho}}{\overline{\tau}^2}, & \overline{\alpha}_5 &= \tfrac{3\overline{\kappa}}{\overline{\tau}^3}R_f - 4R_f\tfrac{\overline{\rho}}{\overline{\tau}^3} + \tfrac{2}{R_f}\tfrac{(1-2\overline{\rho})}{\overline{\tau}^2}
\end{aligned}
$$

3 Empirical Implications

In this section, we develop the empirical implications of the expected return decomposition given in Equations (5) and (7), and the SDFs derived in Equations (6) and (8). In the succeeding sections, we develop the empirical tests of these implications.

3.1 Pricing Coskewness and Systematic Volatility in the Cross-Section

Since the asset k is arbitrary, Equation (5) can be individually applied to R_{M1}. This is obviously not a problem since R_{M1} represents the return on traded assets. On the other hand, the volatility contract r_{M1}^2 and the volatility of future market return $Var_1\left(r_{M1}\right)$ may not be spanned by the traded assets and applying Equation (5) to these quantities involves the use of the returns $r_{M1}^2/\pi_{\mathbb{M}_2}$ and $Var_1\left(r_{M2}\right)/\pi_{\mathbb{V}_2}$, where $\pi_{\mathbb{M}_2}$ represents the fair value of the volatility contract and $\pi_{\mathbb{V}_2}$ is the fair value of the volatility of the future market return.[15] Applying Equation (5) to these returns gives a system of equations for the quantities $\overline{A}_0$, $\overline{B}_0$, and $\overline{D}_0$. Solving for these quantities and substituting back into Equation (5) then produces a multi-beta pricing model that involves covariance risk, coskewness risk and aggregate volatility risk. The above discussion implies that

$$
E_0R_{kt+1} - R_f = \alpha + \beta_1 E_0\left(R_{Mt+1} - R_f\right) + \beta_2 E_0\left(r_{Mt+1}^2 - R_f\right) + \beta_3 E_0 Var_{t+1}\left(r_{Mt+2}\right) \quad (10)
$$

where

$$
\alpha = \beta_2 R_f\left(1 - \pi_{\mathbb{M}_2}\right) - \beta_3 R_f \pi_{\mathbb{V}_2}.
$$

The above model is referred to as the MS_V model. In the case where $\alpha = 0$ and $\beta_3 = 0$, Equation (10) reduces to the Harvey and Siddique (2000) market coskewness model. When $\beta_2 = 0$, Equation (10) reduces to the Ang, Hodrick, Xing, and Zhang (2006) empirical

[15] See Bakshi, Madan and Kapadia (2003) for the definition of the volatility contract.

aggregate volatility risk model

$$E_0 R_{kt+1} - R_f = \alpha + \beta_1 E_0 \left(R_{Mt+1} - R_f\right) + \beta_3 E_0 Var_{t+1} \left(r_{Mt+2}\right). \tag{11}$$

The model described by Equation (11) is referred to as the M_V model. The sample counterpart of Equations (10) and (11) can be used to estimate alpha and the betas. If the excess risk factors are important for explaining risky returns, the estimated betas should be statistically significant. The main prediction from Equation (10) is that stocks with different loadings on the squared excess market return and aggregate volatility risk have different average returns. Equation (10) is consistent with the SDF $m_{t,t+1}$ (with $t = 0$) derived in Proposition 3. This SDF can be written in the form

$$m_{t,t+1} = \Theta_0 + \Theta_1^{\intercal} \mathbf{f}_{t+1}, \tag{12}$$

where $\mathbf{f}_{t+1}$ is the $J \times 1$ factor vector, and Θ_1 is the $J \times 1$ coefficient vector. Nonzero elements of Θ_1 indicate the importance of a factor as a determinant of the SDF. All pricing models have an equivalent representation in terms of multivariate betas and prices of risks

$$E_0 \mathbf{R}_{t+1} - R_f = \boldsymbol{\beta}^{\intercal} \boldsymbol{\lambda} = \sum_{j=1}^{J} \beta_j \lambda_j, \tag{13}$$

where $\boldsymbol{\beta} = Cov_0 \left(\mathbf{f}_{t+1}, \mathbf{f}_{t+1}^{\intercal}\right)^{-1} Cov_0 \left(\mathbf{f}_{t+1}, \mathbf{R}_{t+1}\right)$ and

$$\boldsymbol{\lambda} = -R_f Cov_0 \left(\mathbf{f}_{t+1}, \mathbf{f}_{t+1}^{\intercal}\right) \Theta_1 \tag{14}$$

with $\Theta_1 = \left(\Theta_{1j}\right)_{j=1,\dots,J}$. In Equation (13), the β's are the projections of the returns onto the factors, and the λ's are the prices of beta risks. To determine whether the jth factor significantly influences the expected returns on a particular set of portfolios, we must assess whether the corresponding market price of risk λ_j is significantly different from zero. Notice that $\lambda_j = 0$ does not mean $\Theta_{1j} = 0$ and vice versa. When $Cov_0 \left(\mathbf{f}_{t+1}, \mathbf{f}_{t+1}^{\intercal}\right)$ is diagonal, the two statements are equivalent. The Hansen and Jaganathan (1997) procedure is used to estimate the parameters Θ_1 and then compute the market prices of beta risks.

3.2 Pricing Aggregate Higher Moment Risks in the Cross-Section

We use Equation (7) and follow the methodology described in the above section and deduce the multi-beta pricing model that involves coskewness, cokurtosis, aggregate variance risk, aggregate skewness risk and aggregate correlation risk.

$$\begin{aligned} E_0 R_{kt+1} - R_f &= \alpha + \beta_1 E_0 \left(R_{Mt+1} - R_f \right) + \beta_2 E_0 \left(r_{Mt+1}^2 - R_f \right) + \beta_3 E_0 \left(r_{Mt+1}^3 - R_f \right) \\ &+ \beta_4 E_0 Var_{t+1} \left(r_{Mt+2} \right) + \beta_5 E_0 Skew_{t+1} \left(r_{Mt+2} \right) + \beta_6 E_0 r_{Mt+1} Var_{t+1} \left(r_{Mt+2} \right) \\ &+ \beta_7 E_0 r_{Mt+1}^{(\rho)} + \beta_8 E_0 \left(r_{Mt+1}^{\bar{\varsigma}_0} r_{Mt+1} - R_f \right) + \beta_9 E_0 \left(r_{Mt+1}^{\bar{\varsigma}_{0,\vartheta}} r_{Mt+1} - R_f \right). \end{aligned} \tag{15}$$

The model described by Equation (15) is referred to as the MSC_VS model. The sample counterpart of Equation (15) can be used to estimate alpha and the betas. If the risk factors appearing in Equation (15) are important for explaining risky returns, the estimated betas should be statistically significant. The procedure described in the previous section (see Equations (13) and (14)) is used to estimate the market prices of beta risks that correspond to the risk factors appearing in Equation (15).

Note that the return on the mimicking squared market return $r_{Mt+1}^{\bar{\varsigma}_0}$ is maximally correlated with the squared market return r_{Mt+1}^2. Therefore, if the squared market return r_{Mt+1}^2 is spanned by the set of basis assets, r_{Mt+1}^2 can be well approximated by $r_{Mt+1}^{\bar{\varsigma}_0}$. This implies that r_{Mt+1}^3 can be well approximated by the market-mimicking squared market return product $r_{Mt+1} r_{Mt+1}^{\bar{\varsigma}_0}$. In addition, if the aggregate volatility risk is spanned by the set of basis assets, $Var_1 \left(r_{M2} \right)$ can be well approximated by $r_{Mt+1}^{\bar{\varsigma}_{0,\vartheta}}$; and the market-aggregate volatility product $r_{M1} Var_1 \left(r_{M2} \right)$ can be well approximated by $r_{Mt+1}^{\bar{\varsigma}_{0,\upsilon}} r_{Mt+1}$. Furthermore, if the squared market return r_{Mt+2}^2 is spanned by the set of basis assets, r_{Mt+2}^2 can be approximated by $r_{Mt+2}^{\bar{\varsigma}_1}$; and the aggregate skewness risk factor $Skew_{t+1} \left(r_{Mt+2} \right)$ can be well approximated by the aggregate correlation risk $r_{Mt+1}^{(\rho)}$. The above discussion implies that the multi-beta pricing model in Equation (15) can be simplified to

$$\begin{aligned} E_0 R_{kt+1} - R_f &= \alpha + \beta_1 E_0 \left(R_{Mt+1} - R_f \right) + \beta_2 E_0 \left(r_{Mt+1}^{\bar{\varsigma}_0} - R_f \right) \\ &+ \beta_3 E_0 \left(r_{Mt+1}^{\bar{\varsigma}_0} r_{Mt+1} - R_f \right) + \beta_4 E_0 r_{Mt+1}^{\bar{\varsigma}_{0,\vartheta}} \\ &+ \beta_5 E_0 r_{Mt+1}^{(\rho)} + \beta_6 E_0 r_{Mt+1}^{\bar{\varsigma}_{0,\vartheta}} r_{Mt+1}, \end{aligned} \tag{16}$$

or alternatively to

$$
\begin{aligned}
E_0 R_{kt+1} - R_f &= \alpha + \beta_1 E_0 \left(R_{Mt+1} - R_f\right) + \beta_2 E_0 \left(r_{Mt+1}^2 - R_f\right) \\
&\quad + \beta_3 E_0 \left(r_{Mt+1}^3 - R_f\right) + \beta_4 E_0 Var_{t+1} \left(r_{Mt+2}\right) \\
&\quad + \beta_5 E_0 Skew_{t+1} \left(r_{Mt+2}\right) + \beta_6 E_0 r_{M1} Var_1 \left(r_{M2}\right).
\end{aligned} \tag{17}
$$

It also implies that the alphas and betas loading in Equations (16) and (17) should be nearly identical. The multi-beta models in Equations (16) and (17) are referred to as the Reduced-MSC_VS(1) and Reduced-MSC_VS(2) models respectively.

4 Empirical Results

In this section, we apply the empirical models developed in the previous section to the data by choosing empirical proxies for the risk factors. We then report and discuss the results of the empirical analysis.

4.1 Data

4.1.1 Risk Factors

To investigate how coskewness risk, cokurtosis risk, aggregate volatility risk, aggregate skewness risk, and aggregate correlation risk are priced in the cross-section of equity returns, we use an observable proxy for the market factor and compute observable proxies for the factors representing aggregate volatility risk, aggregate skewness risk, and aggregate correlation risk. We use the CRSP value-weighted market index to proxy the market return. We use the daily value-weighted index from CRSP for the period 1965-2006 to compute the market volatility

$$
v_{Mt}^2 = \sqrt{\sum_{q=1}^{Q-1} \left(R_{M,t+q+1} - R_{M,t+q}\right)^2 / Q} \tag{18}
$$

in each month, where $R_{M,t+q}$ is the daily return on the market portfolio on day q of month t, with Q representing the number of trading days in the month. We compute the market aggregate skewness as

$$
s_{Mt} = \frac{\frac{1}{Q} \sum_{q=1}^{Q} \left(R_{M,t+q} - \mu_{Mt}\right)^3}{\sigma_{Mt}^3} \tag{19}
$$

with

$$\sigma_{Mt}^{2}=\frac{1}{Q}\sum_{q=1}^{Q}\left(R_{M,t+q}-\mu_{Mt}\right)^{2},$$

where μ_{Mt} and σ_{Mt}^{2} are commonly defined as the first and second central moments. We also compute the monthly returns on the mimicking squared return $R_{Mt+1}^{\bar{\varsigma}_t}$ and the mimicking aggregate volatility risk $R_{Mt+1}^{\bar{\varsigma}_{t,\vartheta}}$. To do this, we first use monthly stock returns to compute the sample counterpart of the weights $\bar{\varsigma}_t$ and $\bar{\varsigma}_{t,\vartheta}$ which we denote $\widehat{\bar{\varsigma}}_t$ and $\widehat{\bar{\varsigma}}_{t,\vartheta}$, and we use $R_{Mt+1}^{\bar{\varsigma}_t}=\widehat{\bar{\varsigma}}_t^{\intercal}\mathbf{R}_{t+1}$ and $R_{Mt+1}^{\bar{\varsigma}_{t,\vartheta}}=\widehat{\bar{\varsigma}}_{t,\vartheta}^{\intercal}\mathbf{R}_{t+1}$ as proxies for the return on these mimicking portfolios. To compute the aggregate correlation risk factor $r_{Mt+1}^{(\rho)}$, we use a sample correlation between the daily market return R_{Mt+2} and the return on the mimicking skewness portfolio $R_{Mt+2}^{\bar{\varsigma}_{t+1}}$ within each month. Rather than work with the correlation values, which by construction are bounded by $[-1,1]$, we transform them to Fisher correlation defined by

$$Fcorr=0.5\times\ln\left(\frac{1+r_{Mt+1}^{(\rho)}}{1-r_{Mt+1}^{(\rho)}}\right).$$

This function is continuous and monotonic, and there is one-to-one mapping between the actual correlation $r_{Mt+1}^{(\rho)}$ and the Fisher-transformed correlation. Finally, we transform each risk factor f_{t+1}, except the market factor, as $g_{f_{t+1}}=\left(f_{t+1}-f_t\right)/f_t$.

4.1.2 Stock Returns

We use two sets of independent stock returns dataset. First, we use 30 industry portfolios. The industry portfolios are formed by sorting the NYSE/ AMEX/ NASDAQ equities into groups based on SIC codes. Second, we use the 25 Fama and French portfolios. The Fama and French portfolios are formed by sorting the NYSE/ AMEX/ NASDAQ equities into groups based on size and book-to-market.[16]

4.2 Results

4.2.1 Aggregate Volatility Risk

Panel A of Table 2 summarizes the multivariate regression results for the M_V model using industry portfolios as the dependent variables. The intercepts are significant for far fewer industries, the market betas are significant for all industry portfolios, and the aggregate

[16] We thank Kenneth French for making these data available on his website: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

volatility risk betas are significant for far fewer industries (8 of 30 industries). Panel B of Table 2 shows the cross-sectional average of alphas and betas. The average value of alphas is significant at the 1% level. This suggests that the M_V model might be misspecified. The average value of the market betas is positive and significant at the 1% level, while the average value of the aggregate volatility risk betas is negative and significant at the 1% level. In other words, following periods of unexpected increased market volatility, stock prices drop on average, controlling for the market risk. This is consistent with the literature on asymmetric volatility, especially recent works that examine the effect of aggregate volatility risk on the cross-section of returns [see e.g. Ang et al. (2006) and Boehme et al. (2007)]. They find a negative relation between stock returns and the market aggregate volatility risk. To see the economic impact of the aggregate volatility risk on industry portfolios, consider the Txtls portfolio which has the largest beta in magnitude. For a 1% increase in the excess aggregate volatility risk factor, the portfolio excess return decreases by 0.016%. Panel C evaluates the significance of the betas for each factor. The hypothesis of a zero-beta vector for each factor is rejected. This result reveals that the market and the aggregate volatility risk factors have a significant impact on industry portfolio expected excess returns. We repeat the same analysis in using the 25 Fama and French portfolios. The results are qualitatively similar.

4.2.2 Market Coskewess and Aggregate Volatility Risk

Panel A of Table 3 summarizes the multivariate regression results for the MS_V model using industry portfolios as the dependent variables. The intercepts are significant for far fewer industries, the market betas are significant for all industry portfolios, the squared market return betas are significant for 12 of 30 industries, and the aggregate volatility risk betas are significant for far fewer industries (7 of 30). Panel B of Table 3 shows the cross-sectional average of alphas and betas. The average value of alphas is significant at the 1% level. Similarly, to the M_V model, this suggests that the MS_V model might also be misspecified. The average value of the market betas is positive and significant at the 1% level, the average value of the squared market return betas is negative but not significant. However, the average value of the aggregate volatility risk betas is negative and significant at the 1% level. Panel C evaluates the significance of the betas for each factor. These results reveal that the market, the squared market, and the aggregate volatility risk factors have a significant impact on industry portfolios. It is useful to point out that with the squared market factor, there is nearly no decrease in the number of significant betas for the aggregate volatility risk factor. Moreover, as shown in Panel C, the hypothesis of a zero-beta vector for each factor is rejected. Thus, the squared market return does not drive out the joint significance of the aggregate volatility risk factor. We repeat the same analysis in using the 25 Fama and French portfolios. The results are qualitatively similar.

4.2.3 Market Coskewness, Market Cokurtosis, and Aggregate Higher Moment Risk

Table 4 summarizes the multivariate regression results for the MSC_VS model using industry portfolios as the dependent variables. The intercepts are significant for far fewer industries: Food, Beer, ElcEq, Whlsl, Rtail and Other. The market betas are significant for all industry portfolios. The squared market return betas are significant and negative for far fewer industries: 10 out of 30 industries have significant and negative betas, while one industry's beta is significant and positive. To see the economic impact of the squared market return on industry portfolios, consider the Mines and Coal portfolios, which have the largest beta in magnitude. For a 1% increase in the excess squared market risk factor, the Mines portfolio excess return decreases by 1.37%, and the Coal portfolio excess return decreases by 1.63%. The cubic market return betas are significant and positive for 9 of 30 industries. To see the economic impact of the cubic market return on industry portfolios, consider again the Mines portfolio, which has the largest beta in magnitude. For a 1% increase in the excess cubic market return risk factor, the Mines portfolio excess return increases by 0.90%. On the other hand, the aggregate volatility risk betas are significant and negative for about one-half of industry portfolios. To investigate the economic impact of the aggregate volatility risk factor on industry portfolios, consider again the Mines portfolio, which has the largest beta in magnitude. For a 1% increase in the excess aggregate volatility risk factor, the Mines portfolio excess return decreases by 0.13%. The aggregate skewness risk betas are significant and positive for most industry portfolios (26 out of 30 industries). However, the magnitude of their betas is small, lower than 0.001. Furthermore, the market-aggregate volatility product betas are significant and positive for about half of industry portfolios. To investigate the economic impact of this risk factor on industry portfolios, consider the Mines portfolio, which has the largest beta in magnitude.

For a 1% increase in the excess market-aggregate volatility risk factor, the Mines portfolio excess return increases by 0.13%. The aggregate correlation risk betas are significant for far fewer industries. The magnitude of its betas is similar to the magnitude of the aggregate skewness risk betas. The market-mimicking squared market return product betas are significant for more than half of industry portfolios. To investigate the economic impact of this risk factor on industry portfolios, consider the Servs portfolio, which has the largest beta in magnitude. For a 1% increase in the market-mimicking skewness risk factor, the Servs portfolio excess return increases by 0.44%. The market-aggregate volatility product betas are significant for 16 of 30 industries. To investigate the economic impact of this risk factor on industry portfolios, consider the Coal portfolio which has the largest beta in magnitude. For a 1% increase in the excess market-aggregate volatility risk factor, the Coal portfolio excess return increases by 0.07%.

Panel B of Table 4 shows the cross-sectional average of alphas and betas. The average value of alphas is not significant. This suggests that the MSC_VS model might be correctly specified. The average value of the market betas is positive and significant at the 1% level. The average value of the squared market return betas is negative and significant at the 1% level. On average, for a 1% increase in the excess squared market factor, the average industry portfolios excess return decreases by 0.2%. The average value of the cubic market return beta is positive and not significant, while the average value of the aggregate volatility betas is negative and significant at the 1% level. Note that the cross-sectional average of the aggregate skewness betas is significant at the 5% level and small in magnitude. Panel C shows that the hypothesis of a zero-beta vector for each factor is rejected, except for the aggregate correlation risk factor. This suggests that the market return, squared market return, cubic market return, aggregate volatility risk, aggregate skewness risk, market-aggregate volatility risk product, and market-mimicking squared market return product have a significant impact on industry portfolios. We repeat the same analysis in using the 25 Fama and French portfolios. The results are qualitatively similar.

4.2.4 Are unspanned squared market and unspanned aggregate volatility risk priced?

We investigate whether the component of the squared market and the component of the aggregate volatility risk that cannot be spanned by industry portfolios are priced. Recall that in section 3.2 we show that if the squared market and the aggregate volatility risk are spanned by the basis assets, the estimated alphas and betas in the multi-beta pricing relationships (16) and (17) should be nearly identical. Tables 5 and 6 show the estimated betas for the Reduced-MSC_VS(1) and Reduced-MSC_VS(2) models.

First, the cross-sectional average of alphas is not significant for the Reduced-MSC_VS(1) model while it is significant at the 1% level for the Reduced-MSC_VS(2) model. Table 5 shows that the squared market betas are significant and negative for far fewer industries: 8 of 30 industries have significant and negative betas. The cross-sectional average of betas is negative and significant at the 10% level. In contrast, Table 6 shows that the mimicking squared market return betas are significant for about one-half of 30 industries. Some industries have positive betas while other have negative betas. Furthermore, the cross-sectional average of betas is not significant. This suggests that the component of the squared market return that is not spanned by industry portfolio returns is priced.

Table 5 shows that the aggregate volatility risk betas are significant and negative for 14 of 30 industries. The cross-sectional average of betas is negative and significant at the 1% level. In contrast, Table 6 shows that the mimicking aggregate volatility risk betas are significant and positive for far fewer industries (4 of 30 industries) and the cross-sectional average of aggregate volatility betas is significant and positive at the 1% level. This suggests that both the spanned and unspanned components of the aggregate volatility risk factor are priced. We repeat the same analysis in using the 25 Fama and French portfolios. The results are qualitatively similar.

4.2.5 Market Prices of Risks

Table 7 shows the market prices of beta risks when the 25 Fama and French portfolios are used. We use different subsamples: 1996-2006, 1993-2006, 1990-2006, 1987-2006, 1984-2006, 1981-2006 and the full sample which is from 1965-2006. The Hansen and Jagannathan (1997) distance measure is used to estimate the coefficients of the SDF. The coefficients of the risk factors appearing in the SDF are then used to compute the market prices of beta risks using Equation (14). Panel A presents the results when the M_V model is used. Consistent with our theoretical predictions, the price of the market factor is significant and positive, while the market price of the aggregate volatility risk factor is significant and negative for all sample periods, except for the full sample. This price ranges from -0.497

to -0.291. Panel B presents the results when the MS_V model is used. The price of the market factor is significant and positive for all sample periods. Note that the price of the squared market factor is significant and positive for all sample periods. This result does not contradict our theory because we are computing the market prices of beta risks, which is slightly different from the market prices of covariance risks. The positive sign of this price comes from the correlation between the risk factors. On the other hand, we find that the market price of the aggregate volatility risk factor is significant and negative for all sample periods, except for the full sample. This price ranges from -0.52 to -0.27. Panel C presents the results when the MSC_VS model is used. The price of the market and squared market factors are significant and positive for all sample periods. The price of the cubic market factor is significant and positive for all sample periods, except for the 1981-2006 sample period. Furthermore, the market price of the aggregate volatility risk factor is significant and negative for all sample periods, except for the full sample. This price ranges from -0.94 to -0.32. The market price of the market-aggregate volatility product is significant and negative for all sample periods, except for the full sample. Given the analytical expressions of the market price of aggregate volatility risk (see Proposition 4), we conclude that the cross-sectional variance of investor skew-tolerances has a significant impact on assets' risk premia. Furthermore, the aggregate skewness risk whose market price is 0.24, is significant for the full sample while the market price of correlation risk is not significant. We repeat the same analysis in using the 25 Fama and French portfolios augmented with the 30 industry portfolios. The results are qualitatively similar.

4.2.6 Do the cross-sectional variance and covariance of investor preferences matter for asset pricing?

Tables 4 and 5 show the multivariate regression results for the MSC_VS and Reduced MSC_VS(1) models respectively. As shown in Table 4, the factors that capture the cross-sectional variance and covariance of investor preferences have a significant impact on industry portfolios. For example, the aggregate correlation betas are significant for the Hshld, ElcEq, Autos, Mines, Util, and Paper industries. The market-mimicking squared market product betas and the market-mimicking aggregate volatility product betas are significant for more than half the industries. In addition, the aggregate volatility, aggregate skewness, and market-aggregate volatility betas shown in Tables 4 and 5 are different. This indicates that the cross-sectional variance and covariance of investor preferences have a significant impact on the assets' risk premia via the aggregate volatility, aggregate skewness, and market-aggregate volatility betas. Panel C of Table 5 evaluates the joint significance of the betas for each factor. The hypothesis of a zero-beta vector for each factor is rejected. The factors that capture the cross-sectional variance and covariance of investor preferences have a significant impact on industry portfolios. Note that with the aggregate correlation risk, market-mimicking squared market product, and market-mimicking aggregate volatility product, there is nearly no decrease in the number of significant betas for the market, squared market, cubic market, aggregate volatility risk, aggregate skewness risk, and market-aggregate volatility product factors.

