

Affine General Equilibrium Models

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ABSTRACT

No-arbitrage models are extremely flexible modelling tools, but often lack economic motivation. This paper describes an equilibrium consumption based CAPM framework based on Epstein-Zin preferences, which produces analytic pricing formulas for stocks and bonds under the assumption that macro growth rates follow affine processes. This allows the construction of equilibrium pricing formulas while maintaining the same flexibility of state dynamics as in no-arbitrage models. In demonstrating the approach, the paper presents a version of the Bansal & Yaron (2004) model which maintains a positive volatility process, as well as an example in which the volatility process is allowed to jump. The latter produces endogenous asset stock market crashes as stock prices drop to reflect a higher expected rate of return in response to increased risk. A third generalization is a model of nominal stock and bond prices. The nominal yield curve in this model has positive slope if expected inflation growth negatively impacts real growth. This model also produces asset prices that are consistent with observed data, including a substantial equity premium at moderate levels of risk aversion.

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I Introduction

Models of financial markets can typically use either equilibrium or no-arbitrage arguments to derive explicit pricing formulae. Equilibrium models often impose conditions that are too restrictive to be reconciled with historical data. Models based on no-arbitrage arguments, on the other hand, typically produce very flexible model dynamics which again implies better statistical fit to observed financial market data. Yet, many researchers would agree that the no-arbitrage models offer little insight into the behavior of economic agents because the specification of preferences is not explicit. Indeed, in some no-arbitrage models the specification of risk premia may be difficult to reconcile with equilibrium.

This paper outlines a general framework for deriving explicit asset pricing models in general equilibrium under a very general assumptions of dynamics of exogenous state-variables. Specifically, we assume that there are n 'affine', exogenous state-variables that drive consumption and dividend dynamics. These processes include Gaussian vector auto-regressions (VAR's), affine diffusion processes, and affine jump-diffusion processes. Affine asset pricing models comprise a large fraction of the most popular no-arbitrage models for pricing stocks, bonds and derivatives. A short list of affine no-arbitrage models include the Black and Scholes (1973), Heston (1993), Bates (1996), Bakshi, Cao, and Chen (1997), Vasicek (1977), Duffie and Kan (1996), Dai and Singleton (2000), as well as Duffie, Pan and Singleton (2000). The purpose of this general equilibrium approach is to offer a framework which flexibility rivals that of these no-arbitrage models for pricing stocks and bonds.

In computing equilibrium prices we assume that there exists a representative agent who has Epstein-Zin-Weil preferences over consumption and wealth. Recent work by Bansal and Yaron (2004), Bansal, Dittmar, and Lundblad (2004), and Hansen, Heaton, and Li (2005) show that such models are empirically successful in explaining a number of asset pricing "anomalies." This paper builds on their approach, and their model obtains as a special case of the general framework outlined here. We derive explicit formulae for the market values of securities that pay multi-period dividends, such as stocks and bonds. In the case of stocks, we derive a formula for a perpetual dividend paying stock. For bonds, our framework presents general equilibrium equivalents of the affine no-arbitrage term structure

models of Vasicek, Cox, Ingersoll and Ross, Duffie and Kan, among others. The affine diffusion framework allows us to consider square root diffusive dynamics for processes that reasonably should be positive such as nominal interest rates and market volatility. This poses an advantage over the models in Bansal and Yaron (2004) and Tauchen (2005), in which volatility is a conditionally normal process and thereby may become negative.

The Epstein-Zin framework specifies a pricing kernel in which the return on aggregate wealth enters directly. We follow the Bansal and Yaron analysis and interpret the aggregate wealth to mean the present value of the aggregate consumption stream. While the model is an endowment economy, one may interpret wealth to be the present value of future labor and dividend incomes.

Duffie and Epstein (1992) discuss continuous time equivalent of the recursive preferences in Epstein and Zin (1989). A recent application of this framework is provided in Benzoni, Collin-Dufresne, and Goldstein (2005). The example models presented here are based on the assumption that macro-variables evolve continuously in time. At the same time, we will assume that consumption and investment decisions take place in discrete time. There is no conflict between these assumptions. It is reasonable to interpret the consumption decision as being either one of immediate consumption at discrete (typically monthly) intervals, or one in which the consumption is taking place continuously, but its level is determined at discrete frequencies.

The direct appearance of returns to aggregate wealth in the pricing kernel complicates its computation. We follow the common technique for dealing with this which is to approximate the returns on the aggregate wealth using a linearization suggested in Campbell & Schiller (1987) and applied in the context of recursive Epstein-Zin preferences in Campbell (1993). In applying the Campbell-Schiller approximation we essentially obtain an approximation to the pricing kernel. Prices of all other assets will be computed explicitly, given the pricing kernel. Of course, any approximation used to compute the pricing kernel affects the prices of all assets. It is, however, difficult to see exactly how these types of errors in the pricing kernel differ from specification errors in the consumption process per se. In other words, even if we were to compute the pricing kernel exactly, any misspecification of the process

that drives consumption would lead to pricing errors. Campbell (1993) and Bansal and Yaron (2004) compute the solution to their model numerically and conclude that the log-linearization provides accurate results.

The approach in this paper is illustrated through three examples: First, we consider a version of the Bansal & Yaron (2004) model in which state-variables follow affine diffusions. In particular, this model restricts the stochastic volatility process which driving consumption and dividend processes to be positive. This is accomplished by having the volatility process follow a square root diffusion, as in Heston (1993). This model generates sample paths for stocks and bonds that mimic those of the original Bansal & Yaron model, but avoids the assumption that volatility is normally distributed.

Our next example extends the basic diffusive volatility model to allow for jumps in the volatility process. This extension has shown empirically successful in explaining stock returns, and in equity options data. We show that positive discontinuous jumps in the volatility leads to large negative discontinuous price jumps which mimic market crashes. This can be interpreted as an extended "leverage effect" - as risk increases through a jump in the volatility process, equity becomes less attractive and its price falls dramatically reflecting an increased expected rate of return on stocks. In general, the models considered in this paper produce a negative correlation between monthly stock returns and shocks to volatility. This is true both when the volatility is a diffusion and when it follows a jump diffusion process. In the latter case, jumps in volatility generate negative price shocks which can be made quite large - one example gives crashes that range from -7% to -20%. Thus, these events mimic market "crashes."

In a final example, we confront the issue of a downward sloping real term structure of interest rates in the previous two example models. In constructing a model with a positively sloping yield curve we explicitly introduce inflation, as well as an exogenous mean reverting factor which determines the expected inflation growth. We calibrate the parameters that govern the dynamics of this process to observed inflation to match the negative correlation between real growth rates and inflation in postwar US data. As a result, the model produces nominal stock returns with an even higher equity premium. In fact, risk aversion of $\gamma = 4$

produces an equity premium of 5%. Moreover, the nominal price model produces a flat ($\gamma = 4$), or positively sloping ($\gamma = 5, 6$) term structure of interest rates. The yield curve changes because shocks to expected inflation carry a positive risk premium. Since long term bonds have greater exposure to this additional risk factor, the premium is greater for long dated bonds, leading to an upward sloping nominal yield curve.