Panels C and D of Table 7 present the market prices of risk for the MVC_VS and Reduced MVC_VS(1) models. As shown in Panels C and D, the prices of the market, squared market, and cubic market betas are quite similar. However, the difference between the prices of the aggregate volatility, aggregate skewness, and market-aggregate volatility product betas is economically important. This is due to the cross-sectional variance and covariances of investor preferences. To see this, recall that Section 2.5 theoretically shows that the Reduced MSC_VS(1) model is obtained by setting the cross-sectional variance and covariance parameters to zero. When these parameters are not null, our theoretical result predicts that the cross-sectional variance and covariance of investor preferences have a significant impact on the market prices of aggregate volatility, aggregate skewness, and market-mimicking aggregate volatility risk product. Moreover, if the cross-sectional variance and covariance of investor preferences are important for asset pricing, then the market price of the aggregate correlation risk, market-mimicking squared market, and market-mimicking aggregate volatility product have a significant impact on the assets' risk premia. This prediction is also confirmed by the empirical results obtained in panel C.

5 Summary and Concluding Remarks

We generalize the market coskewness of Rubinstein (1973), Harvey and Siddique (2000), the market cokurtosis of Dittmar (2000) and the empirical aggregate market volatility model of Ang et al. (2006). The equilibrium SDF depends not only on the market return, the squared market return, and the cubic market return, but also on the aggregate volatility risk, aggregate skewness risk, aggregate correlation risk, market-mimicking squared market product, and market-mimicking aggregate volatility product. This implies that the expected return on any risky assets depends not only on its covariance, coskewness, and cokurtosis with the market return, but also on its covariance with the aggregate volatility risk, aggregate skewness risk, aggregate correlation risk, market-mimicking squared market product, and market-mimicking aggregate volatility product. We show that the market prices of the risk factors appearing in the SDF have a structural interpretation in terms of the average value of investor risk-tolerances, skew-tolerances, kurtosis-tolerances, the cross-sectional variance of investor risk-tolerances, the cross-sectional variance of investor skew-tolerances, and finally the cross-sectional covariance of investor risk-tolerances with skew-tolerances. Empirical results with the 30 industry portfolios and the 25 Fama and French portfolios suggest that the market prices of most risk factors appearing in the SDF have a significant impact on the expected excess return on the assets.

Further research in this area could address at least two issues. First, although the theoretical analysis contains some conditioning information due to the rebalancing of investor portfolio weight at each period, the above analysis is not a formal conditional test for asset pricing models with higher moments. Our theory predicts that some of the coefficients of the SDF vary through time. It can be shown that a dynamic model identical to ours in which investors care about preferences beyond kurtosis will lead to a SDF in which the market prices of all risk factors appearing in the SDF vary throughout time. This will also help to identify, in equilibrium, which conditioning variables should be used to estimate the time-varying market prices that correspond to the market return, squared market return, cubic market return, aggregate volatility risk, aggregate skewness risk, aggregate kurtosis risk, aggregate correlation risk, market-mimicking squared return product, and the market-mimicking aggregate volatility risk product. This is an important issue in asset pricing since existing models assume that the coefficients of risk factors appearing in the SDF are linear functions of exogeneous conditioning variables. Second, instead of estimating the market prices of risks via the distribution of investor preferences, we first estimate the coefficients of the SDF that are nonlinear functions of investor preferences, and then use the coefficients to compute the market prices of risk. Further research could use the SDF proposed in this paper to recover the distribution of investor preferences.

In summary, the theory presented here offers a justifiable reason for using both the market return, the squared of the market return, the cubic market return, the aggregate volatility risk, aggregate skewness risk, aggregate correlation risk, the market-mimicking squared return product, and the market-mimicking aggregate volatility risk as explanatory variables.

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6 Appendix

Proof of Propositions 1, 2, and 3. We solve backward investor's portfolio choice.

- First, we solve investor's portfolio choice at date 1. Each investor solves the following optimization problem:

$$\max_{\left\{C_1^{(i)}(\sigma),\, \omega_{1,k}^{(i)}(\sigma)\right\}_{k=1}^{n}} E_1\left[u_i\left(C_0^{(i)}(\sigma)\right)+\beta u_i\left(C_1^{(i)}(\sigma)\right)+\beta^2 u_i\left(W_2^{(i)}(\sigma)\right)\right]. \quad (20)$$

The FOCs with respect to investor's consumption and portfolio weights are:

$$\begin{aligned} C_1^{(i)}(\sigma): &\quad u_i'\left(C_1^{(i)}(\sigma)\right)-\beta E_1 u_i'\left(W_2^{(i)}(\sigma)\right)\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)=0,\\ \omega_{1,k}^{(i)}(\sigma): &\quad E_1 u_i'\left(W_2^{(i)}(\sigma)\right)\left(\sigma a_{k,1}(\sigma)+Y_{k,2}\right)=0. \end{aligned}$$

- Second, given the results derived from Equation (20), we solve investor's portfolio choice at date 0. Each investor solves the following optimization problem:

$$\max_{\left\{C_0^{(i)}(\sigma),\, \omega_{0,k}^{(i)}(\sigma)\right\}} E_0\left[u_i\left(C_0^{(i)}(\sigma)\right)+\beta u_i\left(C_1^{(i)}(\sigma)\right)+\beta^2 u_i\left(W_2^{(i)}(\sigma)\right)\right].$$

The FOCs with respect to investor's consumption and portfolio weights are:

$$\begin{aligned} C_0^{(i)}(\sigma): &\quad u_i'\left(C_0^{(i)}(\sigma)\right)-\beta^2 E_0 u_i'\left(W_2^{(i)}(\sigma)\right)\left(R_f+\omega_0^{(i)\intercal}(\sigma)\,\mathbf{R}_1^e(\sigma)\right)\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)=0.\\ \omega_{0,k}^{(i)}(\sigma): &\quad E_0 u_i'\left(W_2^{(i)}(\sigma)\right)\left(\sigma a_{k,0}(\sigma)+Y_{k,1}\right)\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)=0. \end{aligned}$$

To give a formal proof of Propositions 1, 2 and 3, we first derive $\omega_\tau^{(i)}(0)$, $a_{k,\tau}(0)$, $\omega_\tau^{(i)'}(0)$, $a_{k,\tau}'(0)$ for $\tau=0,1$.

First step: we derive $\omega_\tau^{(i)}(0)$ and $a_{k,\tau}(0)$ for $\tau=0,1$

At date 1, we consider the FOC with respect to the portfolio weights and take the first derivative of this FOC with respect to σ:

$$E_1 u_i''\left(W_2^{(i)}(\sigma)\right)W_2^{(i)'}(\sigma)\left(\sigma a_{k,1}(\sigma)+Y_{k,2}\right)+E_1 u_i'\left(W_2^{(i)}(\sigma)\right)\left(a_{k,1}(\sigma)+\sigma a_{k,1}'(\sigma)\right)=0.$$

Then, we take the limit as σ converges to zero of expression above and get:

$$E_1 \lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)\,Y_{k,2}=\tau_i a_{k,1}(0), \quad (21)$$

with:

$$\tau_i = -\frac{u_i^{'}\left(W_2^{(i)}(0)\right)}{u_i^{''}\left(W_2^{(i)}(0)\right)}.$$

Denote:

$$\mathbf{Y}_\tau = \left(Y_{k\tau}\right)_{k=1,\dots n} \text{ for } \tau = 1, 2.$$

Note that the limit as σ converges to zero of the first derivative of $W_2^{(i)}(\sigma)$ with respect to σ is:

$$\begin{aligned}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)\prime}(\sigma) &= \lim_{\sigma\rightsquigarrow 0}\left(W_0^{(i)}(\sigma) - C_0^{(i)}(\sigma)\right)^{'} R_f^2 \qquad (22)\\ &+ \lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0)\,\mathbf{Y}_1\right) + \frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right) - \lim_{\sigma\rightsquigarrow 0} C_1^{(i)\prime}(\sigma)\,R_f.\end{aligned}$$

Next, we replace Equation (22) in Equation (21) and get:

$$E_1\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right) Y_{k,2} = \tau_i a_{k,1}(0). \qquad (23)$$

Note that the market clearing condition at date 1 is:

$$\sum_{i=1}^{I}\left(W_1^{(i)}(\sigma) - C_1^{(i)}(\sigma)\right)\omega_1^{(i)}(\sigma) = \overline{\theta}_1.$$

Hence, the limit as σ converges to zero of the market clearing condition is

$$\frac{1}{R_f}\sum_{i=1}^{I}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)\omega_1^{(i)}(0) = \overline{\theta}_1.$$

Now, we take the sum of Equation (23) for $i = 1, ..., I$, and utilize of the limit of the market clearing conditions when σ converges to zero to deduce:

$$\frac{1}{\overline{\tau}} Cov_1\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right), Y_{k,2}\right) = a_{k,1}(0) \text{ with } \overline{\omega}_1 = \frac{1}{I}\overline{\theta}_1 \text{ and } \overline{\tau} = \frac{1}{I}\sum_{i=1}^{I}\tau_i.$$

We, thereafter, replace expression above into Equation (23) and get:

$$E_1\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right) Y_{k,2} = \frac{\tau_i}{\overline{\tau}} Cov_1\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right), Y_{k,2}\right).$$

Hence, expression above can be solved for $\omega_1^{(i)}(0)$:

$$\omega_1^{(i)}(0) = \frac{R_f}{W_2^{(i)}(0)}\frac{\tau_i}{\overline{\tau}}\overline{\omega}_1.$$

At date 0, we consider the FOCs with respect to the portfolio weight, take the first derivative of this FOC with respect to σ, then take its limit as σ converges to 0 and get:

$$E_0 \lim_{\sigma \rightsquigarrow 0} W_2^{(i)'}(\sigma) Y_{k,1} = \tau_i a_{k,0}(0). \tag{24}$$

Now, we replace Equation (22) in Equation (24) and get:

$$E_0 \lim_{\sigma \rightsquigarrow 0} W_2^{(i)'}(\sigma) Y_{k,1} = E_0 \lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma) \left(\omega_0^{(i)\intercal}(0) \mathbf{Y}_1\right) Y_{k,1} - E_0 \lim_{\sigma \rightsquigarrow 0} C_1^{(i)'}(\sigma) R_f Y_{k,1}. \tag{25}$$

To solve Equation (25) for $\omega_0^{(i)}(0)$, we first find $\lim_{\sigma \rightsquigarrow 0} C_1^{(i)'}(\sigma)$. To derive the former expression, we take the FOC with respect to consumption at date 1:

$$u_i'\left(C_1^{(i)}(\sigma)\right) - \beta E_1 u_i'\left(W_2^{(i)}(\sigma)\right)\left(R_f + \omega_1^{(i)\intercal}(\sigma) \mathbf{R}_2^e(\sigma)\right) = 0.$$

Then, we take the first derivative of this FOC and take its limit as σ converges to zero:

$$u_i''\left(\lim_{\sigma \rightsquigarrow 0} C_1^{(i)}(\sigma)\right) \lim_{\sigma \rightsquigarrow 0} C_1^{(i)'}(\sigma) - \beta R_f u_i''\left(\lim_{\sigma \rightsquigarrow 0} W_2^{(i)}(\sigma)\right) E_1\left(\lim_{\sigma \rightsquigarrow 0} W_2^{(i)'}(\sigma)\right) = 0 \tag{26}$$

To solve Equation (26) for $\lim_{\sigma \rightsquigarrow 0} C_1^{(i)'}(\sigma)$, we first use Equation (22) to derive:

$$E_1 \lim_{\sigma \rightsquigarrow 0} W_2^{(i)'}(\sigma) = \lim_{\sigma \rightsquigarrow 0}\left(W_0^{(i)}(\sigma) - C_0(\sigma)\right)' R_f^2 + \lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0) \mathbf{Y}_1\right) - \lim_{\sigma \rightsquigarrow 0} C_1'(\sigma) R_f.$$

Then, we replace expression above in Equation (26) and get:

$$\begin{aligned}
& u_i''\left(\lim_{\sigma \rightsquigarrow 0} C_1^{(i)}(\sigma)\right) \lim_{\sigma \rightsquigarrow 0} C_1^{(i)'}(\sigma) \\
& -\beta R_f u_i''\left(\lim_{\sigma \rightsquigarrow 0} W_2^{(i)}(\sigma)\right)\left(\begin{array}{c} \lim_{\sigma \rightsquigarrow 0}\left(W_0^{(i)}(\sigma) - C_0^{(i)}(\sigma)\right)' R_f^2 \\ +\lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0) \mathbf{Y}_1\right) - \lim_{\sigma \rightsquigarrow 0} C_1^{(i)'}(\sigma) R_f \end{array}\right). \\
= \;& 0.
\end{aligned}$$

We divide expression above by $u_i''\left(\lim_{\sigma \rightsquigarrow 0} C_1^{(i)}(\sigma)\right)$ and get:

$$\lim_{\sigma \rightsquigarrow 0} C_1^{(i)'}(\sigma) = \nu_i \lim_{\sigma \rightsquigarrow 0}\left(W_0^{(i)}(\sigma) - C_0^{(i)}(\sigma)\right)' R_f + \frac{1}{R_f} \nu_i\left(\lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0) Y_1\right)\right) \tag{27}$$

with

$$\nu_i = \frac{\beta R_f^2}{\frac{u_i''\left(C_1^{(i)}(0)\right)}{u_i''\left(W_2^{(i)}(0)\right)} + \beta R_f^2}. \tag{28}$$

Next, we replace Equation (27) in Equation (25) and deduce

$$E_0 \lim_{\sigma \rightsquigarrow 0} W_2^{(i)'}(\sigma) Y_{k,1} = (1-\nu_i) E_0 \lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma) \left(\omega_0^{(i)\intercal}(0) \mathbf{Y}_1\right) Y_{k,1}.$$

We, thereafter, use expression above to rewrite Equation (24) as

$$E_0 \lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma) \left(\omega_0^{(i)\intercal}(0) \mathbf{Y}_1\right) Y_{k,1} = \tau_{\nu_i} a_{k,0}(0) \text{ with } \tau_{\nu_i} = \frac{\tau_i}{1-\nu_i}. \tag{29}$$

Note that the market clearing condition at date 0 is:

$$\sum_{i=1}^{I} \left(W_0^{(i)}(\sigma) - C_0^{(i)}(\sigma)\right) \omega_0^{(i)}(\sigma) = \overline{\theta}_0.$$

Hence, the limit as σ converges to zero of the market clearing condition is:

$$\frac{1}{R_f} \sum_{i=1}^{I} \left(\lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma)\right) \omega_0^{(i)}(0) = \overline{\theta}_0. \tag{30}$$

We take the sum of Equation (29) for $i = 1, ..., I$, and utilize the limit of the market clearing conditions in Equation (30) as σ converges to zero and get:

$$\frac{1}{\overline{\tau}_\nu} R_f Cov_0 \left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right), Y_{k,1}\right) = a_{k,0}(0) \text{ with } \overline{\tau}_\nu = \frac{1}{I} \sum_{i=1}^{I} \tau_{\nu_i} \text{ and } \overline{\omega}_0 = \frac{1}{I} \overline{\theta}_0.$$

Now, we replace expression above into Equation (29) and deduce:

$$\omega_0^{(i)}(0) = \frac{R_f}{\lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma)} \frac{\tau_{\nu_i}}{\overline{\tau}_\nu} \overline{\omega}_0.$$

To summarize:

$$\begin{aligned} \omega_1^{(i)}(0) &= \frac{R_f}{W_2^{(i)}(0)} \frac{\tau_i}{\overline{\tau}} \overline{\omega}_1 \text{ and } a_{k,1}(0) = \frac{1}{\overline{\tau}} Cov_1 \left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right), Y_{k,2}\right). \\ \omega_0^{(i)}(0) &= \frac{R_f}{W_1^{(i)}(0)} \frac{\tau_{\nu_i}}{\overline{\tau}_\nu} \overline{\omega}_0 \text{ and } a_{k,0}(0) = \frac{1}{\overline{\tau}_\nu} R_f Cov_0 \left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right), Y_{k,1}\right). \end{aligned}$$

Second step: we derive $\omega_\tau^{(i)'}(0)$ and $a_{k,\tau}^{'}(0)$ for $\tau = 0, 1$

At date 0, we consider the FOCs with respect to the portfolio weight and take the second derivative of this FOC with respect to σ:

$$\begin{aligned} & E_1 \left(u_i^{'''}\left(W_2^{(i)}(\sigma)\right)\left(W_2^{(i)'}(\sigma)\right)^2 + u_i^{''}\left(W_2^{(i)}(\sigma)\right) W_2^{(i)''}(\sigma)\right)\left(\sigma a_{k,1}(\sigma) + Y_{k,2}\right) + \\ & E_1 u_i^{'}\left(W_2^{(i)}(\sigma)\right)\left(2a_{k,1}^{'}(\sigma) + \sigma a_k^{''}(\sigma)\right) + 2E_1\left(u_i^{''}\left(W_2^{(i)}(\sigma)\right) W_2^{(i)'}(\sigma)\right)\left(a_{k,1}(\sigma) + \sigma a_{k,1}^{'}(\sigma)\right) \\ = \;& 0. \end{aligned}$$

The limit as σ converges to zero of expression above is:

$$\begin{aligned}
& E_1 u_i^{'''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)\right)^2 Y_{k,2} \\
& +E_1 u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)\lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma) Y_{k,2} + 2E_1 u_i^{'}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) a_{k,1}^{'}(\sigma) \\
& +2E_1 u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma) a_{k,1}(\sigma) \\
= \; & 0.
\end{aligned} \tag{31}$$

To simplify Equation (31), we compute the limit as σ converges to zero of the second derivative of $W_2^{(i)}(\sigma)$ with respect to σ:

$$\begin{aligned}
\lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma) = & \lim_{\sigma\rightsquigarrow 0}\left(W_0^{(i)}(\sigma) - C_0^{(i)}(\sigma)\right)^{''} R_f^2 \\
& + \lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\begin{pmatrix} 2\omega_0^{(i)\intercal}(0) a_0(0) + 2\omega_0^{(i)'\intercal}(0)\mathbf{Y}_1 \\ 2\omega_1^{(i)\intercal}(0) a_1(0) + 2\omega_1^{(i)'\intercal}(0)\mathbf{Y}_2 \end{pmatrix} \\
& +2\lim_{\sigma\rightsquigarrow 0}\left(W_0^{(i)}(\sigma) - C_0^{(i)}(\sigma)\right)^{'} R_f\left(\omega_0^{(i)\intercal}(0)\mathbf{Y}_1 + \omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right) \\
& +2\lim_{\sigma\rightsquigarrow 0}\left(W_0^{(i)}(\sigma) - C_0^{(i)}(\sigma)\right)\left(\omega_0^{(i)\intercal}(0)\mathbf{Y}_1\right)\left(\omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right) \\
& -\lim_{\sigma\rightsquigarrow 0} C_1^{(i)''}(\sigma) R_f - \lim_{\sigma\rightsquigarrow 0} C_1^{(i)}(\sigma)\left(2\omega_1^{(i)\intercal}(0) a_1(0) + 2\omega_1^{(i)'\intercal}(0)\mathbf{Y}_2\right) \\
& -2\lim_{\sigma\rightsquigarrow 0} C_1^{(i)'}(\sigma)\left(\omega_1^{(i)\intercal}(\sigma)\mathbf{Y}_2\right).
\end{aligned} \tag{32}$$

Hence:

$$\begin{aligned}
E_1\lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma) Y_{k,2} = & \frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma) E_1\left(2\omega_1^{(i)'\intercal}(0)\mathbf{Y}_2\right) Y_{k,2} \\
& -2\lim_{\sigma\rightsquigarrow 0} C_0^{(i)'}(\sigma) R_f E_1\left(\omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right) Y_{k,2} \\
& +2\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0)\mathbf{Y}_1\right) E_1\left(\omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right) Y_{k,2} \\
& -2\lim_{\sigma\rightsquigarrow 0} C_1^{(i)'}(\sigma) E_1\left(\omega_1^{(i)\intercal}(\sigma)\mathbf{Y}_2\right) Y_{k,2}.
\end{aligned} \tag{33}$$

Now, we replace Equation (27) in Equation (33) and get:

$$\begin{aligned}
E_1\lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma) Y_{k,2} = & \frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma) E_1\left(2\omega_1^{(i)'\intercal}(0)\mathbf{Y}_2\right) Y_{k,2} \\
& -2\lim_{\sigma\rightsquigarrow 0} C_0^{'}(\sigma) R_f E_1\left(\omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right) Y_{k,2} \\
& +2\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0)\mathbf{Y}_1\right) E_1\left(\omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right) Y_{k,2} \\
& +2\nu_i\lim_{\sigma\rightsquigarrow 0} C_0^{'}(\sigma) R_f E_1\left(\omega_1^{(i)\intercal}(\sigma)\mathbf{Y}_2\right) Y_{k,2} \\
& -2\nu_i\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0)\mathbf{Y}_1\right) E_1\left(\omega_1^{(i)\intercal}(\sigma)\mathbf{Y}_2\right) Y_{k,2}.
\end{aligned} \tag{34}$$

To simplify expression above, we first derive $\lim_{\sigma\rightsquigarrow 0} C_0^{(i)'}(\sigma)$. We consider the FOC at date 0 with respect to investor's consumption:

$$u_i^{'}\left(C_0^{(i)}(\sigma)\right)-\beta^2 E_0 u_i^{'}\left(W_2^{(i)}(\sigma)\right)\left(R_f+\omega_0^{(i)\intercal}(\sigma)\mathbf{R}_1^e(\sigma)\right)\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right)=0.$$

Then, we take the limit as σ converges to zero of the first derivative of expression above with respect to σ:

$$u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} C_0^{(i)}(\sigma)\right)\lim_{\sigma\rightsquigarrow 0} C_0^{(i)'}(\sigma)-\beta^2 u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)R_f^2 E_0\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)=0. \tag{35}$$

Now, we use Equation (22) to show that:

$$E_0\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)=-\lim_{\sigma\rightsquigarrow 0} C_0^{'}(\sigma)R_f^2-E_0\lim_{\sigma\rightsquigarrow 0} C_1^{'}(\sigma)R_f,$$

and we replace $\lim_{\sigma\rightsquigarrow 0} C_1^{'}(\sigma)$ by its expression (see Equation 27) and simplify expression above to obtain:

$$E_0\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)=(\nu_i-1)\lim_{\sigma\rightsquigarrow 0} C_0^{'}(\sigma)R_f^2.$$

Thereafter, we replace this last expression into Equation (35) and deduce:

$$\left[u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} C_0^{(i)}(\sigma)\right)+\beta^2 u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)R_f^4(1-\nu_i)\right]\lim_{\sigma\rightsquigarrow 0} C_0^{'}(\sigma)=0. \tag{36}$$

Since $0<\nu_i<1$, we have $\left[u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} C_0^{(i)}(\sigma)\right)+\beta^2 u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)R_f^4(1-\nu_i)\right]<0$, and Equation (36) implies:

$$\lim_{\sigma\rightsquigarrow 0} C_0^{'}(\sigma)=0. \tag{37}$$

Next, we replace $\omega_1^{(i)}(0)$, $\omega_0^{(i)}(0)$, and $\lim_{\sigma\rightsquigarrow 0} C_0^{'}(\sigma)$ in Equations (22) and (34) and get:

$$\begin{aligned}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma) &= R_f\frac{\tau_i}{\overline{\tau}_\nu}\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right)+\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right) \\ E_1\lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma)Y_{k,2} &= \frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)E_1\left(2\omega_1^{(i)'\intercal}(0)\mathbf{Y}_2\right)Y_{k,2} \\ &\quad +2\frac{R_f}{\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)}\frac{\tau_i^2}{\overline{\tau}_\nu\overline{\tau}}\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right)E_1\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right)Y_{k,2}.\end{aligned} \tag{38}$$

Now, we replace Equation (38) in Equation (31), then simplify Equation (31) and get:

$$\begin{aligned} & -\frac{\rho_i\tau_i}{\overline{\tau}^2}Cov_1\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^2, Y_{k,2}\right) \\ & +\left(\frac{R_f\left(\tau_i-2\rho_i\tau_i\right)}{\overline{\tau}_\nu\overline{\tau}}+\frac{R_f}{\lim_{\sigma\rightsquigarrow 0}W_2^{(i)}\left(\sigma\right)}\frac{\tau_i^2}{\overline{\tau}_\nu\overline{\tau}}\right)\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)Cov_1\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,2}\right) \\ & +\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0}W_2^{(i)}\left(\sigma\right)E_1\left(\omega_1^{(i)'\intercal}\left(0\right)\mathbf{Y}_2\right)Y_{k,2}-\tau_i a_{k,1}^{'}\left(0\right) \\ = & \ 0 \end{aligned} \tag{39}$$

where:

$$\begin{aligned} \tau_i &= -\frac{u_i^{'}\left(W_2^{(i)}\left(0\right)\right)}{u_i^{''}\left(W_2^{(i)}\left(0\right)\right)},\ \rho_i=\frac{\tau_i^2}{2}\frac{u_i^{'''}\left(W_2^{(i)}\left(0\right)\right)}{u_i^{''}\left(W_2^{(i)}\left(0\right)\right)},\ \nu_i=\frac{\beta R_f^2}{\frac{u_i^{''}\left(C_1^{(i)}(0)\right)}{u_i^{''}\left(W_2^{(i)}(0)\right)}+\beta R_f^2}, \\ \tau_{\nu_i} &= \frac{\tau_i}{1-\nu_i},\ \overline{\tau}=\frac{1}{I}\sum_{i=1}^{I}\tau_i,\ \overline{\tau}_\nu=\frac{1}{I}\sum_{i=1}^{I}\tau_{\nu_i}. \end{aligned}$$

Thereafter, we take the sum of Equation (39) for $i = 1, ..., I$ and divide it by $\sum_{i=1}^{I}\tau_i$ to obtain:

$$\begin{aligned} a_{k,1}^{'}\left(0\right) &= -\frac{\overline{\rho}}{\overline{\tau}^2}Cov_1\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^2, Y_{k,2}\right) \\ & +\left(\frac{R_f\left(1-2\overline{\rho}\right)}{\overline{\tau}_\nu\overline{\tau}}+\sum_{i=1}^{I}\frac{R_f}{W_2^{(i)}\left(0\right)}\frac{\tau_i^2}{\overline{\tau}_\nu\overline{\tau}}/\sum_{i=1}^{I}\tau_i\right)\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)Cov_1\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,2}\right) \\ & +\sum_{i=1}^{I}\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0}W_2^{(i)}\left(\sigma\right)E_1\left(\omega_1^{(i)'\intercal}\left(0\right)\mathbf{Y}_2\right)Y_{k,2}/\sum_{i=1}^{I}\tau_i \end{aligned} \tag{40}$$

where $\overline{\rho}=\sum_{i=1}^{I}\frac{\tau_i}{\sum_{i=1}^{I}\tau_i}\rho_i$.