This paper also examine the critique of standard consumption based asset pricing models based on Euler equation errors (Lettau and Ludvigson (2005)). They show that "leading asset pricing models" cannot fully capture the failure of the standard consumption based power utility model in the US postwar data. We confirm their findings for the model where the stochastic volatility driving consumption and dividend growth follows a square root diffusion. When the volatility process departs significantly from conditional normality, which is the case in the jump models, the Euler equation errors produced by the model match those observed in the data. Thus, we conclude that the presence of non-gaussian innovations in our model explains the standard models' failure to generate non-zero Euler equation errors.

The remainder of the paper is organized as follows: The next section discusses the dynamics of exogenous state variables which are assumed to drive consumption and dividend growth rates and discusses the computation of equilibrium asset prices. Section 3 presents the results from the example models. Section 4 concludes.

II Equilibrium Prices

In the following we outline the general strategy for computing prices under the Epstein-Zin-Weil preference structure. We start by outlining the key assumption about the dynamics of state-variables which produce tractable pricing formulaes.

A Dynamics of Exogenous State Variables

There are n market wide state variables X_t that are assumed to follow a Markov process. The conditional distribution $p(X_{t+s} | X_t)$ is defined through the function $\phi : \mathbb{R}_+ \times \mathbb{C}^n \times \mathbb{R}^n \rightarrow \mathbb{C}^n$ given by

$$\phi(s, u, x) = E\left(\exp\left(uX_{t+s}\right) \mid X_t = x\right) \quad (1)$$

If u is complex and with $\text{Re}(u) = 0$, then ϕ is the conditional characteristic function of the transition density. If u is real, then ϕ is the conditional moment generating function. While the general form of ϕ is that of a laplace transform of $p(y | x)$, we simply refer to ϕ as the "generating function."

We assume here that generating function can be written explicitly as an exponential linear function of the state variable

$$\phi(s, u, X_t) = \exp\left(\alpha(s, u) + \beta(s, u)' X_t\right) \quad (2)$$

where α and β are known explicitly or can be computed from model parameters. There are two immediate examples that characterize the class of processes which have exponential-linear (or affine) generating functions as in (2). The first class are Gaussian Vector Autoregressions (VAR). In other words, if

$$X_t = \mu + AX_{t-1} + \Sigma\epsilon_t$$

the generating function is $\phi(1, u, X_t) = \exp(u'(\mu + AX_t) + \frac{1}{2}u'\Sigma u)$. Our model would coincide with that of Campbell (1993) if X_t follows a gaussian VAR.

An important class of processes that produce exponential affine generating functions is affine diffusion processes. These processes include Gaussian VAR's as special cases¹ An affine diffusion process is a process $X_t \in \mathcal{D}$ for some $\mathcal{D} \subseteq \mathbb{R}^n$ is described by

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t$$

where

$$\mu(x) = \mathcal{M} + \mathcal{K}x \tag{3}$$

$$\Sigma(x) = h + \sum_{k=1}^n x_k H_k \tag{4}$$

where $h, H_k \in \mathbb{R}^{n \times n}$ for $k = 1..n$. In this case, the conditional laplace transform is exponential affine, as in equation (2) with coefficients α and β that satisfy the complex Ricatti ODE's

$$\frac{\partial \beta(s, u)}{\partial s} = \mathcal{K}' \beta(s, u) + \frac{1}{2} \beta(s, u)' H \beta(s, u) \tag{5}$$

$$\frac{\partial \alpha(s, u)}{\partial s} = \mathcal{M}' \beta(s, u) + \frac{1}{2} \beta(s, u)' h \beta(s, u) \tag{6}$$

with boundary conditions $\beta(0, u) = u$, $\alpha(0, u) = 0$. The product $\beta(s, u)' H \beta(s, u)$ denotes the n dimensional vector with k 'th element $\beta(s, u)' H_k \beta(s, u)$.

The diffusion model can be extended to include jumps. Assume now that the dynamics are

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t + dZ_t$$

¹Discrete observations of affine diffusions with constant diffusion matrices generate discrete observations that follow Gaussian VAR's. On the other hand, it is possible to find Gaussian VAR's that are inconsistent with affine diffusion. One such example is an AR(1) process with negative autocorrelation.

with μ and Σ as above and where Z_t is a jump process with intensity $\lambda_0 + \lambda X_t$ and jump sizes ξ_t which distribution $p(\xi)$ has known generating function $\varrho(u) = Ee^{u\xi}$. Then the generating function is exponential affine as in (2) with coefficients that satisfy

$$\begin{aligned}\frac{\partial\beta(s,u)}{\partial s} &= \mathcal{K}'\beta(s,u) + \frac{1}{2}\beta(s,u)'H\beta(s,u) + \lambda_1(\varrho(\beta(s,u)) - 1), \\ \frac{\partial\alpha(s,u)}{\partial s} &= \mathcal{M}'\beta(s,u) + \frac{1}{2}\beta(s,u)'h\beta(s,u) + \lambda_0(\varrho(\beta(s,u)) - 1).\end{aligned}$$

Duffie, Pan, and Singleton (2000) provide a detailed discussion of exponential affine jump diffusion in the context of general no-arbitrage asset pricing models.

B Dividend and consumption processes

We assume that the consumption and dividend growth rates are linear functions of the state variables,

$$g_{c,t} = \gamma'_c X_t, \tag{7}$$

$$g_{d,t} = \gamma'_d X_t. \tag{8}$$

This implies that

$$C_{t+s} = C_t \exp\left(\sum_{u=t+1}^{t+s} \gamma'_c X_u\right), \tag{9}$$

$$D_{t+s} = D_t \exp\left(\sum_{u=t+1}^{t+s} \gamma'_d X_u\right). \tag{10}$$

The vectors γ_c and γ_d may contain zeros in such a way that the respective growth rates are driven by specific state variables, i.e. they may be selection vectors. In the examples considered here, however, these vectors typically contain parameters.

It is important that the consumption and dividend processes themselves are random walks, and hence that the corresponding growth rates are stationary. In some cases, as will be illustrated in the example models, it is convenient to model both consumption and dividends as random walks. In this case, we may transform the state-variables by taking

first differences of the non-stationary components. This produces a re-defined state-vector which is stationary. The generating function corresponding to the differenced variable can be computed from the original one in the following way. Let X_t denote the state variables in levels. Assume for the sake of illustration that the first component of X_t is integrated of order one (random walk), and that the remaining components are stationary. Define $\tilde{X}_{t+1} = (X_{1,t+1} - X_{1,t}, X_{2,t}, \dots, X_{n,t})$. By assumption $E_t \exp(u' X_{t+1}) = \exp(\alpha(u) + \beta(u)' X_t)$ so $E_t \exp(u' \tilde{X}_t) = \exp(\alpha(u) + \tilde{\beta}(u)' X_t)$ where $\tilde{\beta}(u) = \beta(u) - (u(1), 0, 0, \dots, 0)$. For the purpose of simplifying the exposition, the remainder of the paper will simply assume that X_t is stationary.

C Equilibrium

We assume that consumption and dividend payments take place at discrete integer times. The relevant Euler equation is

$$\begin{aligned} 1 &= E_t \left(\delta^\theta G_{t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} R_{i,t+1} \right) \\ &= E_t \exp \left(\theta \ln \delta - \frac{\theta}{\psi} g_{t+1} - (1-\theta) r_{a,t+1} + r_{i,t+1} \right) \end{aligned} \quad (11)$$

where G_t is the consumption growth rate $G_{t+1} = C_{t+1}/C_t$, $R_{a,t+1}$ is the return on an asset that pays dividend equal to aggregate consumption, and $R_{i,t+1}$ is the return on an arbitrary asset with index i .