To derive $\sum_{i=1}^{I}\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0}W_2^{(i)}\left(\sigma\right)E_1\left(\omega_1^{(i)'\intercal}\left(0\right)\mathbf{Y}_2\right)Y_{k,2}/\sum_{i=1}^{I}\tau_i$, we take the first derivative of the market clearing condition at date 1 with respect to σ, and take its limit as σ converges to 0:

$$\sum_{i=1}^{I}\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0}W_2^{(i)}\left(\sigma\right)\omega_1^{(i)'}\left(0\right)+\sum_{i=1}^{I}\left(\lim_{\sigma\rightsquigarrow 0}W_1^{(i)'}\left(\sigma\right)-\lim_{\sigma\rightsquigarrow 0}C_1^{(i)'}\left(\sigma\right)\right)\omega_1^{(i)}\left(0\right)=0. \tag{41}$$

In addition, we replace the value of $\lim_{\sigma\rightsquigarrow 0}C_0^{(i)'}\left(\sigma\right)$ and $\omega_0^{(i)}\left(0\right)$ in Equation (27), then simplify Equation (27) to get:

$$\lim_{\sigma\rightsquigarrow 0}C_1^{(i)'}\left(\sigma\right)=\frac{\nu_i\tau_{\nu_i}}{\overline{\tau}_\nu}\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right). \tag{42}$$

We consider the first derivative of $W_1^{(i)}\left(\sigma\right)$ with respect to σ, then take its limit as σ converges to 0 and get:

$$\lim_{\sigma\rightsquigarrow 0} W_1^{(i)'}(\sigma) = \frac{\tau_{\nu_i}}{\overline{\tau}_\nu}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right). \tag{43}$$

Therefore, we replace $\lim_{\sigma\rightsquigarrow 0} C_1^{(i)'}(\sigma)$ and $\lim_{\sigma\rightsquigarrow 0} W_1^{(i)'}(\sigma)$ in the market clearing condition in Equation (41) and get

$$\sum_{i=1}^{I}\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\,\omega_1^{(i)'}(0) = -\sum_{i=1}^{I}\frac{R_f}{\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)}\frac{\tau_i^2}{\overline{\tau}_\nu\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\overline{\omega}_1.$$

Now, we use expression above and simplify Equation (40), then get:

$$a_{k,1}^{'}(0) = -\frac{\overline{\rho}}{\overline{\tau}^2}Cov_1\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^2, Y_{k,2}\right) + \frac{R_f\left(1-2\overline{\rho}\right)}{\overline{\tau}_\nu\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)Cov_1\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right), Y_{k,2}\right)$$

Next, we replace $a_{k,1}^{'}(0)$ in Equation (39) and deduce:

$$\begin{aligned}\omega_1^{(i)'}(0) \quad = \quad & \frac{1}{\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)}\frac{\left(\rho_i-\overline{\rho}\right)\tau_i R_f}{\overline{\tau}^2}\Sigma_1^{-1}Cov_1\left(\mathbf{Y}_2, \left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^2\right) \\ & +\frac{1}{\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)}\left(\frac{2R_f\tau_i\left(\rho_i-\overline{\rho}\right)R_f}{\overline{\tau}_\nu\overline{\tau}} - \frac{R_f^2}{\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)}\frac{\tau_i^2}{\overline{\tau}_\nu\overline{\tau}}\right)\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\overline{\omega}_1\end{aligned}$$

where $\Sigma_1 = E_1 Y_2 Y_2^\intercal$.

To derive $\omega_0^{(i)'}(0)$, we consider the FOC with respect to investor's consumption at date 0:

$$E_0 u_i^{'}\left(W_2^{(i)}(\sigma)\right)\left(\sigma a_{k,0}(\sigma) + Y_{k,1}\right)\left(R_f + \omega_1^{(i)^\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right) = 0.$$

The second derivative of this FOC with respect to σ is:

$$\begin{aligned} & E_0\left(u_i^{'}\left(W_2^{(i)}(\sigma)\right)\right)^{''}\left(\sigma a_{k,0}(\sigma)+Y_{k,1}\right)\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right) \\ & +E_0 u_i^{'}\left(W_2^{(i)}(\sigma)\right)\left(\sigma a_{k,0}(\sigma)+Y_{k,1}\right)^{''}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right) \\ & +E_0 u_i^{'}\left(W_2^{(i)}(\sigma)\right)\left(\sigma a_{k,0}(\sigma)+Y_{k,1}\right)\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)^{''} \\ & +2E_0\left(u_i^{'}\left(W_2^{(i)}(\sigma)\right)\right)^{'}\left(\sigma a_{k,0}(\sigma)+Y_{k,1}\right)^{'}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right) \\ & +2\left(E_0 u_i^{'}\left(W_2^{(i)}(\sigma)\right)\right)^{'}\left(\sigma a_{k,0}(\sigma)+Y_{k,1}\right)\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)^{'} \\ & +2E_0 u_i^{'}\left(W_2^{(i)}(\sigma)\right)\left(\sigma a_{k,0}(\sigma)+Y_{k,1}\right)^{'}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)^{'} \\ = \quad & 0.\end{aligned}$$

We expand expression above, then take its limit as σ converges to zero and deduce:

$$\begin{aligned}
& u_i^{'''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) E_0\left(\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)\right)^2\right) Y_{k,1} R_f \\
& +u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) E_0\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma)\right) Y_{k,1} R_f \\
& +2u_i^{'}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) a_{k,0}^{'}(0) R_f + 2u_i^{'}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) E_0 Y_{k,1}\left(\omega_1^{(i)\intercal}(0) a_2(0)\right) \\
& +2u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) E_0\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)\right) a_{k,0}(0) R_f \\
& +2u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) E_0\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)\right) Y_{k,1}\left(\omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right) \\
= \quad & 0.
\end{aligned}$$

We divide expression above by $u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)$, then simplify the result and get:

$$\begin{aligned}
& -2\frac{\rho_i}{\tau_i} E_0\left(R_f\frac{\tau_i}{\overline{\tau}_\nu}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right) + \frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)^2 Y_{k,1} R_f \\
& +E_0\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma)\right) Y_{k,1} R_f - 2\tau_i a_{k,0}^{'}(0) R_f \\
& -2\tau_i E_0 Y_{k,1}\left(\omega_1^{(i)\intercal}(0) a_2(0)\right) + 2E_0\left(R_f\frac{\tau_i}{\overline{\tau}_\nu}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right) + \frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right) a_{k,0}(0) R_f \\
& +2E_0\left(R_f\frac{\tau_i}{\overline{\tau}_\nu}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right) + \frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right) Y_{k,1}\left(\omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right) \\
= \quad & 0.
\end{aligned} \tag{44}$$

To simplify Equation (44), we first derive $E_0 \lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma) Y_{k,1}$ and replace its value in (44). To do this, we use (32) to get

$$\begin{aligned}
E_0 \lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma) Y_{k,1} = \; & E_0 \lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(2\omega_0^{(i)'\intercal}(0) Y_1\right) Y_{k,1} \\
& +2\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma) E_0\left(\omega_1^{(i)\intercal}(0) a_1(0)\right) Y_{k,1} \\
& -R_f E_0 \lim_{\sigma\rightsquigarrow 0} C_1^{''}(\sigma) Y_{k,1} \\
& -2\frac{\nu_i \tau_{\nu_i}}{\overline{\tau}_\nu} E_0\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\left(\omega_1^{(i)\intercal}(\sigma)\mathbf{Y}_2\right) Y_{k,1}.
\end{aligned} \tag{45}$$

To simplify Equation (45), it is necessary to derive $\lim_{\sigma\rightsquigarrow 0} C_1^{''}(\sigma)$. We then consider the FOC with respect to investor's consumption at date 1

$$u_i^{'}\left(C_1^{(i)}(\sigma)\right) - \beta E_1 u_i^{'}\left(W_2^{(i)}(\sigma)\right)\left(R_f + \omega_1^{(i)\intercal}(\sigma) R_2^e(\sigma)\right) = 0.$$

The limit as σ converges to zero of the second derivative of this FOC with respect to σ is:

$$\begin{aligned} & u_i^{'''}\left(\lim_{\sigma\rightsquigarrow 0} C_1^{(i)}(\sigma)\right)\left(\lim_{\sigma\rightsquigarrow 0} C_1^{(i)'}(\sigma)\right)^2 + u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} C_1^{(i)}(\sigma)\right)\lim_{\sigma\rightsquigarrow 0} C_1^{(i)''}(\sigma) \\ & -\beta u_i^{'''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) R_f E_1\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)\right)^2 - \beta u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) R_f E_1 \lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma) \\ & -2\beta u_i^{'}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)\left(\omega_1^{(i)\intercal}(0)\, a_1(0)\right) - 2\beta u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) E_1\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)\right)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right) \\ = \; & 0. \end{aligned} \tag{46}$$

To simplify expression above, we take the expectation of Equation (32) and get:

$$\begin{aligned} E_1 \lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma) \;=\; & 2\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0)\, a_0(0)\right) \\ & +2\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)'\intercal}(0)\,\mathbf{Y}_1\right) \\ & +2\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\left(\omega_1^{(i)\intercal}(0)\, a_1(0)\right) \\ & -\lim_{\sigma\rightsquigarrow 0} C_1^{''}(\sigma)\, R_f. \end{aligned}$$

Then, we replace expression above in Equation (46) and get:

$$\begin{aligned} & u_i^{'''}\left(\lim_{\sigma\rightsquigarrow 0} C_1^{(i)}(\sigma)\right)\left(\lim_{\sigma\rightsquigarrow 0} C_1^{(i)'}(\sigma)\right)^2 \\ & +\left[u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} C_1^{(i)}(\sigma)\right) + \beta R_f^2 u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)\right]\lim_{\sigma\rightsquigarrow 0} C_1^{(i)''}(\sigma) \\ & -\beta u_i^{'''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) R_f E_1\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)\right)^2 \\ & -\beta u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) R_f\left(2\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0)\, a_0(0)\right)\right) \\ & -\beta u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) R_f\left(2\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)'\intercal}(0)\,\mathbf{Y}_1\right)\right) \\ & -\beta u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) R_f\left(\frac{2}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\left(\omega_1^{(i)\intercal}(0)\, a_1(0)\right)\right) \\ & -2\beta u_i^{'}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)\left(\omega_1^{(i)\intercal}(0)\, a_1(0)\right) \\ & -2\beta u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right) E_1\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)\right)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right) \\ = \; & 0. \end{aligned}$$

We simplify expression above and deduce:

$$\begin{aligned}\lim_{\sigma\rightsquigarrow 0} C_1^{(i)''}(\sigma) &= (1-\nu_i)\frac{2\rho_{ic}}{\tau_{ic}}\left(\lim_{\sigma\rightsquigarrow 0} C_1^{(i)'}(\sigma)\right)^2 - \frac{2\nu_i}{R_f}\frac{\rho_i}{\tau_i}E_1\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)\right)^2 \\ &+2\frac{\nu_i}{R_f}\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0)\,a_0(0)\right) + \frac{\nu_i}{R_f}\left(2\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)'\intercal}(0)\,\mathbf{Y}_1\right)\right) \\ &+2\frac{\nu_i}{R_f}\left(\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\left(\omega_1^{(i)\intercal}(0)\,a_1(0)\right)\right) - 2\frac{\nu_i\tau_i}{R_f^2}\left(\omega_1^{(i)\intercal}(0)\,a_1(0)\right) \\ &+2\frac{\nu_i}{R_f^2}E_1\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)\right)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right)\end{aligned}$$

where ν_i is given in Equation (28) and $\rho_{ic} = \frac{\tau_{ic}^2}{2}\frac{u_i'''\left(C_1^{(i)}(0)\right)}{u_i'\left(C_1^{(i)}(0)\right)}$. We replace $\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'}(\sigma)$ and $\lim_{\sigma\rightsquigarrow 0} C_1^{(i)'}(\sigma)$ (see Equations (38) and (42)) in expression above and obtain:

$$\begin{aligned}\lim_{\sigma\rightsquigarrow 0} C_1^{(i)''}(\sigma) &= (1-\nu_i)\frac{2\rho_{ic}}{\tau_{ic}}\frac{\nu_i^2\tau_{\nu_i}^2}{\overline{\tau}_\nu^2}\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right)^2 - \frac{2\nu_i}{R_f}\frac{\rho_i\tau_i}{\overline{\tau}^2}Var_1\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right) \\ &-2R_f\frac{\nu_i\rho_i\tau_i}{\overline{\tau}_\nu^2}\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right)^2 + 2\frac{\nu_i}{R_f}\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0)\,a_0(0)\right) \\ &+2\frac{\nu_i}{R_f}\left(\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)'\intercal}(0)\,\mathbf{Y}_1\right)\right) + 2\frac{\nu_i}{R_f}\left(\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\left(\omega_1^{(i)\intercal}(0)\,a_1(0)\right)\right) \\ &-2\frac{\nu_i\tau_i}{R_f^2}\left(\omega_1^{(i)\intercal}(0)\,a_1(0)\right) + 2\frac{\nu_i}{R_f^2}\frac{\tau_i}{\overline{\tau}}E_1\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right).\end{aligned} \tag{47}$$

We multiply expression above by $R_fY_{k,1}$, then take the expectation at date 0 and get:

$$\begin{aligned}&R_fE_0\lim_{\sigma\rightsquigarrow 0} C_1^{(i)''}(\sigma)\,Y_{k,1} \\ =\ &R_f(1-\nu_i)\frac{2\rho_{ic}}{\tau_{ic}}\frac{\nu_i^2\tau_{\nu_i}^2}{\overline{\tau}_\nu^2}E_0\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right)^2Y_{k,1} - R_f\frac{2\nu_i}{R_f}\frac{\rho_i\tau_i}{\overline{\tau}^2}E_0Var_1\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right)Y_{k,1} \\ &-2R_fR_f\frac{\nu_i\rho_i\tau_i}{\overline{\tau}_\nu^2}E_0\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right)^2Y_{k,1} + 2R_f\frac{\nu_i}{R_f}\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\,E_0\left(\omega_0^{(i)'\intercal}(0)\,\mathbf{Y}_1\right)Y_{k,1} \\ &+2R_f\frac{\nu_i}{R_f}\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\,E_0\left(\omega_1^{(i)\intercal}(0)\,a_1(0)\right)Y_{k,1} - 2R_f\frac{\nu_i\tau_i}{R_f^2}E_0\left(\omega_1^{(i)\intercal}(0)\,a_1(0)\right)Y_{k,1} \\ &+2R_f\frac{\nu_i}{R_f^2}\frac{\tau_i}{\overline{\tau}}E_0\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right)Y_{k,1}.\end{aligned}$$

Now, we replace expression above in Equation (45):

$$\begin{aligned}
& E_0 \lim_{\sigma \rightsquigarrow 0} W_2^{(i)''}(\sigma) Y_{k,1} \\
= \; & E_0 \lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma) \left(2\omega_0^{(i)'\intercal}(0) \mathbf{Y}_1\right) Y_{k,1} + 2\frac{1}{R_f} \lim_{\sigma \rightsquigarrow 0} W_2^{(i)}(\sigma) E_0 \left(\omega_1^{(i)\intercal}(0) a_1(0)\right) Y_{k,1} \\
& - R_f (1-\nu_i) \frac{2\rho_{ic}}{\tau_{ic}} \frac{\nu_i^2 \tau_{\nu_i}^2}{\overline{\tau}_\nu^2} E_0 \left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)^2 Y_{k,1} + R_f \frac{2\nu_i}{R_f} \frac{\rho_i \tau_i}{\overline{\tau}^2} E_0 Var_1 \left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right) Y_{k,1} \\
& + 2R_f R_f \frac{\nu_i \rho_i \tau_i}{\overline{\tau}_\nu^2} E_0 \left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)^2 Y_{k,1} - 2R_f \frac{\nu_i}{R_f} \lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma) E_0 \left(\omega_0^{(i)'\intercal}(0) \mathbf{Y}_1\right) Y_{k,1} \\
& - 2R_f \frac{\nu_i}{R_f} \frac{1}{R_f} \lim_{\sigma \rightsquigarrow 0} W_2^{(i)}(\sigma) E_0 \left(\omega_1^{(i)\intercal}(0) a_1(0)\right) Y_{k,1} + 2R_f \frac{\nu_i \tau_i}{R_f^2} E_0 \left(\omega_1^{(i)\intercal}(0) a_1(0)\right) Y_{k,1} \\
& - 2R_f \frac{\nu_i}{R_f^2} \frac{\tau_i}{\overline{\tau}} E_0 \left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right) \left(\omega_1^{(i)\intercal}(0) \mathbf{Y}_2\right) Y_{k,1} - 2\frac{\nu_i \tau_{\nu_i}}{\overline{\tau}_\nu} E_0 \left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right) \left(\omega_1^{(i)\intercal}(\sigma) \mathbf{Y}_2\right) Y_{k,1}.
\end{aligned}$$

We thereafter replace $\omega_1^{(i)}(0)$ and $a_1(0)$ in expression above, then multiply the result by R_f and simplify it to obtain:

$$\begin{aligned}
E_0 \lim_{\sigma \rightsquigarrow 0} W_2^{(i)''}(\sigma) Y_{k,1} R_f \; = \; & 2(1-\nu_i) R_f Cov_0 \left(\lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma) \left(\omega_0^{(i)'\intercal}(0) \mathbf{Y}_1\right), Y_{k,1}\right) \\
& + \left(\frac{2\tau_i (1-\nu_i)}{\overline{\tau}^2} + \frac{2\nu_i \rho_i \tau_i}{\overline{\tau}^2}\right) R_f Cov_0 \left(\left(\overline{\omega}_1 \mathbf{Y}_2\right)^2, Y_{k,1}\right) \\
& + \left(2R_f^2 \frac{\nu_i \rho_i \tau_i}{\overline{\tau}_\nu^2} - 2\frac{(1-\nu_i) \rho_{ic} \nu_i^2 \tau_{\nu_i}^2}{\tau_{ic} \overline{\tau}_\nu^2} R_f\right) R_f Cov_0 \left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)^2, Y_{k,1}\right).
\end{aligned}$$

Now, we replace expression above in Equation (44) and deduce:

$$\begin{aligned}
& 2(1-\nu_i) R_f Cov_0 \left(\lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma) \left(\omega_0^{(i)'\intercal}(0) \mathbf{Y}_1\right), Y_{k,1}\right) \\
& + \left(\frac{2\tau_i (1-\nu_i)}{\overline{\tau}^2} + \frac{2(\nu_i - 1) \rho_i \tau_i}{\overline{\tau}^2}\right) R_f Cov_0 \left(\left(\overline{\omega}_1 \mathbf{Y}_2\right)^2, Y_{k,1}\right) \\
& - \left(\frac{2(1-\nu_i) \rho_i \tau_i R_f^2}{\overline{\tau}_\nu^2} + \frac{2(1-\nu_i) \rho_{ic} \nu_i^2 \tau_{\nu_i}^2}{\tau_{ic} \overline{\tau}_\nu^2} R_f\right) R_f Cov_0 \left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)^2, Y_{k,1}\right) \\
& - 2\tau_i a_{k,0}^{'}(0) R_f \\
= \; & 0.
\end{aligned}$$

Expression above can be rewritten as:

$$\begin{aligned}
& Cov_0 \left(\lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma) \left(\omega_0^{(i)'\intercal}(0) \mathbf{Y}_1\right), Y_{k,1}\right) \qquad (48) \\
& + \frac{\tau_i (1-\rho_i)}{\overline{\tau}^2} Cov_0 \left(\left(\overline{\omega}_1 \mathbf{Y}_2\right)^2, Y_{k,1}\right) - \left(\frac{\rho_i \tau_i R_f^2}{\overline{\tau}_\nu^2} + \frac{\rho_{ic} \nu_i^2 \tau_{\nu_i}^2}{\tau_{ic} \overline{\tau}_\nu^2} R_f\right) Cov_0 \left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)^2, Y_{k,1}\right) \\
& - \tau_{\nu_i} a_{k,0}^{'}(0) \\
= \; & 0.
\end{aligned}$$