C.1 Returns on the Aggregate Wealth

Let z_t denote the log price consumption ratio. We use the Campbell-Schiller approximation,

$$r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1} \quad (12)$$

to approximate the return on aggregate wealth. This allows us to maintain a tractable analytical form of the pricing kernel. Returns on all other assets will be computed explicitly.

We conjecture a solution

$$z_t = A + B'X_t \quad (13)$$

to the log price consumption ratio. The Euler equation for aggregate wealth therefore becomes

$$\begin{aligned} 0 &= \ln E_t \exp\left(\theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + \theta r_{a,t+1}\right) \\ &= \ln E_t \exp\left(\theta \ln \delta - \theta\left(\frac{1}{\psi} - 1\right)g_{t+1} + \theta\kappa_0 + \theta\kappa_1 z_{t+1} - \theta z_t\right) \\ &= \ln E_t \exp\left(\theta \ln \delta - \theta\left(\frac{1}{\psi} - 1\right)\gamma'_c X_{t+1} + \theta\kappa_0 + \theta\kappa_1(A + B'X_{t+1}) - \theta(A + B'X_t)\right) \\ &= \ln E_t \exp\left(\theta \ln \delta + \left(\theta\left(1 - \frac{1}{\psi}\right)\gamma_c + \theta\kappa_1 B\right)' X_{t+1} + \theta(\kappa_0 - A) + \theta\kappa_1 A - \theta B'X_t\right) \\ &= \theta \ln \delta + \theta\kappa_0 + \theta(\kappa_1 - 1)A + \alpha\left(1, \left(\theta\left(1 - \frac{1}{\psi}\right)\gamma_c + \theta\kappa_1 B\right)\right) \\ &\quad + \left[\beta\left(1, \left(\theta\left(1 - \frac{1}{\psi}\right)\gamma_c + \theta\kappa_1 B\right)\right) - \theta B\right]' X_t \end{aligned}$$

The last line uses the expression for $E_t \exp(BX_{t+s})$ in equation (2). Matching terms in this equation leads to

$$0 = \beta\left(1, \theta\left(1 - \frac{1}{\psi}\right)\gamma_c + \theta\kappa_1 B\right) - \theta B, \quad (14)$$

$$A = \frac{\alpha\left(1, \theta\left(1 - \frac{1}{\psi}\right)\gamma_c + \theta\kappa_1 B\right) + \theta(\ln \delta + \kappa_0)}{\theta(1 - \kappa_1)} \quad (15)$$

The solution to these equations can be computed explicitly in a number of cases in which α and β have simple forms. In the case that we can only compute α and β numerically, equation (14) must be solved numerically. This equation may also have multiple roots. Tauchen (2005) shows that the coefficient corresponding to the stochastic volatility variable(s) in his model solves a quadratic equation, and hence has two roots. It is similarly the case in the stochastic volatility examples to be presented below that the element of $\hat{\beta}$ corresponding to the volatility process has two roots. In the example models there is typically one solution which provides economically reasonable behavior of financial prices, and a second solution which does not. For instance, one solution often produces a negative equity premium, or

an equity premium which is decreasing in risk aversion. Thus, for the example models considered here it is easy to find the only economically reasonable solution to (14).

C.2 Prices of Simple Claims

Consider an asset which pays a single dividend, D_{t+s} , at date $t+s$. The value of this claim is

$$\begin{aligned} P_t &= E_t \left(\delta^\theta C_t^{-\frac{\theta}{\psi}} G_{t+1}^{\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} E_{t+1} \left(\delta^\theta G_{t+2}^{-\frac{\theta}{\psi}} R_{a,t+2}^{-(1-\theta)} \dots E_{t+s-1} \left(\delta^\theta G_{t+s}^{-\frac{\theta}{\psi}} R_{a,t+s}^{-(1-\theta)} D_{t+s} \right) \dots \right) \right) \\ &= E_t \left(\delta^{\theta s} C_t^{-\frac{\theta}{\psi}} G_{t+1}^{-\frac{\theta}{\psi}} G_{t+2}^{-\frac{\theta}{\psi}} \dots G_{t+s}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} R_{a,t+2}^{-(1-\theta)} \dots R_{a,t+s}^{-(1-\theta)} D_{t+s} \right) \\ &= E_t \left(\delta^{\theta s} C_t^{-\frac{\theta}{\psi}} G_{t+1:s}^{-\frac{\theta}{\psi}} R_{a,t+1:s}^{-(1-\theta)} D_{t+s} \right). \end{aligned}$$

where

$$G_{t+1:s}^{-\frac{\theta}{\psi}} = \prod_{u=t+1}^{t+s} G_u^{-\frac{\theta}{\psi}}, \quad R_{a,t+1:s}^{-(1-\theta)} = \prod_{u=t+1}^{t+s} R_{a,u}^{-(1-\theta)}.$$

We have used the law of iterated expectations and the Martingale property of the discounted prices, and the fact that the return is defined $R_{t:s} = D_{t+s}/P_t$.

We are interested in the expectations of the form $E_t \exp(\sum_{u=t+1}^{t+s} A_u X_u)$ for $A_u \in \mathbb{R}^n$. The following technical lemma characterizes these expressions explicitly².

Lemma 1. *Let X_t be a process with transition density $p(X_{t+1} | X_t)$ which Laplace transform is $E(\exp(y'X_{t+s}) | X_t = x) = \exp(\alpha(s, y) + \beta(s, y)'x)$. Then for constant vectors $A_1, \dots, A_s \in \mathbb{C}^n$*

$$E\left(\exp\left(\sum_{u=t+1}^{t+s} A'_u X_u\right) | X_t = x\right) = \exp\left(\hat{\alpha}(A) + \hat{\beta}(A)x\right) \quad (16)$$

²This result has been derived independently and in a different context by Darolles, Gourieroux, and Jasiak (2005). An application to no-arbitrage credit risk can be found in Gourieroux, Monfort and Polemis (2005).

where $\hat{\beta}$ and $\hat{\alpha}$ are defined through the recursions

$$\begin{aligned}\hat{\alpha}(A) &= \alpha(1, A_s) + \alpha(1, A_{s-1} + \beta(A_s)) + \dots + \\ &\quad \alpha(1, A_{s-1} + \beta(A_{s-1} + \beta(A_s))) + \dots + \beta(A_1 + \beta(A_2 + \dots + \beta(A_{s-1} + \beta(A_s))))\end{aligned}\tag{17}$$

$$\hat{\beta}(A) = \beta(1, A_1 + \beta(1, \dots, A_{s-2} + \beta(1, A_{s-1} + \beta(1, A_s))))).\tag{18}$$

The expressions for $\hat{\beta}$ and $\hat{\alpha}$ appear messy, but are very easy to compute. The following pseudo computer code illustrates how to compute these functions:

```
betaHat=0 alphaHat=0
For s = T..1
    alphaHat = alphaHat+alpha(A(s) + betaHat)
    betaHat = beta(A(s) + betaHat)
end
```

The main result in this paper is the following:

Proposition 1. *Given a consumption process $C_{t+s} = C_t \exp(\sum_{u=t+1}^{t+s} \gamma'_c X_u)$ the price, P_t , of an asset that pays a single dividend claim $D_{t+s} = D_t \exp(\sum_{u=t+1}^{t+s} \gamma'_d X_u)$ is*

$$P_t = D_t \exp\left(F(s) + \hat{\beta}(\bar{b}_s)' X_t\right)\tag{19}$$

where

$$F(s) = [\theta \ln \delta - (1 - \theta)(\kappa_0 + (\kappa_1 - 1)A)]s + \hat{\alpha}(\bar{b}_s)$$

and the sequence $\bar{b}_s = \{b_u\}_{u=1}^{s+1}$ is given by

$$\begin{aligned}b_1 &= \left(\theta - \frac{\theta}{\psi} - 1\right)\gamma_c - (1 - \theta)(\kappa_1 - 1)B + \gamma_d + (1 - \theta)\kappa_1 B, \\ b_u &= \left(\theta - \frac{\theta}{\psi} - 1\right)\gamma_c - (1 - \theta)(\kappa_1 - 1)B + \gamma_d, \quad \text{for } u = 2, \dots, s \\ b_{s+1} &= -(1 - \theta)\kappa_1 B\end{aligned}\tag{20}$$

and where A and B solve equations (15) and (14), and the functions $\hat{\alpha}$ and $\hat{\beta}$ are defined in lemma 1.

C.3 Prices of Bonds and the Term Structure

The explicit expression for a one period discount bond is found by setting $s = 1$ and $D_{t+s} = 1$ in proposition 1

$$P_t(s) = \exp\left(F(s) + \hat{\beta}(\bar{b}_s)' X_t\right) \quad (21)$$

where $F(s)$ and $\hat{\beta}(\bar{b}_s)$ are obtained through proposition 1 by setting $\gamma_D = 0$. The yield to maturity of a s period zero coupon bond is

$$r_t(s) = -\frac{F(s)}{s} - \frac{\hat{\beta}(\bar{b}_s)'}{s} X_t \quad (22)$$

Since the yields are just linear combinations of affine processes, they are themselves affine processes. We define the "short rate" to be the annualize yield to maturity of a one period (one month) zero coupon bond in the following.

There are a number of recent papers that study the link between macroeconomic and term structure dynamics³. A typical approach in this literature is to specify an exogenous affine model for the macroeconomy as well as latent factors, and then to assume that the short rate process is a linear function of these exogenous processes. The GE modelling framework presented in this paper can possibly be construed as one which imposes parametric equilibrium constraints on such no-arbitrage models. The equilibrium constraints are testable, and possibly poses quite sharp restrictions on the yield curve data. The approach outlined here also clearly addresses the issue of which macroeconomic variables to use in yield curve modelling: all variables, observable or not, that affect dynamics of real consumption consumption growth and inflation matter for the term structure.

Can the yield curve be made to be always positive? The answer to this question is generally no. Yields generally depend on the exogenous state variables X linearly through the

³An incomplete list include Evans and Marshall (1998), Ang and Piazzesi (2003), Ang, Dong and Piazzesi (2005), Diebold, Rudebusch, and Arouba (2005), Duffee (2005) and Bikbov and Chernov (2005).

”factor loadings”, $\hat{\beta}$. Thus, if X_i has support on the real line for some i and its corresponding factor loading is non-zero, the yields will generally take on negative values with some probability. The second possibility is to consider state-variables that are constrained to be, say, positive. Then a negative $\hat{\beta}$ would lead to strictly positive yields, but the sign of $\hat{\beta}$ is determined by preference parameters. Even if the preference parameters could be chosen such that the factor loadings had the desired sign, such restrictions are unlikely to produce economically meaningful results. As an example, in the nominal model considered below the bond yields depend negatively upon volatility. Since the volatility process is unbounded from above, this implies the possibility of negative rates. It should be mentioned that the probability of observing a negative short rate in the example models in the next section is very low.

Negative nominal interest rates do, in the real world, represent a simple arbitrage opportunity. This arbitrage is carried out by borrowing at a negative rate, pocketing the loan, and repaying the loan at less than the principal amount in the future. This strategy require a ”pocket” - a storage technology that allows the money to be costlessly stored over time. In this real model, we could similarly define an arbitrage if there were a way to store consumption goods. There is not. Therefore, the model assumptions do not imply arbitrage inside the model.

C.4 Prices of Stocks

The price of a claim to the perpetual stream D_t obtains as follows.

Proposition 2. *Assume that X_t is stationary. There exists a subjective discount rate $\delta \in (0, 1)$ such that $\sum_{u=t+1}^{\infty} \exp(F(u) + \hat{\beta}(b_u)' X_t) < \infty$. The price of a stock, P_t , that pays a perpetual dividend stream $D_{t+s} = D_t \exp(\sum_{u=t+1}^{t+s} \gamma_d' X_u)$ is then*

$$P_t = D_t \sum_{u=t+1}^{\infty} \exp\left(F(u) + \hat{\beta}(b_u)' X_t\right) \quad (23)$$

where the functions F and $\hat{\beta}$ are as in proposition 1 and b_u is the sequence $b_u = \{b_v\}_{v=1}^{u+1}$ of constant vectors corresponding to a time u dividend payment, as in proposition 1.

III Example models

Example 1: Bansal & Yaron with Square-root volatility

In following we present specific examples of usages of the general methodology in the previous section. For purposes of comparison, we are interested in a model which mimics the Bansal & Yaron (2004) model as closely as possible, but with a volatility process which has support on the positive real line. To accomplish the latter, we specify a model in which the state-variables have stochastic volatility of the square root form as in the Heston (1993) option pricing model. The model is constructed from the following dynamic system

$$\begin{aligned}
 dz_{c,t} &= (\mu - \frac{1}{2}V_t)dt + \sqrt{V_t}dB_t^c, \\
 dz_{d,t} &= (\mu_d - \frac{1}{2}\varphi_d^2V_t)dt + \varphi_d\sqrt{V_t}dB_t^d, \\
 dx_t &= -\kappa_x x_t dt + \varphi_e \sqrt{V_t}dB_t^x, \\
 dV_t &= \kappa_v(\bar{V} - V_t)dt + \sigma_V \sqrt{V_t}dB_t^V.
 \end{aligned}$$

Consumption and dividend growth rates are defined

$$\begin{aligned}
 g_t &= z_{c,t} - z_{c,t-1} + x_t, \\
 g_{d,t} &= z_{d,t} - z_{d,t-1} + \phi x_t.
 \end{aligned}$$

We now define the N dimensional state variable $X_t = (z_{c,t} - z_{c,t-1}, z_{d,t} - z_{d,t-1}, x_t, V_t)$ which is stationary. The selection vectors are given as

$$\gamma_C = (1, 0, 1, 0), \quad \gamma_D = (0, 1, \phi, 0).$$

This completes the characterization of this model. A few remarks are in order. The long run risk factor, x_t , has stochastic volatility. If we instead modelled this process with a homoscedastic diffusion term which would make it an OU process, we would recover an exact expression for the generating function. This may be useful in certain situations. The

presence of stochastic volatility on the other hand, prevents an analytic solution. The model in the way in which it is written as closely as possible mimics the Bansal & Yaron (2004) discrete time model. This includes stochastic volatility in x_t .

[Table 1 about here.]

[Table 2 about here.]

Table II reports moments from simulated samples using the modified Bansal & Yaron model. The parameter values used in the calibration are chosen to match moments found in the data reported in Bansal & Yaron, and are reported in table II as well. Table II illustrates that our modified model generates annualized consumption and dividend growth rates which moments roughly match those calibrated in Bansal & Yaron (2004).