Now, we take the sum of expression above for $i = 1, ..., I$ and divide this sum by $\sum_{i=1}^{I} \tau_{\nu_i}$ to obtain:

$$\begin{aligned} a_{k,0}^{'}(0) &= \sum_{i=1}^{I} Cov_0\left(\lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)'\intercal}(0)\,\mathbf{Y}_1\right), Y_{k,1}\right) / \sum_{i=1}^{I} \tau_{\nu_i} \\ &\quad + \sum_{i=1}^{I} \frac{(\tau_i - \tau_i \rho_i)}{\overline{\tau}^2} Cov_0\left((\overline{\omega}_1 \mathbf{Y}_2)^2, Y_{k,1}\right) / \sum_{i=1}^{I} \tau_{\nu_i} \\ &\quad - \left(\sum_{i=1}^{I}\left(\frac{\rho_i \tau_i R_f^2}{\overline{\tau}_\nu^2} + \frac{\rho_{ic}\nu_i^2 \tau_{\nu_i}^2}{\tau_{ic}\overline{\tau}_\nu^2} R_f\right) / \sum_{i=1}^{I} \tau_{\nu_i}\right) Cov_0\left((\overline{\omega}_0^\intercal \mathbf{Y}_1)^2, Y_{k,1}\right). \end{aligned} \tag{49}$$

Note that the first derivative of the market clearing conditions with respect to σ is:

$$\sum_{i=1}^{I}\left(W_0^{(i)}(\sigma) - C_0^{(i)}(\sigma)\right)^{'} \omega_0^{(i)}(\sigma) + \sum_{i=1}^{I}\left(W_0^{(i)}(\sigma) - C_0^{(i)}(\sigma)\right)\omega_0^{(i)'}(\sigma) = 0.$$

Since we assume the investor's initial wealth $W_0^{(i)}(\sigma)$ is equal to 1, $\lim_{\rightsquigarrow 0} W_0^{(i)}(\sigma)^{'} = 0$ and $\lim_{\sigma \rightsquigarrow 0} C_0^{(i)'}(\sigma) = 0$, the limit as σ converges to zero of expression above is:

$$\sum_{i=1}^{I} \frac{1}{R_f} \lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma)\,\omega_0^{(i)'}(0) = 0. \tag{50}$$

We utilize Equation (50) to simplify Equation (49), then obtain:

$$a_{k,0}^{'}(0) = \frac{(1-\overline{\rho})}{\overline{\tau}_\nu \overline{\tau}} Cov_0\left((\overline{\omega}_1^\intercal \mathbf{Y}_2)^2, Y_{k,1}\right) - \left(\frac{\overline{\tau}.\overline{\rho}}{\overline{\tau}_\nu^3} R_f^2 + \frac{\overline{\rho}_c}{\overline{\tau}_\nu^2} R_f\right) Cov_0\left((\overline{\omega}_0^\intercal \mathbf{Y}_1)^2, Y_{k,1}\right).$$

with:

$$\overline{\rho}_c = \sum_{i=1}^{I} \rho_i . \frac{\tau_i}{\sum_{i=1}^{I} \tau_i},\ \overline{\rho}_c = \sum_{i=1}^{I} \rho_{ic}.\nu_i^2 . \frac{\tau_{\nu_i}}{\tau_{ic}} \frac{\tau_{\nu_i}}{\sum_{i=1}^{I} \tau_{\nu_i}},\ \overline{\tau} = \frac{1}{I}\sum_{i=1}^{I} \tau_i,\ \overline{\tau}_\nu = \frac{1}{I}\sum_{i=1}^{I} \tau_{\nu_i}.$$

Now, we replace $a_{k,0}^{'}(0)$ in Equation (48), and derive:

$$\begin{aligned} &\omega_0^{(i)'}(0) \\ = \;& \frac{1}{W_1^{(i)}(0)}\left[\left(\frac{\tau_{\nu_i}(1-\overline{\rho})}{\overline{\tau}\overline{\tau}_\nu} - \frac{\tau_i(1-\rho_i)}{\overline{\tau}^2}\right)\right] \Sigma_0^{-1} Cov_0\left(\mathbf{Y}_1, (\overline{\omega}_1^\intercal \mathbf{Y}_2)^2\right) \\ & + \frac{1}{W_1^{(i)}(0)}\left[\left(\frac{\rho_i \tau_i}{\overline{\tau}_\nu^2} - \frac{\tau_{\nu_i}\overline{\tau}.\overline{\rho}}{\overline{\tau}_\nu^3}\right) R_f^2 + \left(\frac{\rho_{ic}\nu_i^2 \tau_{\nu_i}^2}{\tau_{ic}\overline{\tau}_\nu^2} - \frac{\tau_{\nu_i}\overline{\rho}_c}{\overline{\tau}_\nu^2}\right) R_f\right] \Sigma_0^{-1} Cov_0\left(\mathbf{Y}_1, (\overline{\omega}_0^\intercal \mathbf{Y}_1)^2\right) \end{aligned}$$

where

$$\Sigma_0 = E_0 \mathbf{Y}_1 \mathbf{Y}_1^\intercal.$$

To summarize:

$$
\begin{aligned}
\omega_1^{(i)}(0) &= \frac{R_f}{W_2^{(i)}(0)}\frac{\tau_i}{\overline{\tau}}\overline{\omega}_1 \text{ and } a_{k,1}(0) = \frac{1}{\overline{\tau}}Cov_1\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,2}\right), \qquad (51)\\
\omega_0^{(i)}(0) &= \frac{R_f}{W_1^{(i)}(0)}\frac{\tau_{\nu_i}}{\overline{\tau}_\nu}\overline{\omega}_0 \text{ and } a_{k,0}(0) = \frac{1}{\overline{\tau}_\nu}R_f Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right), Y_{k,1}\right).
\end{aligned}
$$

and

$$
\begin{aligned}
a_{k,1}^{'}(0) &= -\frac{\overline{\rho}}{\overline{\tau}^2}Cov_1\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^2, Y_{k,2}\right) + \frac{R_f\left(1-2\overline{\rho}\right)}{\overline{\tau}_\nu\overline{\tau}}\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)Cov_1\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,2}\right), \qquad (52)\\
\omega_1^{(i)'}(0) &= \frac{1}{W_2^{(i)}(0)}\frac{\left(\rho_i-\overline{\rho}\right)\tau_i R_f}{\overline{\tau}^2}\Sigma_1^{-1}Cov_1\left(\mathbf{Y}_2, \left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^2\right)\\
&\quad + \frac{1}{W_2^{(i)}(0)}\left(\frac{2R_f\tau_i\left(\rho_i-\overline{\rho}\right)R_f}{\overline{\tau}_\nu\overline{\tau}} - \frac{R_f^2}{W_2^{(i)}(0)}\frac{\tau_i^2}{\overline{\tau}_\nu\overline{\tau}}\right)\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)\overline{\omega}_1,\\
a_{k,0}^{'}(0) &= \frac{\left(1-\overline{\rho}\right)}{\overline{\tau}\tau_\nu}Cov_0\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^2, Y_{k,1}\right) - \left(\frac{\overline{\tau}.\overline{\rho}}{\overline{\tau}_\nu^3}R_f^2 + \frac{\overline{\rho}_c}{\overline{\tau}_\nu^2}R_f\right)Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)^2, Y_{k,1}\right),\\
\omega_0^{(i)'}(0) &= \frac{1}{W_1^{(i)}(0)}\left(\frac{\tau_{\nu_i}\left(1-\overline{\rho}\right)}{\overline{\tau}_\nu\overline{\tau}} - \frac{\tau_i\left(1-\rho_i\right)}{\overline{\tau}^2}\right)\Sigma_0^{-1}Cov_0\left(\mathbf{Y}_1, \left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^2\right)\\
&\quad + \frac{1}{W_1^{(i)}(0)}\left[\left(\frac{\rho_i\tau_i}{\overline{\tau}_\nu^2} - \frac{\tau_{\nu_i}\overline{\tau}.\overline{\rho}}{\overline{\tau}_\nu^3}\right)R_f^2 + \left(\frac{\rho_{ic}\nu_i^2\tau_{\nu_i}^2}{\tau_{ic}\overline{\tau}_\nu^2} - \frac{\tau_{\nu_i}\overline{\rho}_c}{\overline{\tau}_\nu^2}\right)R_f\right]\Sigma_0^{-1}Cov_0\left(\mathbf{Y}_1, \left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)^2\right).
\end{aligned}
$$

Now, we use analytical expressions for $\omega_\tau^{(i)}(0)$, $a_{k,\tau}(0)$, $\omega_\tau^{(i)'}(0)$, $a_{k,\tau}^{'}(0)$ for $\tau = 0, 1$ (see Equations (51) and (52)) to deduce Propositions 1, 2 and 3.

Proposition 1: At date 1, the optimal portfolio weight of agent i is: $\omega_1^{(i)}(\sigma) = \omega_1^{(i)}(0) + \sigma\omega_1^{(i)'}(0)$. We replace $\omega_1^{(i)}(0)$ and $\omega_1^{(i)'}(0)$ by their expressions and get

$$
\begin{aligned}
&\omega_1^{(i)}(\sigma)\\
&= \frac{R_f}{W_2^{(i)}(0)}\left[\frac{\tau_i}{\overline{\tau}} + \left(\frac{2R_f\tau_i\left(\rho_i-\overline{\rho}\right)}{\overline{\tau}_\nu\overline{\tau}} - \frac{R_f}{W_2^{(i)}(0)}\frac{\tau_i^2}{\overline{\tau}_\nu\overline{\tau}}\right)\left(\overline{\omega}_0^{\intercal}\sigma\mathbf{Y}_1\right)\right]\overline{\omega}_1\\
&\quad + \frac{R_f}{W_2^{(i)}(0)}\frac{\left(\rho_i-\overline{\rho}\right)\tau_i}{\overline{\tau}^2}\left(\sigma^2\Sigma_1\right)^{-1}Cov_1\left(\sigma\mathbf{Y}_2, \left(\overline{\omega}_1^{\intercal}\sigma\mathbf{Y}_2\right)^2\right)\\
&= \left(\phi_{1i}^{\intercal}\varkappa_1\right)\overline{\omega}_1 + \psi_{1i}\varsigma_1
\end{aligned}
$$

where:

$$
\begin{aligned}
\varkappa_1^{\intercal} &= \left[1, r_{M1}\right],\\
r_{M2} &= R_{M2} - E_1 R_{M2},\ R_{M2} = \overline{\omega}_1^{\intercal}\mathbf{R}_2 \text{ and } \mathbf{R}_2 - E_1\mathbf{R}_2 = \sigma\mathbf{Y}_2,\\
r_{M1} &= R_{M1} - E_0 R_{M1},\ R_{M1} = \overline{\omega}_0^{\intercal}\mathbf{R}_1 \text{ and } \mathbf{R}_1 - E_0\mathbf{R}_1 = \sigma\mathbf{Y}_1,
\end{aligned}
$$

and

$$\begin{aligned}
\overline{\omega}_1 &= \left(Var_1\left(\mathbf{R}_2\right)\right)^{-1} Cov_1\left(r_{M2}, \mathbf{R}_2\right) \\
\overline{\varsigma}_1 &= \left(Var_1\left(\mathbf{R}_2\right)\right)^{-1} Cov_1\left(r_{M2}^2, \mathbf{R}_2\right), \\
\psi_{1i} &= \frac{R_f}{W_2^{(i)}(0)} \frac{\left(\rho_i - \overline{\rho}\right)\tau_i}{\overline{\tau}^2}, \\
\phi_{1i}^{\intercal} &= \left[\frac{R_f}{W_2^{(i)}(0)} \frac{\tau_i}{\overline{\tau}}, \frac{R_f}{W_2^{(i)}(0)}\left(\frac{2R_f\tau_i\left(\rho_i - \overline{\rho}\right)}{\overline{\tau}_\nu\overline{\tau}} - \frac{R_f}{W_2^{(i)}(0)} \frac{\tau_i^2}{\overline{\tau}_\nu\overline{\tau}}\right)\right],
\end{aligned}$$

At date 0, $\omega_0^{(i)}(\sigma) = \omega_0^{(i)}(0) + \sigma\omega_0^{(i)'}(0)$. We replace $\omega_0^{(i)}(0)$ and $\omega_0^{(i)'}(0)$ by their expressions and get:

$$\begin{aligned}
& \omega_0^{(i)}(\sigma) \\
= \; & \frac{R_f}{W_1^{(i)}(0)} \frac{\tau_{\nu_i}}{\overline{\tau}_\nu}\overline{\omega}_0 \\
& + \frac{1}{W_1^{(i)}(0)}\left(\frac{\tau_{\nu_i}\left(1-\overline{\rho}\right)}{\overline{\tau}\overline{\tau}_\nu} - \frac{\tau_i\left(1-\rho_i\right)}{\overline{\tau}^2}\right)\left(\sigma^2\Sigma_0\right)^{-1} Cov_0\left(\sigma\mathbf{Y}_1, \left(\overline{\omega}_1^{\intercal}\sigma\mathbf{Y}_2\right)^2\right) \\
& + \frac{1}{W_1^{(i)}(0)}\left[\left(\frac{\rho_i\tau_i}{\overline{\tau}_\nu^2} - \frac{\tau_{\nu_i}\overline{\tau}.\overline{\rho}}{\overline{\tau}_\nu^3}\right)R_f^2 + \left(\frac{\rho_{ic}\nu_i^2\tau_{\nu_i}^2}{\tau_{ic}\overline{\tau}_\nu^2} - \frac{\tau_{\nu_i}\overline{\rho}_c}{\overline{\tau}_\nu^2}\right)R_f\right]\left(\sigma^2\Sigma_0\right)^{-1} Cov_0\left(\sigma\mathbf{Y}_1, \left(\overline{\omega}_0^{\intercal}\sigma\mathbf{Y}_1\right)^2\right) \\
= \; & \phi_{0i}\overline{\omega}_0 + \psi_{0i}\overline{\varsigma}_0 + \varphi_{0i}\overline{\varsigma}_{0,\vartheta}
\end{aligned}$$

with

$$\begin{aligned}
\overline{\omega}_0 &= \left(Var_0\left(\mathbf{R}_1\right)\right)^{-1} Cov_0\left(\mathbf{R}_1, r_{M1}\right), \\
\overline{\varsigma}_0 &= \left(Var_0\left(\mathbf{R}_1\right)\right)^{-1} Cov_0\left(\mathbf{R}_1, r_{M1}^2\right), \\
\overline{\varsigma}_{0,\vartheta} &= \left(Var_0\left(\mathbf{R}_1\right)\right)^{-1} Cov_0\left(\mathbf{R}_1, Var_1\left(r_{M2}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
\phi_{0i} &= \frac{R_f}{W_1^{(i)}(0)} \frac{\tau_{\nu_i}}{\overline{\tau}_\nu}, \\
\psi_{0i} &= \frac{1}{W_1^{(i)}(0)}\left[\left(\frac{\rho_i\tau_i}{\overline{\tau}_\nu^2} - \frac{\tau_{\nu_i}\overline{\tau}.\overline{\rho}}{\overline{\tau}_\nu^3}\right)R_f^2 + \left(\frac{\rho_{ic}\nu_i^2\tau_{\nu_i}^2}{\tau_{ic}\overline{\tau}_\nu^2} - \frac{\tau_{\nu_i}\overline{\rho}_c}{\overline{\tau}_\nu^2}\right)R_f\right], \\
\varphi_{0i} &= \frac{1}{W_1^{(i)}(0)}\left(\frac{\tau_{\nu_i}\left(1-\overline{\rho}\right)}{\overline{\tau}\overline{\tau}_\nu} - \frac{\tau_i\left(1-\rho_i\right)}{\overline{\tau}^2}\right).
\end{aligned}$$

Proposition 2: At date 1, the risk premium on asset k is:

$$E_1R_{k,2} - R_f = \sigma^2\left(a_{k,1}(0) + \sigma a_{k,1}^{'}(0)\right)$$

We replace $a_{k,1}(0)$ and $\sigma a_{k,1}^{'}(0)$ by their expressions and get:

$$\begin{aligned} E_1 R_{k,2} - R_f &= \sigma^2 \left(a_{k,1}(0) + \sigma a_{k,1}^{'}(0) \right) \\ &= \frac{1}{\overline{\tau}} Cov_1\left(\left(\overline{\omega}_1^{\intercal} \sigma \mathbf{Y}_2 \right), \sigma Y_{k,2} \right) - \frac{\overline{\rho}}{\overline{\tau}^2} Cov_1 \left(\left(\overline{\omega}_1^{\intercal} \sigma \mathbf{Y}_2 \right)^2, \sigma Y_{k,2} \right) \\ &\quad + \frac{R_f (1-2\overline{\rho})}{\overline{\tau}_\nu \overline{\tau}} \left(\overline{\omega}_0^{\intercal} \sigma \mathbf{Y}_1 \right) Cov_1 \left(\left(\overline{\omega}_1^{\intercal} \sigma \mathbf{Y}_2 \right), \sigma Y_{k,2} \right) \\ &= \overline{A}_1 Cov_1 \left(r_{M2}, R_{k,2} \right) + \overline{B}_1 Cov_1 \left(r_{M2}^2, R_{k,2} \right), \end{aligned}$$

with:

$$\overline{A}_1 = \frac{1}{\overline{\tau}} + \frac{R_f (1-2\overline{\rho})}{\overline{\tau}_\nu . \overline{\tau}} r_{M1}, \text{ and } \ \overline{B}_1 = -\frac{\overline{\rho}}{\overline{\tau}^2}.$$

where

$$\begin{aligned} R_{k,2} - E_1 R_{k,2} &= \sigma Y_{k,2},\ r_{M2} = R_{M2} - E_1 R_{M2},\ R_{M2} = \overline{\omega}_1^{\intercal} \mathbf{R}_2, \\ \mathbf{R}_1 - E_0 \mathbf{R}_1 &= \sigma \mathbf{Y}_1,\ r_{M1} = R_{M1} - E_0 R_{M1},\ R_{M1} = \overline{\omega}_0^{\intercal} \mathbf{R}_1. \end{aligned}$$

At date 0, the risk premium on asset k is

$$E_0 R_{k,1} - R_f = \sigma^2 \left(a_{k,0}(0) + \sigma a_{k,0}^{'}(0) \right)$$

We replace $a_{k,0}(0)$ and $a_{k,0}^{'}(0)$ by their expressions and get:

$$\begin{aligned} E_0 R_{k,1} - R_f &= \frac{1}{\overline{\tau}_\nu} R_f Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \sigma \mathbf{Y}_1 \right), \sigma Y_{k,1} \right) + \frac{(1-\overline{\rho})}{\overline{\tau}\overline{\tau}_\nu} Cov_0 \left(\left(\overline{\omega}_1^{\intercal} \sigma \mathbf{Y}_2 \right)^2, \sigma Y_{k,1} \right) \\ &\quad - \left(\frac{\overline{\tau}.\overline{\rho}}{\overline{\tau}_\nu^3} R_f^2 + \frac{\overline{\rho}_c}{\overline{\tau}_\nu^2} R_f \right) Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \sigma \mathbf{Y}_1 \right)^2, \sigma Y_{k,1} \right) \end{aligned}$$

Hence,

$$E_0 R_{k,1} - R_f = \overline{A}_0 Cov_0 \left(r_{M1}, R_{k,1} \right) + \overline{B}_0 Cov_0 \left(r_{M1}^2, R_{k,1} \right) + \overline{D}_0 Cov_0 \left(Var_1 \left(r_{M2} \right), R_{k,1} \right)$$

with

$$\overline{A}_0 = \frac{1}{\overline{\tau}_\nu} R_f,\ \overline{B}_0 = -\left(\frac{\overline{\tau}.\overline{\rho}}{\overline{\tau}_\nu^3} R_f^2 + \frac{\overline{\rho}_c}{\overline{\tau}_\nu^2} R_f \right), \text{ and } \overline{D}_0 = \frac{(1-\overline{\rho})}{\overline{\tau}.\overline{\tau}_\nu}.$$

Proposition 3 By definition, the risk premium on asset k for the time period $[\tau, \tau+1]$ is given by

$$E_\tau R_{k,\tau+1} - R_f = -Cov_\tau \left(R_f m_{\tau,\tau+1}, R_{k,\tau+1} \right)$$

where $m_{\tau,\tau+1}$ represents the SDF for the period $[\tau, \tau+1]$. We identify expression above with the analytical expression of asset risk premia derived in Proposition 2 and deduce the analytical expression of the SDF.

■

Proof of Proposition 4. Analytical expressions of $\omega_\tau^{(i)}(0)$, $a_{k,\tau}(0)$, $\omega_\tau^{(i)'}(0)$, and $a_{k,\tau}^{'}(0)_{\tau=0,1}$ are derived in the previous propositions. Recall that when there is no intermediate consumption, $C_\tau^{(i)}(\sigma)=0$ which implies that $\nu_i=0$ and $\tau_{\nu_i}=\tau_i$, and the previous results for $\omega_\tau^{(i)}(0)$, $a_{k,\tau}(0)$, $\omega_\tau^{(i)'}(0)$, and $\left(a_{k,\tau}^{'}(0)\right)_{\tau=0,1}$ reduce to:

$$
\begin{aligned}
\omega_1^{(i)}(0) &= \frac{R_f}{W_2^{(i)}(0)}\frac{\tau_i}{\overline{\tau}}\overline{\omega}_1 \text{ and } a_{k,1}(0)=\frac{1}{\overline{\tau}}Cov_1\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right),Y_{k,2}\right), \\
\omega_0^{(i)}(0) &= \frac{R_f}{W_1^{(i)}(0)}\frac{\tau_i}{\overline{\tau}}\overline{\omega}_0 \text{ and } a_{k,0}(0)=\frac{1}{\overline{\tau}}R_f Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right),Y_{k,1}\right).
\end{aligned}
\tag{53}
$$

and

$$
\begin{aligned}
a_{k,1}^{'}(0) &= -\frac{\overline{\rho}}{\overline{\tau}^2}Cov_1\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^2,Y_{k,2}\right)+\frac{R_f\left(1-2\overline{\rho}\right)}{\overline{\tau}^2}\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)Cov_1\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right),Y_{k,2}\right), \\
\omega_1^{(i)'}(0) &= \frac{1}{W_2^{(i)}(0)}\frac{\left(\rho_i-\overline{\rho}\right)\tau_i R_f}{\overline{\tau}^2}\Sigma_1^{-1}Cov_1\left(\mathbf{Y}_2,\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^2\right) \\
&\quad+\frac{1}{W_2^{(i)}(0)}\left(\frac{2R_f\tau_i\left(\rho_i-\overline{\rho}\right)R_f}{\overline{\tau}^2}-\frac{R_f^2}{W_2^{(i)}(0)}\frac{\tau_i^2}{\overline{\tau}^2}\right)\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)\overline{\omega}_1, \\
a_{k,0}^{'}(0) &= \frac{\left(1-\overline{\rho}\right)}{\overline{\tau}^2}Cov_0\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^2,Y_{k,1}\right)-\frac{\overline{\rho}}{\overline{\tau}^2}R_f^2 Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)^2,Y_{k,1}\right), \\
\omega_0^{(i)'}(0) &= \frac{1}{W_1^{(i)}(0)}\frac{\tau_i\left(\rho_i-\overline{\rho}\right)}{\overline{\tau}^2}\Sigma_0^{-1}Cov_0\left(\mathbf{Y}_1,\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^2\right) \\
&\quad+\frac{1}{W_1^{(i)}(0)}\frac{\left(\rho_i-\overline{\rho}\right)\tau_i}{\overline{\tau}^2}R_f^2\Sigma_0^{-1}Cov_0\left(\mathbf{Y}_1,\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)^2\right).
\end{aligned}
\tag{54}
$$