The asset pricing implications are also quite similar. Table III illustrates that the modified Bansal & Yaron model generates equity premiums of the same order of magnitude as in Bansal & Yaron (2004). The model is also successful in generating heavy tailed stock returns (kurtosis in excess of 4.6), and a negative correlation between shocks to volatility and stock returns which is as large as -0.35 for $\gamma = 10, \psi = 0.5$. Overall, our results are quite similar to those of Bansal & Yaron for the aggregate stock market.

Table III also reports the average real yield curve for different preference parameters. The average one month annualized yields are about 2.4% for the $\psi = 1.5$ parameter configurations. This is somewhat larger than numbers reported in Bansal & Yaron (2004). The numbers are very similar in the $\psi = 0.5$ case. We also report the average yields on long dated bonds that are uniformly lower yields of short maturity bonds. The negative slopes of the average yield curve is accentuated with higher values of the risk aversion parameter γ . The slope of the yield curve is also more steeply negative when the volatility of volatility σ_V increases. In fact, increasing σ_V is one easy way to generate a larger equity premium, however, even moderate increases in the value of σ_V may produce long run real yields that are negative on average. Evidence of a negatively sloping real yield curve in this model is not new and was discussed in Bansal & Yaron (2000).

[Table 3 about here.]

Example 2: Volatility Jumps

Why do markets crash? How does the possibility of catastrophic events affect financial prices? It is quite possible and even reasonable that assets which perform exceptionally poorly during periods of financial distress should earn an equilibrium rate of return that exceeds that of other assets. In the following we extend the basic model of the previous section to allow for financial crisis in the form of sudden, exogenous large shocks to the financial uncertainty in our model. Specifically, these shocks to uncertainty are assumed to arrive with an independent poisson arrival rate. The shocks themselves affect only volatility. Specifically, exogenous variables are as in the previous example with the exception of volatility which now is assumed to follow the process

$$dV_t = \kappa_v(\bar{V} - V_t - \lambda_0\mu_V)dt + \sigma_V\sqrt{V_t}dB_t^V + dJ_t$$

where dJ_t is the instantaneous innovation in a jump process with constant arrival intensity λ_0 and exponentially distributed jump sizes $\xi_t \sim \exp(\mu_V)$. Other distributional assumptions for ξ_t are possible. This dynamic specification of volatility was used in Duffie, Pan, & Singleton (2000) for options pricing, and found empirically successful in Eraker, Johannes, & Polson (2003) in describing the volatility of daily S&P 500 returns and fit to options prices (Eraker (2004)). Unlike these models however, we make no assumptions about asset prices being directly influenced by jumps. Rather, jumps in assets prices will occur endogenously. This is because the price/dividend ratio is a continuous function of all the state-variables, including V_t which now includes jumps. Since volatility jumps in this model are always positive, price jumps are always negative whenever $\hat{\beta}(s)_V < 0$ for all s which is typically the case. The magnitude of the impact of a jump to volatility depends on preferences.

In order to isolate the effect of non-gaussian innovations in volatility, we calibrate this model under the assumptions that average volatility and average volatility of volatility remains the same as in the purely diffusive model in example 1. We choose jump parameters

$\lambda_0 = 0.003$ and $\mu_V = 8e - 8$. This implies that jumps occur on average every 27th year, and lead to a variance increase which is about double that of the average variance. This is true for both consumption and dividend volatility since the later is just a multiple of the former.

[Table 4 about here.]

Table IV displays asset price implications of the jump-diffusion model. The introduction of volatility jumps leads to an unconditional equity premium which is substantial. For example, in the case of $\psi = 1.5$ and $\gamma = 7.5$, the premium is almost 9%. As can be seen throughout, high values of γ implies a high equity premium, fat tailed and left skewed unconditional return distributions. The contemporaneous correlation between volatility shocks and returns is of similar magnitudes as the square root volatility model for the same values of ψ and γ .

Table IV also shows mean yields on zero coupon bonds of varying maturities. The results show that the yield curve is downward sloping for all parameter constellations, and sharply downward sloping for high values of γ and low values of ψ . For $\gamma = 7.5$ and $\psi = 1.5$ or 0.5 , yields are negative at long maturities. This is clearly at odds with data, as well as economic intuition. When preference parameters are $\gamma = 5$, $\psi = 30$ and $\delta = .998$, the model seems to generate quite reasonable data with the exception, perhaps, of the still negatively sloping average yield curve. Bansal & Yaron (2000) however argue that the real yield curve could be (mildly) downward sloping citing evidence from inflation indexed bonds in the UK.

[Table 5 about here.]

The jump diffusion model for volatility is interesting because it produces endogenous price jumps. Are these jumps of similar magnitude to observed market crashes? Table V addresses this question by tabulating the probability of various volatility jumps, their probability of occurring, and the corresponding price shock. As can be seen, severe crashes are possible, although not particularly likely. The largest size jump considered is 2.5 multiple the average jump size. This would occur once every 344 years on average. By contrast,

during November 1929 the US stock market fell by about 30%. Other large monthly drops include December of 1931 (-20%) and April 1932 (-27%). In October 1987 the market dropped 12%. On the other hand, the latter is the only observation which we can reasonably classify as a discontinuous event with a the 22% drop in one single day. The 1929-1933 period is perhaps better characterized as a period with extremely high volatility. As such it is still quite possible that the volatility jump model may prevail in giving an adequate description of the data.

Example 3: A model of nominal prices

No-arbitrage term structure models are used to model nominal yield curves. Our modelling framework, so far, has abstracted from inflation and prices are interpreted in units of the consumption good. To introduce a framework which rivals that of no-arbitrage models in applicability, we now turn to a model of nominal bond yields, and nominal stock returns. Part of what motivates this extension is the question of whether we can extend the previous model(s) in such a way that we can recover a positively sloping yield curve, while maintaining essential model characteristics which make the models consistent with observed equity market data.

In pursuing this, we will model inflation as exogenous. If inflation is an exogenously given random walk independent of the real variables in the economy (neutral), it is straightforward to show that inflation has no effect on prices. Thus, under inflation neutrality the nominal term structure equals the real term structure plus expected inflation growth. Inflation neutrality, therefore, does not turn the negatively sloping real yield curve of the previous examples into a positively sloping nominal one.

In the following, we specify a model in which inflation, and long term inflation growth affect financial market prices. We are particularly interested in analyzing a situation in which long term inflation growth correlates negatively with real growth rates. This is consistent with the empirically observed strong negative correlation of -0.52 between inflation and real growth rates in consumption and dividends. Our formulation is also consistent with empirical evidence in Christiano, Eichenbaum, and Evans (1999), Ang and Piazzesi

(2003) among others, showing negative impulse response functions for monetary shocks on real aggregates.

The impact of inflation or monetary shocks on the real economy is a much debated issue in macroeconomics. A model that incorporates money and clears the product market by endogenizing the price level is beyond the scope of this paper⁴. The general consensus in the monetary models is that inflation is neutral, but not super-neutral, absent frictions⁵. Money is defined to be super-neutral if a change in its (expected) growth rate has no impact on the growth or real variables.

In our model, we assume that there is an exogenously given inflation rate and an time-varying, exogenous expected inflation growth rate. Inflation is neutral but not super-neutral so shocks to inflation itself do not impact growth rates, but shocks to expected inflation growth do. In particular, we assume that (log) inflation, i_t , is a random walk with a time varying expected growth rate m_t which follows a square root diffusion process. The model is

$$dz_{c,t} = (\mu + Q_c m_t - Q_c \bar{i} - \frac{1}{2} V_t) dt + \sqrt{V_t} dB_t^c, \quad (24)$$

$$dz_{d,t} = (\mu_d + Q_d m_t - Q_d \bar{i} - \frac{1}{2} \varphi_d^2 V_t) dt + \sqrt{V_t} dB_t^d, \quad (25)$$

$$dx_t = -\kappa_x x_t dt + \varphi_e \sqrt{V_t} dB_t^x, \quad (26)$$

$$dV_t = \kappa_v (\bar{V} - V_t) dt + \sigma_v \sqrt{V_t} dB_t^V + dJ_t, \quad (27)$$

$$di_t = m_t dt + \sigma_i dB_t^i, \quad (28)$$

$$dm_t = \kappa_m (\bar{i} - m_t) dt + \sigma_m \sqrt{m_t} dB_t^m, \quad (29)$$

where Q_c and Q_d should be negative to generate negative correlation between (expected) inflation and real growth. As before, jumps arrive with constant intensity λ_0 and distribute exponentially, $\xi_t \sim \exp(\mu_V)$.

⁴Bakshi and Chen (1996) and Lioui and Poncet (2003) study the effect of monetary shocks on asset prices with endogenous prices.

⁵Friction that generate non-neutrality include price rigidities and limited participation (Christiano, Eichenbaum, and Evans (1997)), and transactions costs (Bansal and Coleman (1996)).

Consider now how inflation impacts prices. First, in pricing stocks, we simply use the pricing equation developed in the previous section. This expression produces a price dividend ratio which does not depend on the commodity price level. Thus, nominal stock prices can be computed by replacing the real dividend, D_t , in equation (23) with the nominal dividend.

In considering the impact on inflation on nominal zero coupon bond prices, consider a general formulation where inflation is a linear function of state-variables,

$$i_t = \gamma_I' X_t.$$

The nominal price of a \$1 zero coupon bond is then

$$P_t(s) = E_t \exp\left(s\theta \ln \delta - \frac{\theta}{\psi} g_{t+1:t+s} - (1 - \theta)r_{a,t+1:t+s} - i_{t+1:t+s}\right) \quad (30)$$

This expression is identical to the expression for a single dividend paying stock with $\gamma_D = -\gamma_I$. Thus, we can interpret nominal bond prices in the presence of inflation as a claim to (negative) future inflation growth.

We calibrate the nominal price model using inflation parameters $\bar{i} = 0.0033$, $\kappa_m = 0.03$, $\sigma_I = 8e - 6$, and $\sigma_m = 0.014$. This generate an annual average inflation of 4%, with a standard deviation of 3.8% and first order autocorrelation of 0.72. These numbers are chosen to match the postwar inflation data in table 1. In addition, we choose $Q_c = -0.5$ and $Q_d = -1.5$ respectively. This generates contemporaneous correlations of -0.54 and -0.51 between inflation and consumption and inflation and dividend growth respectively.

[Table 6 about here.]

Table VI reports the financial market implications of the nominal price model using a value of $\psi = 30$ which was found empirically successful for the real jump model. We report the results assuming $Q_c = Q_d = 0$ (super-neutral), and $Q_c = -.5, Q_d = -1.5$ (non-neutral), respectively. In the case of super-neutrality, the model produces equity data that are consistent with the real model. For example, the equity premium is about 5% for $\gamma = 5$.

The short rates in the neutral case are 6.35 and 6.25 for $\gamma = 4$ and 5 respectively. This compares to 2.35 and 2.26 for the model without inflation. The difference is exactly 4% (absent rounding errors) which equals the average annual inflation rate.

In the case of $Q_c, Q_d < 0$, financial prices change in interesting ways. First, the equity premium increases substantially relative to the neutral case, and large values of γ increases the premium the most. In fact, in order to generate an equity premium of 5% we need to set γ to no more than 4. Relative to the previous two example models the nominal model introduces an additional factor, m_t . This variable is essentially an additional long run risk factor, and as a consequence, assets that are exposed to this risk demand higher premiums. Second, the most interesting aspect of the non-neutral models is that the term structure changes shape from negatively uniformly negatively sloping to flat ($\gamma = 4$) and positively sloping $\gamma = 5, 6$. This happens as short rates decrease and long rates increase relative to the neutral case. Conversely, the exposure of short term bonds to long run inflation risk is small while long bonds similarly are highly sensitive to this risk.

A Euler Equation Errors

In a recent paper, Lettau and Ludvigsen (2005) argue that "leading asset pricing models" do not fully capture the failure of the standard consumption based power utility model. Their argument is simple. The standard power utility consumption capm Euler equation does not hold empirically for US postwar data. That is, there exist no value of γ that will set the average pricing error,

$$e = E(\exp(-\gamma^* g)R)$$

to zero. Here R is a vector (real) stock returns in excess of the (real) risk free rate, and g is real log consumption growth, and γ^* minimizes e^2 . Assume that some model is the "true" model of asset price dynamics, meaning that the model has the same data generating process as in the observed data. Thus if $|e| > 0$ in the data, it must be that $|e| > 0$ for simulated model data as well. Otherwise, the data generating processes cannot be the same.

Lettau and Ludvigson compute e for a number of recent asset pricing models and show that all the models considered produce zero pricing errors. This contrasts with the data, for which the errors are substantial. In conclusion therefore, the DGP of observed US consumption and returns data differ from that of "leading models," including the Campbell and Cochrane (1999) external habit model, Menzly, Santos and Veronesi's (2004) habit model, Guvenen's (2003) limited participation model, and Bansal and Yaron's (2004) long run risk model.

Lettau and Ludvigsen further demonstrate that positive Euler equation errors can be explained by non-gaussian innovations in consumption growth and returns. In particular, they show that when data in limited participation model are generated from (severely) non-gaussian joint distributions, simulated data can produce Euler errors that match that of observed data. An interesting question therefore, is whether the non-gaussian jump models considered here can similarly generate non-zero Euler errors.

[Table 7 about here.]

Table VII reports Euler equation errors for the respective models considered in this paper⁶. As can be seen, the purely diffusive volatility model generates zero errors. By contrast, the jump models all generate non-zero errors and in some cases the errors are substantial. For the second example model without inflation, the errors are 0.099 and 0.292 for $\gamma = 7.5$ and $\gamma = 10$ respectively. The latter is close to that of the data, for which the error is found to be 0.4 using S&P 500 returns in this paper, and 0.48 by Lettau and Ludvigsen (2005) using the value weighted CRSP index. For the nominal price models the numbers are 0.366 and 0.539 under neutral and non-neutral inflationary assumptions. This is close to that observed in the data and in the non-neutral case, actually larger than in the data. Thus we conclude that the jump and inflation extended models are successful in explaining the lack of Euler equation errors reported for other models.