To give a formal proof of propositions 4, we first derive $\omega_0^{(i)''}(0)$ and $a_{k,0}^{''}(0)$. To do this, we consider the FOC at date 0 with respect to investor's portfolio weight $\omega_{0,k}^{(i)}(\sigma)$:

$$
E_0 u_i^{'}\left(W_2^{(i)}(\sigma)\right)\left(\sigma a_{k,0}(\sigma)+Y_{k,1}\right)\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right)=0
$$

and take the third derivative of this FOC with respect to σ:

$$\begin{aligned}
&E_0\left(u_i^{'}\left(W_2^{(i)}(\sigma)\right)\right)^{'''}(\sigma a_{k,0}(\sigma)+Y_{k,1})\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right) \\
&+E_0u_i^{'}\left(W_2^{(i)}(\sigma)\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})^{'''}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right) \\
&+E_0u_i^{'}\left(W_2^{(i)}(\sigma)\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right)^{'''} \\
&+3E_0\left(u_i^{'}\left(W_2^{(i)}(\sigma)\right)\right)^{''}(\sigma a_{k,0}(\sigma)+Y_{k,1})^{'}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right) \\
&+3E_0\left(u_i^{'}\left(W_2^{(i)}(\sigma)\right)\right)^{''}(\sigma a_{k,0}(\sigma)+Y_{k,1})\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right)^{'} \\
&+3E_0u_i^{'}\left(W_2^{(i)}(\sigma)\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})^{''}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right)^{'} \\
&+3E_0\left(u_i^{'}\left(W_2^{(i)}(\sigma)\right)\right)^{'}(\sigma a_{k,0}(\sigma)+Y_{k,1})^{''}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right) \\
&+3E_0u_i^{'}\left(W_2^{(i)}(\sigma)\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})^{'}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right)^{''} \\
&+3E_0\left(u_i^{'}\left(W_2^{(i)}(\sigma)\right)\right)^{'}(\sigma a_{k,0}(\sigma)+Y_{k,1})\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right)^{''} \\
&+6E_0\left(u_i^{'}\left(W_2^{(i)}(\sigma)\right)\right)^{'}(\sigma a_{k,0}(\sigma)+Y_{k,1})^{'}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\mathbf{R}_2^e(\sigma)\right)^{'} \\
=\;&0.
\end{aligned}\tag{55}$$

We expand expression above and get:

$$
\begin{aligned}
& E_0\left(\begin{array}{c} u_i^{''''}\left(W_2^{(i)}(\sigma)\right)\left(W_2^{(i)'}(\sigma)\right)^3 \\ +3u_i^{'''}\left(W_2^{(i)}(\sigma)\right)W_2^{(i)'}(\sigma)\,W_2^{(i)''}(\sigma) \\ +u_i^{''}\left(W_2^{(i)}(\sigma)\right)W_2^{(i)'''}(\sigma) \end{array}\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right) \\
& +E_0 u_i^{'}\left(W_2^{(i)}(\sigma)\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})^{'''}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right) \\
& +E_0 u_i^{'}\left(W_2^{(i)}(\sigma)\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)^{'''} \\
& +3E_0\left(\begin{array}{c} u_i^{'''}\left(W_2^{(i)}(\sigma)\right)\left(W_2^{(i)'}(\sigma)\right)^2 \\ +u_i^{''}\left(W_2^{(i)}(\sigma)\right)W_2^{(i)''}(\sigma) \end{array}\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})^{'}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right) \\
& +3E_0\left(\begin{array}{c} u_i^{'''}\left(W_2^{(i)}(\sigma)\right)\left(W_2^{(i)'}(\sigma)\right)^2 \\ +u_i^{''}\left(W_2^{(i)}(\sigma)\right)W_2^{(i)''}(\sigma) \end{array}\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)^{'} \\
& +3E_0 u_i^{'}\left(W_2^{(i)}(\sigma)\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})^{''}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)^{'} \\
& +3E_0\left(u_i^{''}\left(W_2^{(i)}(\sigma)\right)W_2^{(i)'}(\sigma)\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})^{''}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right) \\
& +3E_0 u_i^{'}\left(W_2^{(i)}(\sigma)\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})^{'}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)^{''} \\
& +3E_0\left(u_i^{''}\left(W_2^{(i)}(\sigma)\right)W_2^{(i)'}(\sigma)\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)^{''} \\
& +6E_0\left(u_i^{''}\left(W_2^{(i)}(\sigma)\right)W_2^{(i)'}(\sigma)\right)(\sigma a_{k,0}(\sigma)+Y_{k,1})^{'}\left(R_f+\omega_1^{(i)\intercal}(\sigma)\,\mathbf{R}_2^e(\sigma)\right)^{'} \\
= \; & 0.
\end{aligned}
$$

We take the limit of expression above as σ converges to zero and divide the result by $u_i^{''}\left(\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\right)$, then simplify it to get

$$
\begin{aligned}
& \frac{3\kappa_i}{\tau_i^2}E_0\left(\lim_{\sigma\rightsquigarrow 0}W_2^{(i)'}(\sigma)\right)^3 Y_{k,1}R_f-6\frac{\rho_i}{\tau_i}E_0\left(\lim_{\sigma\rightsquigarrow 0}W_2^{(i)'}(\sigma)\right)\left(\lim_{\sigma\rightsquigarrow 0}W_2^{(i)''}(\sigma)\right)Y_{k,1}R_f \\
& +E_0\lim_{\sigma\rightsquigarrow 0}W_2^{(i)'''}(\sigma)\,Y_{k,1}R_f-3\tau_i E_0 a_{k,0}^{''}(0)\,R_f-\tau_i E_0 Y_{k,1}\left(6\omega_1^{(i)\intercal}(0)\,a_1(0)+6\omega_1^{(i)'\intercal}(\sigma)\,a_1(0)\right) \\
& -6\frac{\rho_i}{\tau_i}E_0\left(\lim_{\sigma\rightsquigarrow 0}W_2^{(i)'}(\sigma)\right)^2 a_{k,0}(0)\,R_f+3E_0\lim_{\sigma\rightsquigarrow 0}W_2^{(i)''}(\sigma)\,a_{k,0}(0)\,R_f \\
& -6\frac{\rho_i}{\tau_i}E_0\left(\lim_{\sigma\rightsquigarrow 0}W_2^{(i)'}(\sigma)\right)^2 Y_{k,1}\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right)+3E_0\lim_{\sigma\rightsquigarrow 0}W_2^{(i)''}(\sigma)\,Y_{k,1}\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right) \\
& -3\tau_i E_0 a_{k,0}(0)\left(2\omega_1^{(i)\intercal}(0)\,a_1(0)\right)+3E_0\lim_{\sigma\rightsquigarrow 0}W_2^{(i)'}(\sigma)\,Y_{k,1}\left(2\omega_1^{(i)\intercal}(0)\,a_1(0)+2\omega_1^{(i)'\intercal}(0)\,\mathbf{Y}_2\right) \\
& +6E_0\lim_{\sigma\rightsquigarrow 0}W_2^{(i)'}(\sigma)\,a_{k,0}(0)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right) \\
= \; & 0.
\end{aligned} \tag{56}
$$

Note that we have assumed there is no intermediate consumption, hence Equation (22) reduces to:

$$\lim_{\sigma \rightsquigarrow 0} W_2^{(i)'}(\sigma) = \lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma)\left(\omega_0^{(i)\intercal}(0)\,\mathbf{Y}_1\right) + \frac{1}{R_f}\lim_{\sigma \rightsquigarrow 0} W_2^{(i)}(\sigma)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right).$$

We, thereafter, replace $\lim_{\sigma \rightsquigarrow 0} W_2^{(i)'}(\sigma)$ in (56), then expand the result and get:

$$\begin{aligned}
& +3\frac{\kappa_i\tau_i}{\overline{\tau}^3}R_f^4 Cov_0\left(\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right)^3, Y_{k,1}\right) + 3\frac{\kappa_i\tau_i}{\overline{\tau}^3}R_f Cov_0\left(\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right)^3, Y_{k,1}\right) \\
& +9\frac{\kappa_i\tau_i}{\overline{\tau}^3}R_f^2 Cov_0\left(\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right)\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right)^2, Y_{k,1}\right) \\
& -6\frac{\rho_i}{\tau_i}E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right) + \frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right)\right)\left(\lim_{\sigma \rightsquigarrow 0} W_2^{(i)''}(\sigma)\right)Y_{k,1}R_f \\
& +E_0\lim_{\sigma \rightsquigarrow 0} W_2^{(i)'''}(\sigma)\,Y_{k,1}R_f - 3\tau_i a_{k,0}^{''}(0)\,R_f - \tau_i E_0 Y_{k,1}\left(6\omega_1^{(i)\intercal}(0)\,a_1(0) + 6\omega_1^{(i)'\intercal}(\sigma)\,a_1(0)\right) \\
& -6\frac{\rho_i}{\tau_i}E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right) + \frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right)\right)^2 a_{k,0}(0)\,R_f + 6\frac{\tau\nu_i}{\overline{\tau}^2}R_f Var_0\left(\overline{\omega}_0\mathbf{Y}_1\right)a_{k,0}(0)\,R_f \\
& -6\frac{\rho_i}{\tau_i}E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right) + \frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right)\right)^2 Y_{k,1}\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right) \\
& +3E_0\lim_{\sigma \rightsquigarrow 0} W_2^{(i)''}(\sigma)\,Y_{k,1}\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right) - 3\tau_i E_0 a_{k,0}(0)\left(2\omega_1^{(i)\intercal}(0)\,a_1(0)\right) \\
& +3E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right) + \frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right)\right)Y_{k,1}\left(2\omega_1^{(i)\intercal}(0)\,a_1(0) + 2\omega_1^{(i)'\intercal}(0)\,\mathbf{Y}_2\right) \\
& +6E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal\mathbf{Y}_1\right) + \frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal\mathbf{Y}_2\right)\right)a_{k,0}(0)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right) \\
= \;& 0.
\end{aligned} \tag{57}$$

To simplify expression above, it is necessary to find $\lim_{\sigma \rightsquigarrow 0} W_2^{(i)'''}(\sigma)$ and $\lim_{\sigma \rightsquigarrow 0} W_2^{(i)''}(\sigma)$. We first derive $\lim_{\sigma \rightsquigarrow 0} W_2^{(i)''}(\sigma)$. To do this, we utilize (32) and assume there is no intermediate consumption. This implies that $\lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma) = R_f$, $\lim_{\sigma \rightsquigarrow 0} W_2^{(i)}(\sigma) = R_f^2$ and

$$\begin{aligned}
\lim_{\sigma \rightsquigarrow 0} W_2^{(i)''}(\sigma) \;=\; & R_f\begin{pmatrix} 2\omega_0^{(i)\intercal}(0)\,a_0(0) + 2\omega_0^{(i)'\intercal}(0)\,\mathbf{Y}_1 \\ 2\omega_1^{(i)\intercal}(0)\,a_1(0) + 2\omega_1^{(i)'\intercal}(0)\,\mathbf{Y}_2 \end{pmatrix} \\
& +2\left(\omega_0^{(i)\intercal}(0)\,\mathbf{Y}_1\right)\left(\omega_1^{(i)\intercal}(0)\,\mathbf{Y}_2\right)
\end{aligned}$$

We then replace expressions

$$\omega_\tau^{(i)}(0),\ \omega_\tau^{(i)'}(0),\ a_{k,\tau}(0),\ a_{k,\tau}^{'}(0)\ \text{ for } \tau = 0, 1$$

(see Equations (53) and (54)) in Equation (32). Thus, we obtain:

$$\lim_{\sigma\rightsquigarrow 0} W_2^{(i)''}(\sigma) = \frac{2\tau_i}{\overline{\tau}^2} R_f^2 Var_0\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right) + \frac{2\tau_i}{\overline{\tau}^2} Var_1\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right) \tag{58}$$
$$+2\frac{\tau_i\left(\rho_i-\overline{\rho}\right)}{\overline{\tau}^2}\left(\left(\Sigma_0^{-1}Cov_0\left(\mathbf{Y}_1,\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^2\right)\right)^\intercal \mathbf{Y}_1\right)$$
$$+2\frac{\left(\rho_i-\overline{\rho}\right)\tau_i}{\overline{\tau}^2}R_f^2\left(\left(\Sigma_0^{-1}Cov_0\left(\mathbf{Y}_1,\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)^2\right)^\intercal\right)\mathbf{Y}_1\right)$$
$$+\frac{2}{R_f}\frac{\left(\rho_i-\overline{\rho}\right)\tau_i R_f}{\overline{\tau}^2}\left(\left(\Sigma_1^{-1}Cov_1\left(\mathbf{Y}_2,\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^2\right)\right)^\intercal \mathbf{Y}_2\right)$$
$$+\frac{4R_f\tau_i\left(\rho_i-\overline{\rho}\right)}{\overline{\tau}^2}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)$$

We replace Equation (58) in Equation (57) and obtain:

$$3\frac{\kappa_i\tau_i}{\overline{\tau}^3}R_f^4 Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)^3, Y_{k,1}\right) \tag{59}$$
$$+3\frac{\kappa_i\tau_i}{\overline{\tau}^3}R_f Cov_0\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^3, Y_{k,1}\right)+9\frac{\kappa_i\tau_i}{\overline{\tau}^3}R_f^2 Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^2, Y_{k,1}\right)$$
$$-6\frac{\rho_i}{\tau_i}R_f E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)+\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)\left(\frac{2\tau_i}{\overline{\tau}^2}R_f^2 Var_0\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\right)Y_{k,1}$$
$$-12\frac{\rho_i}{\tau_i}R_f\frac{\tau_i}{\overline{\tau}^2}E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)+\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)Var_1\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)Y_{k,1}$$
$$-12\frac{\rho_i}{\tau_i}E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)+\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)\frac{\tau_i\left(\rho_i-\overline{\rho}\right)}{\overline{\tau}^2}\left(\frac{1}{\sigma}\overline{\varsigma}_{0,\vartheta}^\intercal \mathbf{Y}_1\right)Y_{k,1}R_f$$
$$-6\frac{\rho_i}{\tau_i}E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)+\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)\left(2\left(\frac{\rho_i\tau_i}{\overline{\tau}^2}-\frac{\tau_i\overline{\rho}}{\overline{\tau}^2}\right)R_f^2\left(\frac{1}{\sigma}\overline{\varsigma}_0^\intercal \mathbf{Y}_1\right)\right)Y_{k,1}R_f$$
$$-6\frac{\rho_i}{\tau_i}E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)+\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)\left(\frac{2}{R_f}\frac{\left(\rho_i-\overline{\rho}\right)\tau_i R_f}{\overline{\tau}^2}\left(\frac{1}{\sigma}\overline{\varsigma}_1^\intercal \mathbf{Y}_2\right)\right)Y_{k,1}R_f$$
$$-6\frac{\rho_i}{\tau_i}E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)+\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)\left(\frac{4R_f\tau_i\left(\rho_i-\overline{\rho}\right)}{\overline{\tau}^2}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)Y_{k,1}R_f$$
$$+E_0\lim_{\sigma\rightsquigarrow 0}W_2^{(i)'''}(\sigma)Y_{k,1}R_f-3\tau_i a_{k,0}^{''}(0)R_f-\tau_i E_0 Y_{k,1}\left(6\omega_1^{(i)\intercal}(0)a_1(0)+6\omega_1^{(i)'\intercal}(\sigma)a_1(0)\right)$$
$$-6\frac{\rho_i}{\tau_i}E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)+\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)^2 a_{k,0}(0)R_f+6\frac{\tau_i}{\overline{\tau}^2}R_f Var_0\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)a_{k,0}(0)R_f$$
$$-6\frac{\rho_i}{\tau_i}E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)+\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)^2 Y_{k,1}\left(\omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right)$$
$$+3E_0\left(\frac{2}{R_f}\frac{\left(\rho_i-\overline{\rho}\right)\tau_i R_f}{\overline{\tau}^2}\left(\frac{1}{\sigma}\overline{\varsigma}_1^\intercal \mathbf{Y}_2\right)\right)Y_{k,1}\left(\omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right)$$
$$+3E_0\left(\frac{4R_f\tau_i\left(\rho_i-\overline{\rho}\right)}{\overline{\tau}^2}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)Y_{k,1}\left(\omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right)-3\tau_i E_0 a_{k,0}(0)\left(2\omega_1^{(i)\intercal}(0)a_1(0)\right)$$
$$+3E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)+\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)Y_{k,1}\left(2\omega_1^{(i)\intercal}(0)a_1(0)+2\omega_1^{(i)'\intercal}(0)\mathbf{Y}_2\right)$$
$$+6E_0\left(R_f\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)+\frac{\tau_i}{\overline{\tau}}\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right)a_{k,0}(0)\left(\omega_1^{(i)\intercal}(0)\mathbf{Y}_2\right)$$
$$= 0.$$

where

$$
\begin{aligned}
\bar{\varsigma}_0 &= \left(\sigma^2\Sigma_0\right)^{-1} Cov_0\left(\sigma\mathbf{Y}_1, \left(\overline{\omega}_0^{\intercal}\sigma\mathbf{Y}_1\right)^2\right) = \left(Var_0\left(\mathbf{R}_1\right)\right)^{-1} Cov_0\left(\mathbf{R}_1, r_{M1}^2\right), \\
\bar{\varsigma}_{0,\vartheta} &= \left(\sigma^2\Sigma_0\right)^{-1} Cov_0\left(\sigma\mathbf{Y}_1, \left(\overline{\omega}_1^{\intercal}\sigma\mathbf{Y}_2\right)^2\right) = \left(Var_0\left(\mathbf{R}_1\right)\right)^{-1} Cov_0\left(\mathbf{R}_1, Var_1\left(r_{M2}\right)\right), \\
\bar{\varsigma}_1 &= \left(\sigma^2\Sigma_1\right)^{-1} Cov_1\left(\sigma\mathbf{Y}_2, \left(\overline{\omega}_1^{\intercal}\sigma\mathbf{Y}_2\right)^2\right) = \left(Var_1\left(\mathbf{R}_2\right)\right)^{-1} Cov_1\left(\mathbf{R}_2, r_{M2}^2\right).
\end{aligned}
$$

To expand (59), we need to find $E_0 \lim_{\sigma\rightsquigarrow 0} W_2^{(i)'''}(\sigma)\, Y_{k,1} R_f$. To derive this expression, we first find $\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'''}(\sigma)$:

$$
\begin{aligned}
\lim_{\sigma\rightsquigarrow 0} W_2^{(i)'''}(\sigma) &= \frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left(6\omega_0^{(i)\intercal}(0)\, a_0(0) + 3\omega_0^{(i)''\intercal}(\sigma)\,\mathbf{Y}_1 + 6\omega_0^{(i)'\intercal}(\sigma)\, a_0(0)\right) R_f \\
&+ \frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_2^{(i)}(\sigma)\left(6\omega_1^{(i)\intercal}(0)\, a_1(0) + 3\omega_1^{(i)''\intercal}(\sigma)\,\mathbf{Y}_2 + 6\omega_1^{(i)'\intercal}(\sigma)\, a_1(0)\right) \\
&+ 3\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left[\left(2\omega_0^{(i)\intercal}(0)\, a_0(0) + 2\omega_0^{(i)'\intercal}(0)\,\mathbf{Y}_1\right)\left(\omega_1^{(i)\intercal}(\sigma)\,\mathbf{Y}_2\right)\right] \\
&+ 3\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\left[\left(\omega_0^{(i)\intercal}(\sigma)\,\mathbf{Y}_1\right)\left(2\omega_1^{(i)\intercal}(0)\, a_1(0) + 2\omega_1^{(i)'\intercal}(0)\,\mathbf{Y}_2\right)\right].
\end{aligned} \tag{60}
$$

Now, we consider Equations (53), (54), and replace expressions

$$
\omega_\tau^{(i)}(0),\ \omega_\tau^{(i)'}(0),\ a_{k,\tau}(0),\ a_{k,\tau}^{'}(0) \text{ for } \tau = 0, 1
$$

in Equation (60), then multiply Equation (60) by $Y_{k,1}R_f$ and take the expectation at date 0 to get:

$$
\begin{aligned}
E_0 \lim_{\sigma\rightsquigarrow 0} W_2^{(i)'''}(\sigma)\, Y_{k,1} R_f &= 3\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\, E_0\left(\omega_0^{(i)''\intercal}(\sigma)\,\mathbf{Y}_1\right) Y_{k,1} R_f \\
&+ 6R_f\frac{\tau_i}{\overline{\tau}^2} Cov_0\left(Var_1\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,1}\right) \\
&+ 6E_0\left(\frac{\left(\rho_i - \overline{\rho}\right)\tau_i R_f}{\overline{\tau}^3}\right) Cov_1\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)\left(\frac{1}{\sigma}\bar{\varsigma}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,1}\right) \\
&+ 12\frac{\tau_i\left(\rho_i - \overline{\rho}\right) R_f^2}{\overline{\tau}^3} Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right) Var_1\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,1}\right).
\end{aligned} \tag{61}
$$

Now, we replace $E_0 \lim_{\sigma \rightsquigarrow 0} W_2^{(i)'''}(\sigma) Y_{k,1} R_f$ by its expression in Equation (59) and get:

$$
\begin{aligned}
& 3\frac{\kappa_i \tau_i}{\overline{\tau}^3} R_f^4 Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)^3, Y_{k,1}\right) + 3\frac{\kappa_i \tau_i}{\overline{\tau}^3} R_f Cov_0\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^3, Y_{k,1}\right) \\
& + \left(9\frac{\kappa_i \tau_i}{\overline{\tau}^3} R_f^2 - 12 R_f^2 \frac{\rho_i \tau_i}{\overline{\tau}^3}\right) Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right) Var_1\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right), Y_{k,1}\right) \\
& - \frac{12\rho_i \tau_i}{\overline{\tau}^3} R_f^4 Var_0\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right) Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right), Y_{k,1}\right) - 12 R_f^2 \frac{\rho_i \tau_i \left(\rho_i - \overline{\rho}\right)}{\overline{\tau}^3} Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\left(\frac{1}{\sigma}\overline{\varsigma}_{0,\vartheta}^\intercal \mathbf{Y}_1\right), Y_{k,1}\right) \\
& -12\frac{\rho_i \tau_i \left(\rho_i - \overline{\rho}\right)}{\overline{\tau}^3} R_f^4 Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\left(\frac{1}{\sigma}\overline{\varsigma}_0^\intercal \mathbf{Y}_1\right), Y_{k,1}\right) - 12\frac{\left(\rho_i - \overline{\rho}\right)\rho_i \tau_i}{\overline{\tau}^3} R_f Cov_0\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\left(\frac{1}{\sigma}\overline{\varsigma}_1^\intercal \mathbf{Y}_2\right), Y_{k,1}\right) \\
& -6\frac{\rho_i}{\tau_i}\frac{4R_f^2 \tau_i \left(\rho_i - \overline{\rho}\right)}{\overline{\tau}^2}\frac{\tau_i}{\overline{\tau}} Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right) Var_1\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right), Y_{k,1}\right) \\
& +3\lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma) E_0\left(\omega_0^{(i)''\intercal}(\sigma)\mathbf{Y}_1\right) Y_{k,1} R_f + 6R_f\frac{\tau_i}{\overline{\tau}^2} Cov_0\left(Var_1\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right), Y_{k,1}\right) \\
& +6\left(\frac{\left(\rho_i - \overline{\rho}\right)\tau_i R_f}{\overline{\tau}^3}\right) Cov_0\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\left(\frac{1}{\sigma}\overline{\varsigma}_1^\intercal \mathbf{Y}_2\right), Y_{k,1}\right) \\
& +12\frac{\tau_i \left(\rho_i - \overline{\rho}\right) R_f^2}{\overline{\tau}^3} Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right) Var_1\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right), Y_{k,1}\right) - 3\tau_i a_{k,0}^{''}(0) R_f \\
& -6\frac{R_f}{R_f^2}\frac{\tau_i^2}{\overline{\tau}^2} Cov_0\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^2, Y_{k,1}\right) - \frac{6}{R_f^2}\frac{\left(\rho_i - \overline{\rho}\right)\tau_i^2 R_f}{\overline{\tau}^3} Cov_0\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\left(\frac{1}{\sigma}\overline{\varsigma}_1^\intercal Y_2\right), Y_{k,1}\right) \\
& -\frac{6}{R_f^2}\left(\frac{2R_f \tau_i^2 \left(\rho_i - \overline{\rho}\right) R_f}{\overline{\tau}^3} - \frac{R_f^2}{R_f^2}\frac{\tau_i^3}{\overline{\tau}^3}\right) Cov_0\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^2\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right), Y_{k,1}\right) \\
& -6R_f^4\frac{\rho_i \tau_i}{\overline{\tau}^3} Var_0\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right) Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right), Y_{k,1}\right) - 6\frac{\rho_i \tau_i}{\overline{\tau}^3} R_f^2 Var_0\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right) Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right), Y_{k,1}\right) \\
& +6\frac{\tau_i}{\overline{\tau}^3} R_f^3 Var_0\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right) Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right), Y_{k,1}\right) - 12\frac{R_f^2}{R_f^2}\frac{\rho_i \tau_i^2}{\overline{\tau}^3} Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^2, Y_{k,1}\right) \\
& -6\frac{\rho_i}{\tau_i}\frac{R_f}{R_f^2}\frac{\tau_i^3}{\overline{\tau}^3} Cov_0\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^3, Y_{k,1}\right) + 6\frac{\left(\rho_i - \overline{\rho}\right)\tau_i^2 R_f}{\overline{\tau}^3}\frac{1}{R_f^2} Cov_0\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\left(\frac{1}{\sigma}\overline{\varsigma}_1^\intercal \mathbf{Y}_2\right), Y_{k,1}\right) \\
& +\frac{1}{R_f^2}\frac{12R_f^2 \tau_i^2 \left(\rho_i - \overline{\rho}\right)}{\overline{\tau}^3} Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^2, Y_{k,1}\right) \\
& -6\frac{R_f^2}{R_f^2}\frac{\tau_i^2}{\overline{\tau}^3} E_0 Var_1\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right) Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right), Y_{k,1}\right) + 6\frac{1}{R_f^2}\frac{\left(\rho_i - \overline{\rho}\right)\tau_i^2 R_f}{\overline{\tau}^3} Cov_0\left(\left(\frac{1}{\sigma}\overline{\varsigma}_1^\intercal \mathbf{Y}_2\right)\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right), Y_{k,1}\right) \\
& +6\frac{1}{R_f^2}\left(\frac{2R_f \tau_i^2 \left(\rho_i - \overline{\rho}\right) R_f}{\overline{\tau}^3} - \frac{R_f^2}{R_f^2}\frac{\tau_i^3}{\overline{\tau}^3}\right) Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right)\left(\overline{\omega}_1 \mathbf{Y}_2\right)^2, Y_{k,1}\right) \\
& +6\frac{R_f^2}{R_f^2}\frac{\tau_i^2}{\overline{\tau}^3} Cov_0\left(\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)^2\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right), Y_{k,1}\right) + 6\frac{R_f^2}{R_f^2}\frac{\tau_i^2}{\overline{\tau}^3}\left(E_0 Var_1\left(\overline{\omega}_1^\intercal \mathbf{Y}_2\right)\right) Cov_0\left(\left(\overline{\omega}_0^\intercal \mathbf{Y}_1\right), Y_{k,1}\right) \\
= \; & 0.
\end{aligned}
$$