⁶The EEE is defined

$$EEE = \min_{\gamma} \sqrt{\frac{1}{T} \sum_t \exp(-\gamma g_t) (r_t^s - r_t^f)^2 / \frac{1}{T} \sum_t (r_t^s - r_t^f)^2}.$$

IV Conclusion

This paper has presented a general framework for valuation of stocks and bonds based upon an Epstein-Zin preference structure, and under general assumptions about the dynamics of state-variables that affect consumption and dividend growth. The three example models presented demonstrate that this framework is capable of explaining well known asset pricing puzzles. The Epstein-Zin preference structure represents an important component in explaining these puzzles as it disentangles the elasticity of substitution from the temporal resolution of uncertainty. This produces low bond yields, high equity returns, as well as higher order moments that are in line with observed data.

This paper delivers a framework for analyzing stock and bond prices which effectively can be seen as an equilibrium version of no-arbitrage factor models. The advantages relative to standard no-arbitrage models is that 1) the links to macro economic time series is explicit, and 2) the factor "loadings" that determine the various assets' sensitivity to changes in the economic variables are explicit functions parameters that determine the dynamic behavior of macro quantities, and more importantly, preferences. This allows a fairly rich framework for analyzing the link between macro and financial market variables. It also allows for a fair amount of flexibility in allowing for unobserved components such as expected growth rates and volatility. This is similar in spirit to the no-arbitrage models, but has the advantage that these latent factors have economic interpretations. Again, this facilitates a closer examination of the links between financial market dynamics and the macro economy. In building on this framework it is possible to construct quite flexible models of yield curve dynamics while maintaining an equilibrium foundation. This is likely to produce an interesting equilibrium based alternative to the growing body of papers that study the link between macroeconomic dynamics and the term structure.

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Appendix A. Proofs

Proof of lemma 1. Using iterated expectations,

$$\begin{aligned}
& E_t \left(\exp \left(\sum_{u=t+1}^{t+s} A'_u X_u \right) \right) \\
&= E_t \left(e^{A'_{t+1} X_{t+1}} E_{t+1} \left[e^{A' X_{t+2}} \times \dots \times E_{t+s-1} \left[e^{A'_{t+s} X_{t+s}} \right] \dots \right] \right) \\
&= E_t \left(e^{A'_{t+1} X_{t+1}} E_{t+1} \left[e^{A'_{t+2} X_{t+2}} \times \dots \times E_{t+s-2} \left[e^{A'_{t+s-1} X_{t+s-1}} E_{t+s-1} \left[e^{A'_{t+s} X_{t+s}} \right] \dots \right] \right] \right) \\
&= E_t \left(e^{A'_{t+1} X_{t+1}} E_{t+1} \left[e^{A' X_{t+2}} \times \dots \times E_{t+s-2} \left[e^{\alpha(1, A_{t+s}) + (A_{t+s-1} + \beta(1, A_{t+s}))' X_{t+s-1}} \right] \dots \right] \right) \\
&= E_t \left(e^{A'_{t+1} X_{t+1}} E_{t+1} \left[e^{A' X_{t+2}} \times \dots \times e^{\alpha(1, A_{t+s})} E_{t+s-2} \left[e^{(A_{t+s-1} + \beta(1, A_{t+s}))' X_{t+s-1}} \right] \dots \right] \right) \\
&= E_t \left(\dots \times E_{t-s-3} \left[e^{\alpha(1, A_{t+s}) + A'_{t+s-2} X_{t+s-2} + \alpha(1, A_{t+s-1} + \beta(1, A_{t+s})) + \beta(1, A_{t+s-1} + \beta(1, A_{t+s}))' X_{t+s-1}} \right] \dots \right) \\
&= E_t \left(\dots \times e^{\alpha(1, A_{t+s}) + \alpha(1, A_{t+s-1} + \beta(1, A_{t+s}))} E_{t-s-3} \left[e^{(A_{t+s-2} + \beta(1, A_{t+s-1} + \beta(1, A_{t+s}))' X_{t+s-1}} \right] \dots \right).
\end{aligned}$$

□

Proof of proposition 1.

$$\begin{aligned}
P_t/D_t &= E_t \exp \left(s\theta \ln \delta - \frac{\theta}{\psi} g_{t+1:s} - (1-\theta)r_{a,t+1:s} + \ln D_{t+s} \right) \\
&= E_t \exp \left(s\theta \ln \delta + \left(\gamma_d - \frac{\theta}{\psi} \gamma_c \right)' \sum_{u=t+1}^{t+s} X_u \right. \\
&\quad \left. - (1-\theta) \sum_{u=t+1}^{t+s} \left(\kappa_0 + \kappa_1 A + \kappa_1 B' X_{u+1} - A - B' X_u + \gamma_c' X_u \right) \right) \\
&= E_t \exp \left(\left[\theta \ln \delta - (1-\theta)(\kappa_0 + (\kappa_1 - 1)A) \right] s + \left(\gamma_d - \frac{\theta}{\psi} \gamma_c \right)' \sum_{u=t+1}^{t+s} X_u \right. \\
&\quad \left. - (1-\theta) \sum_{u=t+1}^{t+s} \left(\kappa_1 B' X_u - B' X_u + \gamma_c' X_u \right) - \kappa_1 B' X_{t+1} + \kappa_1 B' X_{t+s+1} \right) \\
&= E_t \exp \left(\left[\theta \ln \delta - (1-\theta)(\kappa_0 + (\kappa_1 - 1)A) \right] s + \right. \\
&\quad \left[\left(\theta - \frac{\theta}{\psi} - 1 \right) \gamma_c + \gamma_d - (1-\theta)(\kappa_1 - 1)B \right]' \sum_{u=t+1}^{t+s} X_u \\
&\quad \left. - (1-\theta)(\kappa_1 B' X_{t+s+1} - \kappa_1 B' X_{t+1}) \right)
\end{aligned}$$

This is of the form given in lemma 1 with coefficients as suggested in the proposition. \square

Proof of proposition 2. Stationarity of X_t implies that $(\alpha(s), \beta(s)) \rightarrow (\alpha, 0)$ as $s \rightarrow \infty$, i.e., the limiting generating function does not depend on the initial state, X_t . This implies that for finite constants $a, b \in \mathbb{R} \times \mathbb{R}^n$,

$$\lim_{s \rightarrow \infty} E_t e^{\sum_u^{t+s} b'_u X_u} = e^{as + sb' X_t},$$

with b_u as given in proposition 1. Thus

$$\sum_{s=1}^{\infty} e^{[\theta \ln \delta - (1-\theta)(\kappa_0 + (\kappa_1 - 1)A)]s + \hat{\alpha}_s + \hat{\beta}'_s X_t} = \sum_{s=1}^{\infty} e^{[\theta \ln \delta^* - (1-\theta)(\kappa_0 + (\kappa_1 - 1)A) + a + b' X_t]s} < \infty$$

for some $\delta^* < \exp \{[(1 - \theta)(\kappa_0 + (\kappa_1 - 1)A) + a + b' X_t] / \theta\}$.

\square

Tables

Table I: Real logarithmic growth rates from 1945-2004

	mean	std	ac(1)	ac(2)	ac(3)	ac(5)	ac(10)
consumption	2.22	1.74	0.19	-0.04	-0.11	-0.04	0.07
dividends	1.63	6.21	0.29	-0.24	-0.15	0.06	0.18
earnings	2.93	18.99	0.07	-0.35	-0.20	0.05	0.27
inflation	4.21	3.82	0.72	0.31	0.14	0.39	-0.01

correlations				
consumption	1.00	0.40	0.42	-0.52
dividends	0.40	1.00	0.49	-0.52
earnings	0.42	0.49	1.00	-0.21
inflation	-0.52	-0.52	-0.21	1.00

Table II: Modified BY model Growth Rates

Moments calibrated using parameter values $\kappa_x = 0.025, \mu = \mu_d = 0.0015, \phi = 5, \bar{v} = 0.025^2, \kappa_v = 0.04, \varphi_e = 0.07, \varphi_d = 4.5, \sigma_v = 0.0011$.

	mean	std	ac(1)	ac(2)	ac(3)	ac(5)	ac(10)
consumption	1.92	2.65	0.49	0.34	0.26	0.14	0.03
dividends	1.85	10.53	0.38	0.25	0.19	0.10	0.02
correlations							
	consumption	1.00	0.47				
	dividends	0.47	1.00				

Table III: Asset Price Implications - Modified BY Model

Moments calibrated using parameter values $\kappa_x = 0.025, \mu = \mu_d = 0.0015, \phi = 5, \bar{v} = 0.025^2, \kappa_v = 0.04, \varphi_e = 0.07, \varphi_d = 4.5, \sigma_v = 0.0011$.

	$\gamma = 4$	$\psi = 0.5$ $\gamma = 7$	$\gamma = 10$	$\gamma = 4$	$\psi = 1.5$ $\gamma = 7$	$\gamma = 10$
Real Stock Returns						
Equity Prem.	0.40	1.24	2.51	1.18	2.51	4.07
Mean (annual)	5.22	6.10	7.44	3.57	4.92	6.51
Std. (annual)	11.36	11.28	10.79	15.25	14.82	14.47
Skew (month)	-0.01	-0.01	0.01	-0.02	0.00	0.03
Kurt (month)	4.84	4.84	4.63	4.81	4.86	4.67
Corr(dV, r)	-0.01	-0.15	-0.32	-0.05	-0.19	-0.35
Real Yield Curve						
1 month mean	4.82	4.86	4.93	2.40	2.41	2.44
1 month std	3.75	3.88	4.01	1.37	1.54	1.66
1 year mean	4.80	4.86	4.96	2.36	2.35	2.37
1 year std	3.28	3.40	3.49	1.20	1.36	1.47
10 year mean	4.80	4.86	4.96	2.36	2.35	2.37
10 year std	3.28	3.40	3.49	1.20	1.36	1.47
30 year mean	3.76	2.71	1.30	1.78	1.17	0.28
30 year std	0.43	0.46	0.51	0.16	0.19	0.23

Table IV: Asset Price Implications - Jump Diffusion Volatility

The table report financial price dynamics using growth rate parameters $\kappa_x = 0.025, \mu = \mu_d = 0.0015, \phi = 5, \bar{v} = 0.025^2, \kappa_v = 0.04, \varphi_e = 0.07, \varphi_d = 4.5, \sigma_v = 0.0011/3, \lambda_0 = 0.003, \mu_V = 8e - 8$.

	$\delta = 0.999$		$\delta = 0.998$			
	$\psi = 0.5$		$\psi = 1.5$		$\psi = 30$	
	$\gamma = 4$	$\gamma = 7.5$	$\gamma = 4$	$\gamma = 7.5$	$\gamma = 4$	$\gamma = 5$
Real Stock Returns						
Equity Prem.	0.93	5.10	2.76	8.88	3.22	5.02
Mean (annual)	5.82	10.89	5.14	11.06	5.57	7.27
Std. (annual)	11.13	10.80	14.68	12.83	16.16	15.36
Skew (month)	-0.03	-0.13	0.03	-0.28	-0.05	-0.03
Kurt (month)	5.02	8.06	5.48	8.42	5.53	6.02
Corr(dV, r)	-0.05	-0.19	-0.08	-0.22	-0.09	-0.13
Real Yield Curve						
1 month mean	4.88	5.79	2.39	2.18	2.35	2.26
1 month std	3.76	4.14	1.36	1.57	0.41	0.46
1 year mean	4.82	5.84	2.25	1.88	2.12	1.94
1 year std	3.29	3.60	1.19	1.39	0.35	0.39
10 year mean	2.78	-0.97	0.93	-2.64	1.18	0.63
10 year std	1.22	1.35	0.45	0.55	0.12	0.14
30 year mean	1.07	-7.33	0.14	-5.66	0.87	0.17
30 year std	0.43	0.48	0.16	0.20	0.04	0.05

Table V: Market Crashes and their Probabilities

The table gives the price impact of volatility jumps. Volatility jump sizes are measured in units of $\mu_V = 8e - 8$. Conditional probability denotes the chance of observing a volatility jump of at least the size given conditional upon the jump occurring. The unconditional probability the combined probability of observing the particular shock in a given year. Parameter values as in table IV and $\gamma = 5, \psi = 30$.

Volatility jump	1	1.5	2	2.5
Log price impact (%)	-7.87	-11.72	-15.55	-19.34
Conditional Prob. (%)	36.79	22.31	13.53	8.21
Unconditional Prob. (%)	1.32	0.80	0.48	0.29

Table VI: Asset Price Implications - Nominal Price Model

The table report financial price dynamics using growth rate parameters $\kappa_x = 0.025, \mu = \mu_d = 0.0015, \phi = 5, \bar{v} = 0.025^2, \kappa_v = 0.04, \varphi_e = 0.07, \varphi_d = 4.5, \sigma_v = 0.0011/3, \lambda_0 = 0.003, \mu_V = 8e - 8, \bar{i} = 0.0033, \kappa_m = 0.03, \sigma_I = 8e - 6, \sigma_m = 0.014$.

	$Q_c = Q_d = 0$			$Q_c = -0.5, Q_d = -1.5$		
	$\gamma = 4$	$\gamma = 5$	$\gamma = 6$	$\gamma = 4$	$\gamma = 5$	$\gamma = 6$
Nominal Stock Returns						
Equity Prem.	3.23	5.10	6.94	5.07	7.39	10.89
Mean (annual)	5.50	7.28	8.92	6.89	8.90	11.80
Std. (annual)	16.14	15.27	14.70	20.35	19.80	19.12
Skew (month)	-0.01	-0.07	-0.13	-0.07	-0.07	-0.08
Kurt (month)	5.66	6.04	6.88	4.23	4.23	4.27
corr(dV,r)	-0.08	-0.13	-0.17	-0.06	-0.09	-0.11
Nominal Yield Curve						
1 month mean	6.35	6.25	6.06	5.91	5.59	4.99
1 month std	3.96	3.93	4.03	3.19	3.02	2.52
1 year mean	6.10	5.93	5.64	5.92	5.64	5.07
1 year std	3.36	3.34	3.42	2.93	2.88	2.55
10 year mean	4.94	4.39	3.58	5.90	5.80	5.69
10 year std	1.00	1.00	1.02	1.24	1.44	1.70
30 year mean	4.55	3.85	2.81	5.89	5.98	6.54
30 year std	0.34	0.34	0.35	0.44	0.54	0.71

Table VII: Euler Equation Errors

	data	Models					
		Diffusive		Jump Diffusion		Nom.	Jump Diffusion
γ/ψ		4/1.5	7.5/1.5	7.5/1.5	10/1.5	5/30 ^a	5/30 ^b
EEE	0.4	0.0	0.0	0.099	0.292	0.366	0.539

a) Neutrality, $Q_c = Q_d = 0$.

b) Non-neutrality, $Q_c = -0.5, Q_d = -1.5$.