Expression above can be simplified to:

$$
\begin{aligned}
& \frac{\kappa_i \tau_i}{\overline{\tau}^3} R_f^4 Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right)^3, Y_{k,1} \right) \\
& + \left(\frac{\kappa_i \tau_i}{\overline{\tau}^3} R_f - \frac{2}{R_f} \frac{\rho_i \tau_i^2}{\overline{\tau}^3} \right) Cov_0 \left(\left(\overline{\omega}_1^{\intercal} \mathbf{Y}_2 \right)^3, Y_{k,1} \right) \\
& + \left(\begin{array}{c} 3 \frac{\kappa_i \tau_i}{\overline{\tau}^3} R_f^2 - 4 R_f^2 \frac{\rho_i \tau_i}{\overline{\tau}^3} - \frac{8 R_f^2 \rho_i \tau_i (\rho_i - \overline{\rho})}{\overline{\tau}^3} \\ -4 \frac{\rho_i \tau_i^2}{\overline{\tau}^3} + 4 \frac{\tau_i (\rho_i - \overline{\rho}) R_f^2}{\overline{\tau}^3} + \frac{4 \tau_i^2 (\rho_i - \overline{\rho})}{\overline{\tau}^3} + 2 \frac{\tau_i^2}{\overline{\tau}^3} \end{array} \right) Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right) Var_1 \left(\overline{\omega}_1^{\intercal} \mathbf{Y}_2 \right), Y_{k,1} \right) \\
& \left(-\frac{6 \rho_i \tau_i}{\overline{\tau}^3} R_f^4 - 2 \frac{\rho_i \tau_i}{\overline{\tau}^3} R_f^2 + 2 \frac{\tau_i}{\overline{\tau}^3} R_f^3 \right) Var_0 \left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right) Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right), Y_{k,1} \right) \\
& -4 R_f^2 \frac{\rho_i \tau_i \left(\rho_i - \overline{\rho} \right)}{\overline{\tau}^3} Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right) \left(\frac{1}{\sigma} \overline{\varsigma}_{0,\vartheta}^{\intercal} \mathbf{Y}_1 \right), Y_{k,1} \right) \\
& -4 \frac{\rho_i \tau_i \left(\rho_i - \overline{\rho} \right)}{\overline{\tau}^3} R_f^4 Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right) \left(\frac{1}{\sigma} \overline{\varsigma}_0^{\intercal} \mathbf{Y}_1 \right), Y_{k,1} \right) \\
& \left(2 \frac{\left(\rho_i - \overline{\rho} \right) \tau_i R_f}{\overline{\tau}^3} - 4 \frac{\left(\rho_i - \overline{\rho} \right) \rho_i \tau_i}{\overline{\tau}^3} R_f + 2 \frac{1}{R_f} \frac{\left(\rho_i - \overline{\rho} \right) \tau_i^2}{\overline{\tau}^3} \right) Cov_0 \left(\left(\overline{\omega}_1^{\intercal} \mathbf{Y}_2 \right) \left(\frac{1}{\sigma} \overline{\varsigma}_1^{\intercal} \mathbf{Y}_2 \right), Y_{k,1} \right) \\
& + \lim_{\sigma \rightsquigarrow 0} W_1^{(i)} \left(\sigma \right) E_0 \left(\omega_0^{(i)'' \intercal} \left(\sigma \right) \mathbf{Y}_1 \right) Y_{k,1} R_f + \left(2 R_f \frac{\tau_i}{\overline{\tau}^2} - \frac{2}{R_f} \frac{\tau_i^2}{\overline{\tau}^2} \right) Cov_0 \left(Var_1 \left(\overline{\omega}_1^{\intercal} \mathbf{Y}_2 \right), Y_{k,1} \right) \\
& - \tau_i a_{k,0}^{''} \left(0 \right) R_f \\
= \quad & 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& D_{7i} Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right)^3, Y_{k,1} \right) + D_{6i} Cov_0 \left(\left(\overline{\omega}_1^{\intercal} \mathbf{Y}_2 \right)^3, Y_{k,1} \right) \\
& + D_{5i} Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right) Var_1 \left(\overline{\omega}_1^{\intercal} \mathbf{Y}_2 \right), Y_{k,1} \right) + D_{4i} Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right) \left(\frac{1}{\sigma} \overline{\varsigma}_{0,\vartheta}^{\intercal} \mathbf{Y}_1 \right), Y_{k,1} \right) \\
& + D_{0i} Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right), Y_{k,1} \right) + D_{3i} Cov_0 \left(\left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right) \left(\frac{1}{\sigma} \overline{\varsigma}_0^{\intercal} \mathbf{Y}_1 \right), Y_{k,1} \right) \\
& + D_{2i} Cov_0 \left(\left(\overline{\omega}_1^{\intercal} \mathbf{Y}_2 \right) \left(\frac{1}{\sigma} \overline{\varsigma}_1^{\intercal} \mathbf{Y}_2 \right), Y_{k,1} \right) + \lim_{\sigma \rightsquigarrow 0} W_1^{(i)} \left(\sigma \right) E_0 \left(\omega_0^{(i)'' \intercal} \left(\sigma \right) \mathbf{Y}_1 \right) Y_{k,1} \\
& + D_{1i} Cov_0 \left(Var_1 \left(\overline{\omega}_1^{\intercal} \mathbf{Y}_2 \right), Y_{k,1} \right) \\
& - \tau_i a_{k,0}^{''} \left(0 \right) \\
= \quad & 0
\end{aligned} \tag{62}
$$

where:

$$
\begin{aligned}
D_{7i} &= \frac{\kappa_i \tau_i}{\overline{\tau}^3} R_f^3 \\
D_{6i} &= \frac{\kappa_i \tau_i}{\overline{\tau}^3} - 2\frac{1}{R_f^2}\frac{\rho_i \tau_i^2}{\overline{\tau}^3} \\
D_{5i} &= \begin{pmatrix} 3\frac{\kappa_i \tau_i}{\overline{\tau}^3} R_f - 4R_f \frac{\rho_i \tau_i}{\overline{\tau}^3} - 2\frac{4R_f \rho_i \tau_i (\rho_i - \overline{\rho})}{\overline{\tau}^3} - 4\frac{1}{R_f}\frac{\rho_i \tau_i^2}{\overline{\tau}^3} \\ +4\frac{\tau_i (\rho_i - \overline{\rho}) R_f}{\overline{\tau}^3} + \frac{4\tau_i^2 (\rho_i - \overline{\rho})}{R_f \overline{\tau}^3} + 2\frac{1}{R_f}\frac{\tau_i^2}{\overline{\tau}^3} \end{pmatrix} \\
D_{4i} &= -4R_f \frac{\rho_i \tau_i \left(\rho_i - \overline{\rho}\right)}{\overline{\tau}^3} \\
D_{3i} &= -4\frac{\rho_i \tau_i \left(\rho_i - \overline{\rho}\right)}{\overline{\tau}^3} R_f^3 \\
D_{2i} &= \left(2\frac{\left(\rho_i - \overline{\rho}\right)\tau_i}{\overline{\tau}^3} - 4\frac{\left(\rho_i - \overline{\rho}\right)\rho_i \tau_i}{\overline{\tau}^3} + \frac{2}{R_f^2}\frac{\left(\rho_i - \overline{\rho}\right)\tau_i^2}{\overline{\tau}^3}\right) \\
D_{1i} &= \frac{2\tau_i}{\overline{\tau}^2} - \frac{2}{R_f^2}\frac{\tau_i^2}{\overline{\tau}^2} \\
D_{0i} &= .\left(-\frac{6\rho_i \tau_i}{\overline{\tau}^3} R_f^3 - 2\frac{\rho_i \tau_i}{\overline{\tau}^3} R_f + 2\frac{\tau_i}{\overline{\tau}^3} R_f^2\right) Var_0\left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1\right)
\end{aligned}
$$

Now, we take the sum of expression above for $i = 1, ..., I$ and divide the result by $\sum_{i=1}^{I} \tau_i$, then deduce

$$
\begin{aligned}
& a_{k,0}^{''}(0) = \overline{D}_7 Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)^3, Y_{k,1}\right) + \overline{D}_6 Cov_0\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^3, Y_{k,1}\right) \\
& +\overline{D}_5 Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right) Var_1\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,1}\right) + \overline{D}_4 Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)\left(\frac{1}{\sigma}\overline{\varsigma}_{0,\vartheta}^{\intercal}\mathbf{Y}_1\right), Y_{k,1}\right) \\
& +\overline{D}_0 Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right), Y_{k,1}\right) + \overline{D}_3 Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)\left(\frac{1}{\sigma}\overline{\varsigma}_0^{\intercal}\mathbf{Y}_1\right), Y_{k,1}\right) \\
& +\overline{D}_2 Cov_0\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)\left(\frac{1}{\sigma}\overline{\varsigma}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,1}\right) + \sum_{i=1}^{I} \lim_{\sigma \rightsquigarrow 0} W_1^{(i)}(\sigma) E_0\left(\omega_0^{(i)''\intercal}(\sigma)\mathbf{Y}_1\right) Y_{k,1} / \sum_{i=1}^{I} \tau_i \\
& +\overline{D}_1 Cov_0\left(Var_1\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,1}\right)
\end{aligned}
\tag{63}
$$

with:

$$
\begin{aligned}
\overline{D}_7 &= \frac{\overline{\kappa}}{\overline{\tau}^3} R_f^3 \qquad (64)\\
\overline{D}_6 &= \frac{\overline{\kappa}}{\overline{\tau}^3} - \frac{2}{R_f^2} \frac{\overline{\rho\tau}}{\overline{\tau}^3}\\
\overline{D}_5 &= \left(\frac{3\overline{\kappa}}{\overline{\tau}^3} R_f - 4R_f \frac{\overline{\rho}}{\overline{\tau}^3} - 2\frac{4R_f\left(\overline{\rho^2} - \overline{\rho}^2\right)}{\overline{\tau}^3} - \frac{4}{R_f}\frac{\overline{\rho}}{\overline{\tau}^2} + \frac{2\left(1-2\overline{\rho}\right)}{R_f}\frac{\left(\overline{\tau^2} - \overline{\tau}^2\right)}{\overline{\tau}^4} + \frac{2}{R_f}\frac{1}{\overline{\tau}^2} \right)\\
\overline{D}_4 &= -4R_f \frac{\left(\overline{\rho^2} - \overline{\rho}^2\right)}{\overline{\tau}^3}\\
\overline{D}_3 &= -4\frac{\left(\overline{\rho^2} - \overline{\rho}^2\right)}{\overline{\tau}^3} R_f^3\\
\overline{D}_2 &= \left(-4\frac{\left(\overline{\rho^2} - \overline{\rho}^2\right)}{\overline{\tau}^3} + \frac{2}{R_f^2}\frac{\left(\overline{\rho\tau} - \overline{\rho\tau}\right)}{\overline{\tau}^3} - \frac{2}{R_f^2}\frac{\overline{\rho}\left(\overline{\tau^2} - \overline{\tau}^2\right)}{\overline{\tau}^4} \right)\\
\overline{D}_1 &= \frac{2}{\overline{\tau}^2} - \frac{2}{R_f^2}\frac{1}{\overline{\tau}} - \frac{2}{R_f^2}\frac{\left(\overline{\tau^2} - \overline{\tau}^2\right)}{\overline{\tau}^3}\\
\overline{D}_0 &= \left(-\frac{6\overline{\rho}}{\overline{\tau}^3} R_f^3 - 2\frac{\overline{\rho}}{\overline{\tau}^3} R_f + \frac{2}{\overline{\tau}^3} R_f^2 \right) Var_0\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)
\end{aligned}
$$

where:

$$
\overline{\tau} = \frac{1}{I}\sum_{i=1}^{I}\tau_i, \qquad \overline{\rho} = \frac{\sum_{i=1}^{I}\rho_i\tau_i}{\sum_{i=1}^{I}\tau_i}, \qquad \overline{\kappa} = \frac{\sum_{i=1}^{I}\kappa_i\tau_i}{\sum_{i=1}^{I}\tau_i},
$$

$$
\overline{\tau^2} - \overline{\tau}^2 = \frac{1}{I}\sum_{i=1}^{I}\left(\tau_i - \overline{\tau}\right)^2, \qquad \overline{\rho\tau} - \overline{\rho}.\overline{\tau} = \frac{\sum_{i=1}^{I}(\rho_i - \overline{\rho})(\tau_i - \overline{\tau})\tau_i}{\sum_{i=1}^{I}\tau_i}, \qquad \overline{\rho^2} - \overline{\rho}^2 = \frac{\sum_{i=1}^{I}(\rho_i - \overline{\rho})^2\tau_i}{\sum_{i=1}^{I}\tau_i}.
$$

In case, there is no intermediate consumption, note that at date 0, the limit as σ converges to zero of the second derivative of the market clearing condition with respect to σ is:

$$
\sum_{i=1}^{I}\frac{1}{R_f}\lim_{\sigma\rightsquigarrow 0} W_1^{(i)}(\sigma)\,\omega_0^{(i)''}(0) = 0. \qquad (65)
$$

We utilize (65) to simplify (63) and get

$$
\begin{aligned}
& a_{k,0}^{''}(0) \qquad (66)\\
= \; & \overline{D}_7 Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)^3, Y_{k,1}\right) + \overline{D}_6 Cov_0\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)^3, Y_{k,1}\right)\\
& + \overline{D}_5 Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right) Var_1\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,1}\right) + \overline{D}_4 Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)\left(\frac{1}{\sigma}\overline{\varsigma}_{0,\vartheta}^{\intercal}\mathbf{Y}_1\right), Y_{k,1}\right)\\
& + \overline{D}_3 Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right)\left(\frac{1}{\sigma}\overline{\varsigma}_0^{\intercal}\mathbf{Y}_1\right), Y_{k,1}\right) + \overline{D}_2 Cov_0\left(\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right)\left(\frac{1}{\sigma}\overline{\varsigma}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,1}\right)\\
& + \overline{D}_1 Cov_0\left(Var_1\left(\overline{\omega}_1^{\intercal}\mathbf{Y}_2\right), Y_{k,1}\right) + \overline{D}_0 Cov_0\left(\left(\overline{\omega}_0^{\intercal}\mathbf{Y}_1\right), Y_{k,1}\right)
\end{aligned}
$$

where $\overline{D}_i$ are defined in Equation (64). With the knowledge of $a_{k,0}(0)$, $a_{k,0}^{'}(0)$ and $a_{k,0}^{''}(0)$, we next deduce the risk premium on risky assets. At date 0, the risk premium on asset k is:

$$
E_0 R_{k,1} - R_f = \sigma^2\left(a_{k,0}(0) + \sigma a_{k,0}^{'}(0) + \frac{1}{2}\sigma^2 a_{k,0}^{''}(0)\right)
$$

We replace $a_{k,0}(0)$, $a_{k,0}^{'}(0)$ and $a_{k,0}^{''}(0)$ by their expressions and get:

$$
\begin{aligned}
E_0 R_{k,1} - R_f \;=\; & \overline{\alpha}_0 Cov_0\left(r_{M1}, R_{k,1}\right) + \overline{\alpha}_1 Cov_0\left(r_{M1}^2, R_{k,1}\right) + \overline{\alpha}_2 Cov_0\left(r_{M1}^3, R_{k,1}\right)\\
& + \overline{\alpha}_3 Cov_0\left(Var_1\left(r_{M2}\right), R_{k,1}\right) + \overline{\alpha}_4 Cov_0\left(Skew_1\left(r_{M2}\right), R_{k,1}\right)\\
& \overline{\alpha}_5 Cov_0\left(r_{M1} Var_1\left(r_{M2}\right), R_{k,1}\right) + \overline{\alpha}_6 Cov_0\left(R_{M1}^{(\rho)}, R_{k,1}\right)\\
& + \overline{\alpha}_7 Cov_0\left(R_{M1}^{\overline{\varsigma}_0}.r_{M1}, R_{k,1}\right) + \overline{\alpha}_8 Cov_0\left(r_{M1} R_{M1}^{\overline{\varsigma}_{0,\vartheta}}, Y_{k,1}\right)
\end{aligned}
$$

with

$$
Skew_1\left(r_{M2}\right) = E_1\left(R_{M2} - E_1 R_{M2}\right)^3 \text{ and } R_{M1}^{(\rho)} = Cov_1\left(r_{M2}, R_{M2}^{\overline{\varsigma}_1}\right)
$$

and

$$
\begin{aligned}
R_{M1}^{\overline{\varsigma}_0} &= \overline{\varsigma}_0^{\intercal}\mathbf{R}_1,\; R_{M2}^{\overline{\varsigma}_1} = \overline{\varsigma}_1^{\intercal}\mathbf{R}_2,\; R_{M1}^{\overline{\varsigma}_{0,\vartheta}} = \overline{\varsigma}_{0,\vartheta}^{\intercal}\mathbf{R}_2\\
\overline{\varsigma}_1 &= \left(Var_1\left(\mathbf{R}_2\right)\right)^{-1} Cov_1\left(\mathbf{R}_2, r_{M2}^2\right),\\
\overline{\varsigma}_0 &= \left(Var_0\left(\mathbf{R}_1\right)\right)^{-1} Cov_0\left(\mathbf{R}_1, r_{M1}^2\right),\\
\overline{\varsigma}_{0,\vartheta} &= \left(Var_0\left(\mathbf{R}_1\right)\right)^{-1} Cov_0\left(\mathbf{R}_1, Var_1\left(r_{M2}\right)\right)
\end{aligned}
$$

where the parameters $\overline{\alpha}_j$, $j = 0, ..., 8$ are given below:

$$\begin{aligned}
\overline{\alpha}_0 &= \frac{1}{\overline{\tau}} R_f + \left(\frac{2}{\overline{\tau}^3} R_f^2 - 2\left(1 + 3R_f^2\right) \frac{\overline{\rho}}{\overline{\tau}^3} R_f \right) Var_0 \left(\overline{\omega}_0^{\intercal} \mathbf{Y}_1 \right) \\
\overline{\alpha}_1 &= -\frac{\overline{\rho}}{\overline{\tau}^2} R_f^2 \\
\overline{\alpha}_2 &= \frac{\overline{\kappa}}{\overline{\tau}^3} R_f^3 \\
\overline{\alpha}_3 &= \frac{(3 - \overline{\rho})}{\overline{\tau}^2} - \frac{2}{R_f^2} \frac{\left(\overline{\tau^2} - \overline{\tau}^2 \right)}{\overline{\tau}^3} - \frac{2}{R_f^2} \frac{1}{\overline{\tau}} \\
\overline{\alpha}_4 &= \frac{\overline{\kappa}}{\overline{\tau}^3} - \frac{2}{R_f^2} \frac{\overline{\rho \tau}}{\overline{\tau}^3} \\
\overline{\alpha}_5 &= \frac{3\overline{\kappa}}{\overline{\tau}^3} R_f - 4R_f \frac{\overline{\rho}}{\overline{\tau}^3} - \frac{8R_f \left(\overline{\rho^2} - \overline{\rho}^2 \right)}{\overline{\tau}^3} \\
&\quad + \frac{2\left(1 - 2\overline{\rho}\right)}{R_f} \frac{\left(\overline{\tau^2} - \overline{\tau}^2 \right)}{\overline{\tau}^4} + \frac{2}{R_f} \frac{\left(1 - 2\overline{\rho}\right)}{\overline{\tau}^2} \\
\overline{\alpha}_6 &= -4 \frac{\left(\overline{\rho^2} - \overline{\rho}^2 \right)}{\overline{\tau}^3} + \frac{2}{R_f^2} \frac{\left(\overline{\rho \tau} - \overline{\rho \tau} \right)}{\overline{\tau}^3} - \frac{2}{R_f^2} \frac{\overline{\rho} \left(\overline{\tau^2} - \overline{\tau}^2 \right)}{\overline{\tau}^4} \\
\overline{\alpha}_7 &= -4 \frac{\left(\overline{\rho^2} - \overline{\rho}^2 \right)}{\overline{\tau}^3} R_f^3 \\
\overline{\alpha}_8 &= -4R_f \frac{\left(\overline{\rho^2} - \overline{\rho}^2 \right)}{\overline{\tau}^3}
\end{aligned}$$

Now, if we replace (66) in (62), then use the portfolio weight expression

$$\omega_0^{(i)}(\sigma) = \omega_0^{(i)}(0) + \sigma \omega_0^{(i)'}(0) + \frac{1}{2} \sigma^2 \omega_0^{(i)''}(0)$$

it can be shown that the portfolio weight includes a skewness hedging component, an intertemporal volatility hedging component and other higher-order related intertemporal hedging components. The hedging components depend on the correlation between the aggregate volatility risk with risky assets, the correlation between the aggregate skewness risk with risky asset, and the distribution of investor preferences. ■

Proof of Proposition 5.. By definition, the risk premium on asset k for the time period $[0, 1]$ is given by

$$E_0 R_{k,1} - R_f = -Cov_0 \left(R_f m_{0,1}, R_{k,1} \right)$$

where $m_{0,1}$ represents the SDF for the period $[0, 1]$. We identify expression above with the analytical expression of asset risk premia derived in Proposition 4 and deduce the analytical expression of the SDF. ■

Table 1
Multi-beta Pricing Models

Risk Factors	M_V	MS_V	MSC_VS	Reduced-MSC_VS(1)	Reduced-MSC_VS(2)
R_{Mt+1}	x	x	x	x	x
r_{Mt}^{2}		x	x	x	x
r_{Mt}^{3}			x	x	
$Var_{t+1}\left(r_{Mt+2}\right)$	x	x	x	x	
$Skew_{t+1}\left(r_{Mt+2}\right)$			x	x	
$r_{Mt+1}Var_{t+1}\left(r_{Mt+2}\right)$			x	x	
$r_{Mt+1}^{(\rho)}$			x		x
$r_{Mt+1}^{\bar{\varsigma}_0}r_{Mt+1}$			x		x
$r_{Mt+1}^{\bar{\varsigma}_{0,\vartheta}}r_{Mt+1}$			x		x
$r_{Mt+1}^{\bar{\varsigma}_{0,\vartheta}}$					x

This table shows the multi-beta pricing models used in the empirical section. The symbol X indicates that the model contains the risk factor.

Table 2
Multi-beta regression for the 30 industry portfolios

			M_V		
Panel A		α	β on $R_M - R_f$	β on $g_{Var(R_M)} - R_f$	Adjusted R^2
Food	Coeff	0.004**	0.706***	-0.007***	0.517
	t-stat	2.265	12.653	-2.605	
Beer	Coeff	0.003*	0.795***	-0.006	0.432
	t-stat	1.718	11.704	-1.347	
Smoke	Coeff	0.007**	0.679***	0.000	0.222
	t-stat	2.317	9.005	0.003	
Games	Coeff	0.002	1.221***	-0.009**	0.670
	t-stat	0.860	22.322	-2.060	
Books	Coeff	0.001	0.996***	-0.008***	0.693
	t-stat	0.833	19.197	-2.785	
Hshld	Coeff	0.000	0.853***	0.001	0.610
	t-stat	0.270	16.495	0.334	
Clths	Coeff	0.001	1.105***	-0.007*	0.565
	t-stat	0.687	17.006	-1.877	
Hlth	Coeff	0.002	0.864***	0.002	0.572
	t-stat	1.139	15.882	0.694	
Chems	Coeff	0.000	0.981***	-0.002	0.674
	t-stat	-0.066	22.747	-0.458	
Txtls	Coeff	0.000	0.912***	-0.016***	0.513
	t-stat	0.159	15.962	-3.799	
Cnstr	Coeff	0.001	1.123***	-0.004	0.751
	t-stat	0.602	24.955	-1.305	
Steel	Coeff	-0.001	1.191***	-0.008**	0.599
	t-stat	-0.671	19.726	-2.005	
FabPr	Coeff	0.000	1.153***	-0.007**	0.775
	t-stat	-0.166	41.845	-2.445	
ElcEq	Coeff	0.002*	1.171***	-0.004	0.740
	t-stat	1.792	32.715	-1.230	
Autos	Coeff	-0.001	1.001***	-0.001	0.515
	t-stat	-0.558	16.900	-0.255	
Carry	Coeff	0.001	1.105***	-0.004	0.592
	t-stat	0.632	18.549	-0.992	
Mines	Coeff	0.001	0.839***	-0.011	0.297
	t-stat	0.440	10.481	-1.347	
Coal	Coeff	0.005	1.068***	-0.003	0.254
	t-stat	1.248	10.501	-0.330	
Oil	Coeff	0.003	0.785***	0.004	0.411
	t-stat	1.530	14.300	1.047	
Util	Coeff	0.001	0.540***	0.004	0.328
	t-stat	0.951	11.047	1.165	
Telcm	Coeff	0.001	0.760***	0.001	0.516
	t-stat	0.590	16.187	0.277	
Servs	Coeff	0.000	1.383***	-0.002	0.782
	t-stat	0.212	32.950	-0.583	
BusEq	Coeff	-0.002	1.325***	0.005	0.677
	t-stat	-0.771	21.153	1.208	
Paper	Coeff	0.001	0.916***	-0.001	0.671
	t-stat	0.899	21.806	-0.272	
Trans	Coeff	0.000	1.062***	-0.002	0.652
	t-stat	-0.269	22.124	-0.429	
Whlsl	Coeff	0.001	1.089***	-0.002	0.726
	t-stat	0.501	25.099	-0.483	
Rtail	Coeff	0.002	1.028***	-0.004	0.665
	t-stat	1.092	22.991	-1.040	
Meals	Coeff	0.003	1.111***	-0.005	0.569
	t-stat	1.244	15.363	-1.409	
Fin	Coeff	0.002*	1.009***	0.001	0.756
	t-stat	1.781	28.728	0.601	
Other	Coeff	-0.002	1.065***	-0.009***	0.707
	t-stat	-1.114	26.323	-3.086	

Panel B	α	β on $R_M - R_f$	β on $g_{Var(R_M)} - R_f$
Average of betas	0.001***	0.995***	-0.003***
t-stat	3.134	25.751	-3.528

Panel C	β on $R_M - R_f$	β on $g_{Var(R_M)} - R_f$
Number of Significant betas at 10%	30	8
χ^2 test (beta vector=0)	26913.072	48.074
p-values	(<0.001)	(0.004)

Panel A shows the multivariate regression results using the 30 industry portfolios as the dependent variables. The market return is value-weighted of NYSE/AMEX/NASDAQ equities. The estimated regression is $R_{k,t} - R_{f,t} = \alpha_{k,t} + \beta\left(R_{M,t} - R_{f,t}\right) + \beta_k^{\intercal}\left(g_{f_t} - R_f\right) + \varepsilon_{k,t}$, where $g_{f_t} = \left(f_t - f_{t-1}\right)/f_{t-1}$. For each industry portfolio, the first line shows the estimated beta and the second line shows the t-stat. The numbers with *** indicates significance at 1% level, ** indicates significance at 5 % level, and * indicates significance at 10% level. Panel B shows the cross-sectional average of betas and their t-stat. Panel C evaluates the significance of the beta for each factor. The test statistics is $\widehat{\beta}\Sigma_{\widehat{\beta}}^{-1}\widehat{\beta}$ and has 30 degrees of freedom. The p-values are in parentheses. All tests use Newey and West (1987) standard errors that are robust to heteroskedasticity and autocorrelation (3lags).

Table 3
Multi-beta regression for the 30 industry portfolios

		MS_V				
Panel A		α	β on $R_M - R_f$	β on $g_{R_M^2} - R_f$	β on $g_{Var(R_M)} - R_f$	Adjusted R^2
Food	Coeff	0.004**	0.661***	0.023	-0.007***	0.518
	t-stat	2.349	10.187	1.368	-2.628	
Beer	Coeff	0.003*	0.805***	-0.005	-0.005	0.431
	t-stat	1.652	9.424	-0.227	-1.328	
Smoke	Coeff	0.007**	0.609***	0.035	-0.001	0.223
	t-stat	2.388	5.652	1.095	-0.095	
Games	Coeff	0.001	1.320***	-0.050**	-0.008*	0.675
	t-stat	0.666	19.823	-2.422	-1.903	
Books	Coeff	0.001	1.100***	-0.053***	-0.007**	0.700
	t-stat	0.595	18.780	-2.978	-2.619	
Hshld	Coeff	0.000	0.880***	-0.014	0.001	0.610
	t-stat	0.202	13.586	-0.781	0.413	
Clths	Coeff	0.001	1.214***	-0.055**	-0.006	0.570
	t-stat	0.513	14.546	-2.355	-1.614	
Hlth	Coeff	0.002	0.815***	0.025	0.002	0.573
	t-stat	1.255	14.218	1.216	0.589	
Chems	Coeff	0.000	0.940***	0.021	-0.002	0.675
	t-stat	0.029	16.335	1.281	-0.543	
Txtls	Coeff	0.000	1.075***	-0.082***	-0.015***	0.528
	t-stat	-0.090	15.068	-3.498	-3.282	
Cnstr	Coeff	0.001	1.207***	-0.042**	-0.003	0.755
	t-stat	0.394	19.756	-2.462	-1.051	
Steel	Coeff	-0.001	1.198***	-0.004	-0.008*	0.598
	t-stat	-0.671	13.575	-0.133	-1.931	
FabPr	Coeff	0.000	1.189***	-0.018	-0.007**	0.775
	t-stat	-0.241	24.672	-0.956	-2.227	
ElcEq	Coeff	0.002*	1.185***	-0.007	-0.004	0.739
	t-stat	1.737	26.619	-0.401	-1.175	
Autos	Coeff	-0.002	1.143***	-0.072***	0.000	0.526
	t-stat	-0.784	17.360	-2.641	-0.031	
Carry	Coeff	0.001	1.215***	-0.056***	-0.003	0.597
	t-stat	0.448	18.529	-2.820	-0.778	
Mines	Coeff	0.001	0.868***	-0.015	-0.011	0.296
	t-stat	0.397	7.003	-0.298	-1.289	
Coal	Coeff	0.006	0.884***	0.093*	-0.004	0.261
	t-stat	1.372	5.665	1.846	-0.470	
Oil	Coeff	0.003*	0.703***	0.042*	0.003	0.415
	t-stat	1.665	8.989	1.888	0.871	
Util	Coeff	0.002	0.471***	0.035**	0.004	0.333
	t-stat	1.102	7.560	2.007	0.987	
Telcm	Coeff	0.001	0.725***	0.018	0.001	0.516
	t-stat	0.658	11.513	0.982	0.196	
Servs	Coeff	0.000	1.374***	0.004	-0.002	0.781
	t-stat	0.229	24.902	0.242	-0.607	
BusEq	Coeff	-0.002	1.366***	-0.021	0.005	0.677
	t-stat	-0.835	18.786	-1.008	1.277	
Paper	Coeff	0.001	0.892***	0.012	-0.001	0.670
	t-stat	0.942	16.523	0.781	-0.317	
Trans	Coeff	-0.001	1.084***	-0.011	-0.002	0.651
	t-stat	-0.314	18.662	-0.649	-0.385	
Whlsl	Coeff	0.001	1.125***	-0.018	-0.002	0.726
	t-stat	0.408	20.641	-1.019	-0.404	
Rtail	Coeff	0.001	1.086***	-0.030*	-0.003	0.667
	t-stat	0.953	20.469	-1.799	-0.894	
Meals	Coeff	0.002	1.215***	-0.053***	-0.004	0.573
	t-stat	1.100	13.937	-2.595	-1.219	
Fin	Coeff	0.002*	1.017***	-0.005	0.001	0.755
	t-stat	1.744	21.099	-0.317	0.634	
Other	Coeff	-0.002	1.108***	-0.022	-0.009***	0.708
	t-stat	-1.220	21.105	-1.465	-3.001	

Panel B	α	β on $R_M - R_f$	β on $g_{R_M^2} - R_f$	β on $g_{Var(R_M)} - R_f$
Average of betas	0.001	1.016***	-0.011	-0.003***
t-stat	2.778	20.131	-1.349	-3.628

Panel C	β on $R_M - R_f$	β on $g_{R_M^2} - R_f$	β on $g_{Var(R_M)} - R_f$
Number of Significant betas at 10%	30	12	7
χ^2 test (beta vector=0)	15009.062	86.593	46.024
p-values	(<0.001)	(<0.001)	(0.007)

Panel A shows the multivariate regression results using the 30 industry portfolios as the dependent variables. The market return is value-weighted of NYSE/AMEX/NASDAQ equities. The estimated regression is $R_{k,t} - R_{f,t} = \alpha_{k,t} + \beta\left(R_{M,t} - R_{f,t}\right) + \beta_k^{\intercal}\left(g_{f_t} - R_f\right) + \varepsilon_{k,t}$, where $g_{f_t} = \left(f_t - f_{t-1}\right)/f_{t-1}$. For each industry portfolio, the first line shows the estimated beta and the second line shows the t-stat. The numbers with *** indicates significance at 1% level, ** indicates significance at 5 % level, and * indicates significance at 10% level. Panel B shows the cross-sectional average of betas and their t-stat. Panel C evaluates the significance of the beta for each factor. The test statistics is $\widehat{\beta}\Sigma_{\widehat{\beta}}^{-1}\widehat{\beta}$ and has 30 degrees of freedom. The p-values are in parentheses. All tests use Newey and West (1987) standard errors that are robust to heteroskedasticity and autocorrelation (3lags).

Table 4
Multi-beta regression for the 30 industry portfolios

Panel A		α	β on $R_M - R_f$	β on $g_{R_M^2} - R_f$	β on $g_{R_M^3} - R_f$	β on $g_{Var(R_M)} - R_f$	$\beta \times 10^2$ on $g_{Skew_1} - R_f$	β on $g_{r_M.VarR_M} - R_f$	$\beta \times 10^2$ on $g_{R_M^{(\rho)}} - R_f$	β on $g_{r_M^{\bar{\varsigma}_0}.r_M} - R_f$	β on $g_{r_M^{\bar{\varsigma}_{0,\vartheta}}.r_M} - R_f$	Adjusted R^2
						MSC_VS						
Food	Coeff	0.003*	0.669***	-0.010	0.245**	-0.034	0.010****	0.029	0.021	-0.291***	-0.019***	0.556
	t-stat	1.642	9.730	-0.051	2.013	-1.274	11.939	1.049	1.463	-5.683	-3.320	
Beer	Coeff	0.006**	0.790***	0.610*	-0.179	0.015	0.013***	-0.018	0.010	-0.257***	-0.028***	0.451
	t-stat	2.427	9.014	1.950	-0.910	0.507	10.410	-0.581	0.518	-4.418	-3.953	
Smoke	Coeff	0.003	0.620***	-0.587*	0.427**	-0.098***	0.005***	0.098***	0.020	-0.080	0.022*	0.237
	t-stat	1.051	5.782	-1.648	2.182	-2.795	3.054	2.652	1.139	-0.823	1.700	
Games	Coeff	0.001	1.323***	-0.280	-0.053	-0.001	-0.001	-0.007	0.000	0.256***	0.000	0.685
	t-stat	0.295	20.204	-0.792	-0.239	-0.044	-0.613	-0.201	0.001	4.199	0.053	
Books	Coeff	0.002	1.089***	0.188	-0.153	-0.006	0.006***	0.001	0.007	0.002	-0.011	0.701
	t-stat	0.965	16.519	0.506	-0.615	-0.225	6.690	0.025	0.693	0.030	-1.520	
Hshld	Coeff	0.002	0.857***	0.392	-0.117	-0.039*	0.003***	0.044**	0.019**	-0.194***	-0.019***	0.623
	t-stat	1.121	12.499	1.341	-0.641	-1.829	3.269	2.018	2.449	-3.159	-3.049	
Clths	Coeff	0.002	1.188***	0.055	-0.165	-0.064**	0.006***	0.063*	0.015	0.098	-0.009	0.574
	t-stat	0.782	13.355	0.120	-0.574	-2.041	4.311	1.911	0.984	1.271	-1.038	
Hlth	Coeff	-0.002	0.845***	-0.640**	0.432**	-0.029	0.010***	0.029	0.013	-0.024	0.016**	0.591
	t-stat	-0.876	14.416	-1.980	2.057	-1.438	8.108	1.342	0.682	-0.397	2.037	
Chems	Coeff	0.000	0.920***	-0.012	-0.021	-0.067***	0.001***	0.070***	0.007	0.028	-0.010	0.678
	t-stat	0.263	15.969	-0.049	-0.161	-2.868	0.624	2.809	1.070	0.418	-1.327	
Txtls	Coeff	0.002	1.041***	0.166	-0.003	-0.074**	0.013***	0.066***	-0.013	-0.212***	-0.046***	0.556
	t-stat	0.516	13.768	0.354	-0.010	-2.466	8.087	2.037	-0.788	-3.227	-5.351	
Cnstr	Coeff	0.000	1.198***	-0.310	0.046	-0.041	0.003***	0.041	-0.004	0.146***	-0.017***	0.770
	t-stat	0.045	20.664	-1.177	0.299	-1.477	3.349	1.419	-0.276	3.062	-2.862	
Steel	Coeff	0.001	1.183***	0.219	-0.213	0.021	-0.007***	-0.026	-0.012	0.105	-0.035***	0.620
	t-stat	0.193	14.077	0.786	-1.287	0.640	-4.092	-0.728	-0.694	1.169	-3.211	
FabPr	Coeff	-0.002	1.200***	-0.459*	0.139	-0.008	-0.010***	0.002	-0.005	0.180***	-0.004	0.788
	t-stat	-1.155	27.646	-1.865	0.942	-0.384	-9.984	0.104	-0.971	3.302	-0.640	
ElcEq	Coeff	0.004**	1.164***	0.038	-0.155	-0.043	-0.006***	0.045***	0.011*	0.145***	-0.021***	0.760
	t-stat	2.013	25.651	0.181	-1.285	-2.411	-4.240	2.244	1.850	2.607	-3.537	
Autos	Coeff	0.004	1.087***	0.833**	-0.500*	-0.059*	0.000	0.067*	0.028***	-0.120*	-0.024**	0.542
	t-stat	1.217	15.526	1.984	-1.846	-1.745	0.220	1.799	2.848	-1.722	-2.399	
Carry	Coeff	0.001	1.187***	-0.196	0.013	-0.092**	0.016***	0.095**	0.003	0.067	-0.025**	0.616
	t-stat	0.484	17.202	-0.698	0.075	-2.340	7.717	2.316	0.242	1.005	-2.435	
Mines	Coeff	-0.005	0.883***	-1.368**	0.901**	-0.133***	-0.006**	0.125**	-0.051*	-0.096	-0.011	0.317
	t-stat	-1.269	8.049	-2.129	2.340	-2.788	-2.330	2.495	-1.937	-0.894	-0.874	
Coal	Coeff	-0.003	0.945***	-1.633*	0.804	-0.062	-0.010***	0.051	-0.008	0.344*	0.066***	0.286
	t-stat	-0.628	6.051	-1.931	1.574	-0.819	-3.495	0.662	-0.304	1.926	2.706	
Oil	Coeff	0.002	0.705***	-0.131	0.208	-0.025	0.010***	0.026	-0.020	-0.136**	0.019**	0.439
	t-stat	0.724	9.541	-0.525	1.500	-0.847	8.432	0.818	-1.495	-2.084	2.261	
Util	Coeff	-0.001	0.499***	-0.184	0.357**	0.023	0.013***	-0.025	-0.016**	-0.269***	0.024***	0.452
	t-stat	-0.391	8.411	-0.697	2.095	1.163	11.972	-1.206	-2.334	-5.014	2.950	

Table continues on next page...

						MSC_VS					
Panel A		α	β on $R_M - R_f$	β on $g_{R_M^2} - R_f$	β on $g_{R_M^3} - R_f$	β on $g_{Var(R_M)} - R_f$	$\beta \times 10^2$ on $g_{Skew_1} - R_f$	β on $g_{r_M \cdot VarR_M} - R_f$	$\beta \times 10^2$ on $g_{R_M^{(\rho)}} - R_f$	β on $g_{r_M^{\bar{\varsigma}_0} \cdot r_M} - R_f$	β on $g_{r_M^{\bar{\varsigma}_{0,\vartheta}} \cdot r_M} - R_f$	Adjusted R^2
Telcm	Coeff	0.002	0.718***	0.245	0.023	-0.007	-0.002	0.009	-0.003	-0.215***	-0.004	0.529
	t-stat	0.874	11.339	0.863	0.137	-0.256	-1.514	0.298	-0.227	-3.389	-0.590	
Servs	Coeff	-0.003	1.408***	-0.733***	0.137	0.045**	0.000	-0.050***	0.002	0.442***	0.004	0.820
	t-stat	-1.291	25.290	-2.641	0.797	2.548	-0.255	-2.624	0.158	7.058	0.486	
BusEq	Coeff	-0.002	1.379***	-0.150	-0.196	0.069*	-0.013***	-0.065*	0.008	0.378***	-0.004	0.713
	t-stat	-0.737	19.374	-0.524	-1.199	1.954	-7.891	-1.812	0.579	5.399	-0.527	
Paper	Coeff	0.001	0.883***	-0.143	0.115	-0.074***	-0.002*	0.077***	0.026**	-0.049	-0.008	0.676
	t-stat	0.543	16.035	-0.672	0.922	-3.241	-1.906	3.218	2.434	-0.875	-1.141	
Trans	Coeff	-0.001	1.072***	-0.183	0.004	-0.053*	0.005***	0.053	0.001	0.114*	0.004	0.652
	t-stat	-0.461	17.658	-0.748	0.025	-1.658	3.257	1.540	0.071	1.837	0.585	
Whlsl	Coeff	-0.003**	1.143***	-0.863***	0.401***	-0.067***	0.011***	0.066***	0.014	0.150**	0.001	0.742
	t-stat	-2.090	22.058	-4.222	3.516	-2.873	9.468	2.669	0.897	2.278	0.117	
Rtail	Coeff	0.003*	1.058***	0.288	-0.255**	-0.044*	0.011***	0.043*	0.011	0.052	-0.001	0.673
	t-stat	1.815	18.947	1.289	-1.898	-1.894	13.444	1.772	1.064	0.959	-0.192	
Meals	Coeff	0.003	1.212***	-0.106	0.090	-0.010	0.010***	0.009	-0.005	-0.067	-0.032***	0.582
	t-stat	1.030	13.547	-0.379	0.555	-0.290	5.172	0.243	-0.188	-1.110	-4.710	
Fin	Coeff	0.001	1.024***	-0.242	0.082	0.000	0.009***	0.000	-0.009	0.088*	0.007	0.759
	t-stat	0.683	21.053	-0.991	0.562	0.009	10.091	-0.004	-1.465	1.752	1.251	
Other	Coeff	-0.004**	1.129***	-0.502**	0.356***	-0.011	0.005***	0.003	-0.001	-0.058	-0.029***	0.723
	t-stat	-2.336	21.745	-2.377	2.815	-0.624	5.266	0.172	-0.106	-0.901	-4.717	
Panel B		α	β on $R_M - R_f$	β on $g_{R_M^2} - R_f$	β on $g_{R_M^3} - R_f$	β on $g_{Var(R_M)} - R_f$	$\beta \times 10^2$ on $g_{Skew_1} - R_f$	β on $g_{r_M \cdot VarR_M} - R_f$	$\beta \times 10^2$ on $g_{R_M^{(\rho)}} - R_f$	β on $g_{r_M^{\bar{\varsigma}_0} \cdot r_M} - R_f$	β on $g_{r_M^{\bar{\varsigma}_{0,\vartheta}} \cdot r_M} - R_f$	
Average of betas		0.001	1.014***	-0.190**	0.092	-0.032***	0.004**	0.031***	0.002	0.018	-0.006	
t-stat		1.122	21.237	-2.242	1.578	-3.388	2.228	3.204	0.629	0.583	-1.388	
Panel C			β on $R_M - R_f$	β on $g_{R_M^2} - R_f$	β on $g_{R_M^3} - R_f$	β on $g_{Var(R_M)} - R_f$	β on $g_{Skew_1} - R_f$	β on $g_{r_M \cdot VarR_M} - R_f$	β on $g_{R_M^{(\rho)}} - R_f$	β on $g_{r_M^{\bar{\varsigma}_0} \cdot r_M} - R_f$	β on $g_{r_M^{\bar{\varsigma}_{0,\vartheta}} \cdot r_M} - R_f$	
Number of Significant betas at 10%			30	10	9	14	26	14	6	18	16	
χ^2 test (beta vector=0)			16697.190	135.731	96.917	48.038	73.384	48.104	30.332	494.579	350.718	
p-values			(<0.001)	(<0.001)	(<0.001)	(0.005)	(<0.001)	(0.004)	(0.051)	(<0.001)	(<0.001)	

Panel A shows the multivariate regression results using the 30 industry portfolios as the dependent variables. The market return is value-weighted of NYSE/AMEX/NASDAQ equities. The estimated regression is $R_{k,t} - R_{f,t} = \alpha_{k,t} + \beta\left(R_{M,t} - R_{f,t}\right) + \beta_k^{\intercal}\left(g_{f_t} - R_f\right) + \varepsilon_{k,t}$, where $g_{f_t} = \left(f_t - f_{t-1}\right)/f_{t-1}$. For each industry portfolio, the first line shows the estimated beta and the second line shows the t-stat. The numbers with *** indicates significance at 1% level, ** indicates significance at 5 % level, and * indicates significance at 10% level. Panel B shows the cross-sectional average of betas and their t-stat. Panel C evaluates the significance of the beta for each factor. The test statistics is $\widehat{\beta}\Sigma_{\widehat{\beta}}^{-1}\widehat{\beta}$ and has 30 degrees of freedom. The p-values are in parentheses. All tests use Newey and West (1987) standard errors that are robust to heteroskedasticity and autocorrelation (3lags).

Table 5

Multi-beta regression for the 30 industry portfolios

Reduced-MSC_VS(1)									
Panel A		α	β on $R_M - R_f$	β on $g_{R_M^2} - R_f$	β on $g_{R_M^3} - R_f$	β on $g_{Var(R_M)} - R_f$	$\beta \times 10^2$ on $g_{Skew_1} - R_f$	β on $g_{r_M \cdot VarR_M} - R_f$	Adjusted R^2
Food	Coeff	0.002	0.670***	-0.344	0.226	-0.036	0.011***	0.029	0.526
	t-stat	0.918	9.427	-1.447	1.507	-1.252	13.041	0.995	
Beer	Coeff	0.005*	0.796***	0.324	-0.201	0.017	0.014***	-0.024	0.436
	t-stat	1.882	9.035	1.004	-0.981	0.550	11.584	-0.694	
Smoke	Coeff	0.003	0.610***	-0.699**	0.435**	-0.108***	0.006***	0.111***	0.229
	t-stat	1.054	5.643	-1.998	2.017	-3.117	3.377	3.035	
Games	Coeff	0.002	1.321***	0.028	-0.046	0.006	-0.002*	-0.014	0.673
	t-stat	0.654	19.521	0.072	-0.185	0.159	-1.702	-0.375	
Books	Coeff	0.002	1.089***	0.200	-0.159	-0.002	0.006***	-0.004	0.701
	t-stat	0.880	16.628	0.504	-0.624	-0.085	7.338	-0.159	
Hshld	Coeff	0.002	0.858***	0.176	-0.133	-0.038*	0.004***	0.041*	0.611
	t-stat	0.651	12.403	0.544	-0.646	-1.653	4.306	1.744	
Clths	Coeff	0.003	1.185***	0.183	-0.168	-0.058*	0.006***	0.056*	0.571
	t-stat	0.827	13.125	0.375	-0.545	-1.883	3.947	1.713	
Hlth	Coeff	-0.002	0.837***	-0.681*	0.440**	-0.036*	0.010***	0.037*	0.585
	t-stat	-0.760	13.879	-1.927	1.956	-1.725	8.744	1.715	
Chems	Coeff	0.000	0.920***	0.032	-0.026	-0.063***	0.000	0.064***	0.677
	t-stat	0.261	15.867	0.150	-0.200	-2.716	0.345	2.610	
Txtls	Coeff	0.000	1.057***	-0.054	-0.032	-0.064**	0.013***	0.051	0.532
	t-stat	0.049	13.875	-0.119	-0.113	-2.032	8.174	1.504	
Cnstr	Coeff	0.000	1.202***	-0.120	0.041	-0.031	0.002**	0.028	0.754
	t-stat	0.184	19.402	-0.458	0.247	-1.135	2.026	1.004	
Steel	Coeff	0.000	1.194***	0.374	-0.228	0.037	-0.008***	-0.045	0.599
	t-stat	0.127	13.325	1.217	-1.157	1.126	-4.596	-1.326	
FabPr	Coeff	-0.002	1.201***	-0.240	0.142	-0.002	-0.011***	-0.005	0.778
	t-stat	-0.881	25.231	-1.129	1.053	-0.085	-10.815	-0.205	
ElcEq	Coeff	0.004**	1.164***	0.231	-0.162	-0.032*	-0.008***	0.031	0.742
	t-stat	2.278	24.476	1.301	-1.460	-1.747	-4.473	1.538	
Autos	Coeff	0.003	1.086***	0.712*	-0.517**	-0.054	0.001	0.059	0.536
	t-stat	0.991	15.425	1.709	-1.955	-1.616	0.326	1.618	
Carry	Coeff	0.001	1.191***	-0.094	0.001	-0.081**	0.015***	0.081**	0.605
	t-stat	0.432	17.096	-0.310	0.007	-2.173	7.185	2.098	
Mines	Coeff	-0.006	0.901***	-1.480**	0.896**	-0.131***	-0.006**	0.123**	0.313
	t-stat	-1.455	8.160	-2.503*	2.340	-2.799	-2.364	2.450	
Coal	Coeff	-0.001	0.928***	-1.274	0.848*	-0.076	-0.010***	0.072	0.267
	t-stat	-0.210	5.722	-1.663	1.720	-0.994	-3.669	0.938	
Oil	Coeff	0.001	0.708***	-0.313	0.215	-0.035	0.011***	0.038	0.419
	t-stat	0.600	9.181	-1.285	1.391	-1.279	8.991	1.305	
Util	Coeff	-0.001	0.500***	-0.529**	0.363**	0.008	0.015***	-0.006	0.357
	t-stat	-0.760	8.274	-2.271	2.504	0.299	15.100	-0.229	
Telcm	Coeff	0.001	0.722***	-0.012	0.015	-0.012	-0.001	0.013	0.513
	t-stat	0.518	11.132	-0.053	0.105	-0.388	-0.789	0.431	
Servs	Coeff	-0.001	1.403***	-0.203	0.151	0.056***	-0.003**	-0.061***	0.783
	t-stat	-0.415	23.843	-0.521	0.601	2.677	-2.210	-2.737	
BusEq	Coeff	0.000	1.375***	0.309	-0.188	0.081***	-0.015***	-0.078**	0.683
	t-stat	-0.106	19.075	0.962	-0.932	2.063	-10.213	-1.968	
Paper	Coeff	0.001	0.878***	-0.193	0.107	-0.073***	-0.002*	0.075***	0.674
	t-stat	0.410	16.109	-0.966	0.861	-3.180	-1.787	3.107	
Trans	Coeff	0.000	1.070***	-0.049	0.009	-0.051	0.005***	0.052	0.652
	t-stat	-0.236	17.585	-0.228	0.067	-1.626	2.826	1.514	
Whlsl	Coeff	-0.003*	1.138***	-0.681***	0.405***	-0.063***	0.010***	0.062***	0.736
	t-stat	-1.664	21.588	-3.468	3.271	-2.833	8.809	2.662	
Rtail	Coeff	0.004*	1.055***	0.352	-0.255*	-0.042*	0.011***	0.041*	0.673
	t-stat	1.919	18.903	1.605	-1.821	-1.869	14.018	1.750	
Meals	Coeff	0.002	1.222***	-0.161	0.071	-0.001	0.010***	-0.004	0.574
	t-stat	0.763	13.578	-0.597	0.431	-0.028	5.247	-0.107	
Fin	Coeff	0.001	1.024***	-0.143	0.089	0.000	0.009***	0.001	0.758
	t-stat	0.949	21.050	-0.664	0.649	0.004	10.456	0.028	
Other	Coeff	-0.005***	1.137***	-0.549***	0.339***	-0.002	0.005***	-0.009	0.712
	t-stat	-2.777	21.610	-2.689	2.682	-0.116	5.248	-0.474	

Panel B	α	β on $R_M - R_f$	β on $g_{R_M^2} - R_f$	β on $g_{R_M^3} - R_f$	β on $g_{Var(R_M)} - R_f$	$\beta \times 10^2$ on $g_{Skew_1} - R_f$	β on $g_{r_M \cdot VarR_M} - R_f$
Average of betas	0.001	1.015***	-0.163*	0.089	-0.030***	0.003**	0.027***
t-stat	1.233	21.125	-1.798	1.482	-3.035	1.964	2.763

Panel C	β on $R_M - R_f$	β on $g_{R_M^2} - R_f$	β on $g_{R_M^3} - R_f$	β on $g_{Var(R_M)} - R_f$	β on $g_{Skew_1} - R_f$	β on $g_{r_M \cdot VarR_M} - R_f$
Number of Significant betas at 10%	30	8	9	14	27	12
χ^2 test (beta vector=0)	14614.786	87.308	93.370	49.852	74.583	50.941
p-values	(<0.001)	(<0.001)	(<0.001)	(0.003)	(<0.001)	(0.002)

Panel A shows the multivariate regression results using the 30 industry portfolios as the dependent variables. The market return is value-weighted of NYSE/AMEX/NASDAQ equities. The estimated regression is $R_{k,t} - R_{f,t} = \alpha_{k,t} + \beta\left(R_{M,t} - R_{f,t}\right) + \beta_k^{\intercal}\left(g_{f_t} - R_f\right) + \varepsilon_{k,t}$, where $g_{f_t} = \left(f_t - f_{t-1}\right)/f_{t-1}$. For each industry portfolio, the first line shows the estimated beta and the second line shows the t-stat. The numbers with *** indicates significance at 1% level, ** indicates significance at 5 % level, and * indicates significance at 10% level. Panel B shows the cross-sectional average of betas and their t-stat. Panel C evaluates the significance of the beta for each factor. The test statistics is $\widehat{\beta}\Sigma_{\widehat{\beta}}^{-1}\widehat{\beta}$ and has 30 degrees of freedom. The p-values are in parentheses. All tests use Newey and West (1987) standard errors that are robust to heteroskedasticity and autocorrelation (3lags).

Table 6
Multi-Beta Regression for the 30 industry portfolios

Reduced-MSC_VS(2)									
Panel A		α	β on $R_M - R_f$	β on $g_{r_M^{\bar{\varsigma}0}} - R_f$	β on $g_{r_M^{\bar{\varsigma}0}\cdot r_M} - R_f$	β on $g_{r_M^{\bar{\varsigma}0,\vartheta}} - R_f$	$\beta \times 10^2$ on $g_{R_M(\rho)} - R_f$	β on $g_{r_M^{\bar{\varsigma}0,\vartheta}\cdot r_M} - R_f$	Adjusted R^2
Food	Coeff	0.001	0.713***	-0.644***	0.339***	-0.111	0.023	0.091	0.550
	t-stat	0.614	11.190	-4.920	3.468	-1.307	1.543	1.067	0.000
Beer	Coeff	0.002	0.839***	-0.412**	0.195	-0.086	0.011	0.060	0.442
	t-stat	0.936	10.029	-2.192	1.336	-0.706	0.513	0.484	0.000
Smoke	Coeff	0.005	0.639***	-0.368	0.224	-0.026	0.022	0.045	0.236
	t-stat	1.529	6.176	-1.588	1.298	-0.166	1.256	0.289	0.000
Games	Coeff	0.002	1.340***	0.427***	-0.201	0.205	-0.001	-0.210	0.685
	t-stat	1.002	21.258	2.576	-1.591	1.612	-0.095	-1.634	0.000
Books	Coeff	0.001	1.129***	0.007	-0.012	0.088	0.006	-0.101	0.700
	t-stat	0.296	19.553	0.041	-0.111	1.033	0.615	-1.193	0.000
Hshld	Coeff	0.000	0.881***	-0.209	0.059	-0.124*	0.020**	0.111	0.619
	t-stat	0.159	13.788	-1.524	0.629	-1.646	2.415	1.464	0.000
Clths	Coeff	0.002	1.216***	0.279	-0.204	-0.008	0.015	-0.002	0.572
	t-stat	1.027	14.805	1.481	-1.546	-0.069	1.001	-0.020	0.000
Hlth	Coeff	0.000	0.826***	-0.250	0.165	0.010	0.014	0.003	0.583
	t-stat	0.205	15.049	-1.278	1.197	0.094	0.760	0.027	0.000
Chems	Coeff	0.001	0.933***	0.114	-0.080	-0.088	0.007	0.079	0.677
	t-stat	0.589	17.008	0.831	-0.891	-1.253	1.123	1.118	0.000
Txtls	Coeff	-0.002	1.129***	-0.385**	0.162	0.016	-0.012	-0.064	0.546
	t-stat	-0.778	17.138	-2.091	1.288	0.166	-0.756	-0.646	0.000
Cnstr	Coeff	0.002	1.198***	0.243**	-0.131	0.091	-0.004	-0.110	0.768
	t-stat	1.148	22.579	2.323	-1.586	1.105	-0.292	-1.344	0.000
Steel	Coeff	-0.001	1.207***	0.197	-0.067	0.109	-0.013	-0.145	0.621
	t-stat	-0.308	15.456	1.021	-0.544	1.215	-0.797	-1.607	0.000
FabPr	Coeff	0.000	1.204***	0.220	-0.075	0.157*	-0.006	-0.166**	0.783
	t-stat	-0.148	31.011	1.495	-0.722	1.909	-1.164	-2.007	0.000
ElcEq	Coeff	0.004***	1.177***	0.362***	-0.197***	0.025	0.010	-0.044	0.759
	t-stat	2.924	27.977	3.256	-2.659	0.355	1.781	-0.643	0.000
Autos	Coeff	-0.001	1.150***	0.043	-0.080	0.010	0.027**	-0.026	0.531
	t-stat	-0.395	18.887	0.190	-0.555	0.100	2.440	-0.271	0.000
Carry	Coeff	0.002	1.209***	0.134	-0.099	0.041	0.004	-0.067	0.606
	t-stat	0.884	19.792	0.771	-0.835	0.407	0.271	-0.656	0.000
Mines	Coeff	-0.001	0.902***	-0.476*	0.269	0.015	-0.050*	-0.035	0.302
	t-stat	-0.455	9.008	-1.765	1.335	0.111	-1.846	-0.256	0.000
Coal	Coeff	0.003	0.930***	0.102	0.115	0.265	0.008	-0.211	0.277
	t-stat	0.746	6.349	0.249	0.379	0.770	-0.351	-0.640	0.000
Oil	Coeff	0.001	0.709***	-0.378**	0.216**	-0.074	-0.019	0.092	0.439
	t-stat	0.667	9.696	-2.533	1.977	-0.765	-1.419	0.949	0.000
Util	Coeff	-0.003*	0.517***	-0.845***	0.538***	0.095	-0.015**	-0.074	0.454
	t-stat	-1.888	9.825	-6.302	5.500	1.049	-2.193	-0.821	0.000
Telcm	Coeff	0.000	0.738***	-0.371***	0.186**	-0.093	-0.003	0.092	0.529
	t-stat	-0.151	13.149	-3.030	2.334	-1.423	-0.191	1.417	0.000
Servs	Coeff	0.002	1.355***	0.611***	-0.246**	0.231***	0.001	-0.235***	0.812
	t-stat	1.244	25.125	4.493	-2.535	3.104	0.054	-3.199	0.000
BusEq	Coeff	0.002	1.315***	0.732***	-0.361***	0.165	0.006	-0.170	0.707
	t-stat	0.910	20.019	4.323	-2.793	1.444	0.454	-1.491	0.000
Paper	Coeff	0.000	0.914***	-0.160	0.117	0.043	0.026**	-0.050	0.673
	t-stat	0.262	17.904	-1.224	1.300	0.639	2.378	-0.742	0.000
Trans	Coeff	0.001	1.074***	0.271**	-0.171**	-0.024	0.001	0.028	0.652
	t-stat	0.364	19.295	2.040	-2.010	-0.326	0.079	0.391	0.000
Whlsl	Coeff	0.000	1.143***	0.007	0.066	0.210*	0.014	-0.215*	0.732
	t-stat	-0.160	22.537	0.045	0.578	1.771	0.913	-1.830	0.000
Rtail	Coeff	0.003*	1.079***	0.272*	-0.216*	-0.097	0.011	0.097	0.669
	t-stat	1.826	19.953	1.885	-1.945	-0.859	1.102	0.855	0.000
Meals	Coeff	0.002	1.221***	-0.141	0.051	0.022	-0.005	-0.055	0.582
	t-stat	1.021	14.426	-0.977	0.463	0.186	-0.172	-0.463	0.000
Fin	Coeff	0.003**	1.005***	0.096	-0.044	0.031	-0.009	-0.026	0.755
	t-stat	2.039	21.708	0.812	-0.532	0.465	-1.420	-0.387	0.000
Other	Coeff	-0.004**	1.148***	-0.308**	0.205**	0.111*	-0.001	-0.145**	0.720
	t-stat	-2.270	24.307	-1.965	2.062	1.792	-0.102	-2.359	0.000

Panel B	α	β on $R_M - R_f$	β on $g_{r_M^{\bar{\varsigma}0}} - R_f$	β on $g_{r_M^{\bar{\varsigma}0}\cdot r_M} - R_f$	β on $g_{r_M^{\bar{\varsigma}0,\vartheta}} - R_f$	$\beta \times 10^2$ on $g_{R_M(\rho)} - R_f$	β on $g_{r_M^{\bar{\varsigma}0,\vartheta}\cdot r_M} - R_f$
Average of betas	0.001***	1.028***	-0.028	0.024	0.040***	0.002	-0.048***
t-stat	3.477	21.770	-0.393	0.581	2.534	0.661	-3.035

Panel C	β on $R_M - R_f$	β on $g_{r_M^{\bar{\varsigma}0}} - R_f$	β on $g_{r_M^{\bar{\varsigma}0}\cdot r_M} - R_f$	β on $g_{r_M^{\bar{\varsigma}0,\vartheta}} - R_f$	β on $g_{R_M(\rho)} - R_f$	β on $g_{r_M^{\bar{\varsigma}0,\vartheta}\cdot r_M} - R_f$
Number of Significant betas at 5%	30	13	10	5	5	4
χ^2 test (beta vector=0)	17384.162	279.926	178.793	65.137	29.493	64.060
p-values	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(0.052)	(<0.001)

Panel A shows the multivariate regression results using the 30 industry portfolios as the dependent variables. The market return is value-weighted of NYSE/AMEX/NASDAQ equities. The estimated regression is $R_{k,t} - R_{f,t} = \alpha_{k,t} + \beta\left(R_{M,t} - R_{f,t}\right) + \beta_k^{\intercal}\left(g_{f_t} - R_f\right) + \varepsilon_{k,t}$, where $g_{f_t} = \left(f_t - f_{t-1}\right)/f_{t-1}$. For each industry portfolio, the first line shows the estimated beta and the second line shows the t-stat. The numbers with *** indicates significance at 1% level, ** indicates significance at 5 % level, and * indicates significance at 10% level. Panel B shows the cross-sectional average of betas and their t-stat. Panel C evaluates the significance of the beta for each factor. The test statistics is $\widehat{\beta}\Sigma_{\widehat{\beta}}^{-1}\widehat{\beta}$ and has 30 degrees of freedom. The p-values are in parentheses. All tests use Newey and West (1987) standard errors that are robust to heteroskedasticity and autocorrelation (3lags).

Table 7

Market prices of beta risks using the 25 Fama and French portfolios

Pane A

M_V

		$\lambda(R_M)$	$\lambda(Var(R_M))$
1996-2006	Coeff	0.007*	-0.326***
	se	0.004	0.108
1993-2006	Coeff	0.008***	-0.407***
	se	0.003	0.112
1990-2006	Coeff	0.007**	-0.291***
	se	0.003	0.110
1987-2006	Coeff	0.007**	-0.497***
	se	0.003	0.108
1984-2006	Coeff	0.006***	-0.439***
	se	0.002	0.089
1981-2006	Coeff	0.006***	-0.439***
	se	0.002	0.089
1965-2006	Coeff	0.005**	-0.127
	se	0.002	0.081

Panel B

MS_V

		$\lambda(R_M)$	$\lambda\left(R_M^2\right)$	$\lambda(Var(R_M)$
1996-2006	Coeff	0.007*	0.014*	-0.324***
	se	0.004	0.008	0.109
1993-2006	Coeff	0.008***	0.014**	-0.406***
	se	0.003	0.006	0.113
1990-2006	Coeff	0.007**	0.014**	-0.271***
	se	0.003	0.006	0.114
1987-2006	Coeff	0.007**	0.013**	-0.517***
	se	0.003	0.006	0.118
1984-2006	Coeff	0.006***	0.013***	-0.518***
	se	0.002	0.005	0.095
1981-2006	Coeff	0.006***	0.013***	-0.518***
	se	0.002	0.005	0.095
1965-2006	Coeff	0.005**	0.009**	-0.122
	se	0.002	0.004	0.085

Panel C

MSC_VS

		$\lambda(R_M)$	$\lambda\left(R_M^2\right)$	$\lambda\left(R_M^3\right)$	$\lambda(Var(R_M))$	$\lambda(Skew(R_M))$	$\lambda(R_M Var(R_M))$	$\lambda\left(R_M^{(\rho)}\right)$	$\lambda\left(R_M R_M^{\varsigma 0}\right)$	$\lambda\left(R_M R_M^{\varsigma 0,\vartheta}\right.$
1996-2006	Coeff	0.008**	0.014*	0.020*	-0.495***	-0.005	-0.495***	-0.152	0.005	0.054***
	se	0.004	0.008	0.012	0.125	0.118	0.126	0.139	0.010	0.019
1993-2006	Coeff	0.009***	0.015**	0.021**	-0.467***	0.111	-0.460***	0.025	0.006	0.046***
	se	0.003	0.006	0.010	0.140	0.127	0.141	0.142	0.008	0.016
1990-2006	Coeff	0.009***	0.015**	0.021**	-0.315***	-0.074	-0.303***	-0.124	0.007	0.045***
	se	0.003	0.006	0.009	0.151	0.146	0.152	0.158	0.008	0.014
1987-2006	Coeff	0.008***	0.013**	0.017*	-0.680***	-0.027	-0.631***	-0.100	0.005	0.048***
	se	0.003	0.006	0.009	0.182	0.160	0.176	0.178	0.007	0.013
1984-2006	Coeff	0.008***	0.013**	0.017*	-0.800***	-0.049	-0.753***	-0.127	0.005	0.045***
	se	0.003	0.005	0.008	0.156	0.169	0.151	0.185	0.007	0.012
1981-2006	Coeff	0.007***	0.012**	0.014*	-0.940***	-0.067	-0.880***	-0.063	0.002	0.049***
	se	0.002	0.005	0.008	0.161	0.156	0.156	0.185	0.006	0.011
1965-2006	Coeff	0.004**	0.011*	0.021	0.625	0.242*	0.603	0.170	0.008	0.008
	se	0.002	0.006	0.017	1.784	0.136	1.718	0.183	0.010	0.018

Panel D

Reduced-MSC_VS(1)

		$\lambda(R_M)$	$\lambda\left(R_M^2\right)$	$\lambda\left(R_M^3\right)$	$\lambda(Var(R_M))$	$\lambda(Skew(R_M))$	$\lambda(R_M Var(R_M))$
1996-2006	Coeff	0.007*	0.014*	0.020*	-0.351***	0.094	-0.353***
	se	0.004	0.008	0.012	0.111	0.111	0.112
1993-2006	Coeff	0.007**	0.014**	0.020**	-0.358***	0.203*	-0.360***
	se	0.003	0.006	0.010	0.122	0.123	0.123
1990-2006	Coeff	0.007**	0.014**	0.021**	-0.231***	0.134	-0.229*
	se	0.003	0.006	0.009	0.125	0.138	0.127
1987-2006	Coeff	0.007**	0.013**	0.019**	-0.382***	0.260*	-0.344***
	se	0.003	0.006	0.009	0.122	0.146	0.120
1984-2006	Coeff	0.007**	0.013**	0.020***	-0.501***	0.313**	-0.470***
	se	0.003	0.005	0.008	0.104	0.154	0.102
1981-2006	Coeff	0.006***	0.013**	0.019**	-0.460***	0.355**	-0.416***
	se	0.002	0.005	0.008	0.098	0.143	0.096
1965-2006	Coeff	0.005**	0.008**	0.012**	-0.123	0.298**	-0.122
	se	0.002	0.004	0.006	0.085	0.125	0.084

Panel A shows the market prices of beta risks using the M_V model using different sample period. The Hansen and Jagannathan (1997) distance measure is used to estimate the coefficients of the pricing kernels. The coefficients are then used to compute the market prices of beta risk using the expected return decomposition. For each period the first line shows the estimated market prices of beta risks and the second line shows the standard errors. The numbers with *** indicates significance at 1% level, ** indicates significance at 5 % level, and * indicates significance at 10% level. Panel B shows the market prices of beta risks using the MS_V model. Panel C shows the market prices of beta risks using the MSC_VS model. Panel D shows the market prices of beta risks using the Reduced-MSC_VS(1) model.