Model Uncertainty and Option Markets in Heterogeneous Economies

Andrea Buraschi

Columbia University and London Business School

Alexei Jiltsov

Lehman Brothers, Fixed Income Research

First version: September 21^{st}, 2003

Abstract: This paper provides option pricing and volume implications for an incomplete market economy with heterogenous agents who face model uncertainty and disagree on the dividend growth rate. Market incompleteness makes options non-redundant while heterogeneity creates a link between differences in beliefs and option volumes. We solve for both option prices and volumes and test the joint empirical implications using SP500 index option data. We use survey data to build an Index of Dispersion in Beliefs and find that a model which takes into account information heterogeneity can explain the dynamics of option volume better than reduced-form models with stochastic volatility. Moreover, its hedging performance is superior. Finally, we find that the Index of Dispersion in Beliefs is correlated with changes in the shape of the smile and it forecasts future realized volatility even after controlling for the current implied volatility.

JEL classification: D51, G12, G13.

Keywords: Options, Heterogeneity, Beliefs, Trading Volume, Open Interest.

We would like to thank Gurdip Bakshi, Suleyman Basak, Francisco Gomes, Dennis Gromb, Randi Rosenblatt, Raman Uppal, seminar participants at Columbia Business School, London Business School, and Stockholm School of Economics. The usual disclaimer applies. The project was supported by an ESRC Research Grant.

* *Corresponding author. Tel: +44-207-431 3771; Email address: aburaschi@london.edu*

Model Uncertainty and Option Markets in Heterogeneous Economies

Abstract: This paper provides option pricing and volume implications for an incomplete market economy with heterogenous agents who face model uncertainty and disagree on the dividend growth rate. Market incompleteness makes options non-redundant while heterogeneity creates a link between differences in beliefs and option volumes. We solve for both option prices and volumes and test the joint empirical implications using SP500 index option data. We use survey data to build an Index of Dispersion in Beliefs and find that a model which takes into account information heterogeneity can explain the dynamics of option volume better than reduced-form models with stochastic volatility. Moreover, its hedging performance is superior. Finally, we find that the Index of Dispersion in Beliefs is correlated with changes in the shape of the smile and it forecasts future realized volatility even after controlling for the current implied volatility.

JEL classification: D51, G12, G13.

Keywords: Options, Heterogeneity, Beliefs, Trading Volume, Open Interest.

*

1 Introduction

THE NOTION THAT OPTIONS CAN BE REPLICATED via a dynamic trading strategy in the underlying asset (Black and Scholes (1973)) has been one of the most influential innovations in financial theory. The traditional no-arbitrage approach with deterministic volatility is, however, silent about option trading volume: since options are redundant securities, agents are indifferent about holding them so that their trading volume is indeterminate. Nonetheless, the last two decades have witnessed an impressive proliferation of options and other derivative securities. Option open interest in all main option exchanges has increased tenfold in fifteen years. This suggests that options, far from being redundant, provide an economic value that at least exceeds the cost of maintaining option exchanges. The main contribution of this paper is to link heterogeneity in beliefs to option open interest and to provide a model that delivers realistic *joint* restrictions on both option open interest and prices.

Our model suggests a simple generalization of the standard general equilibrium Lucas economy with rational agents, identical preferences and endowments, but incomplete and heterogeneous information. We allow the dividend growth rate to be stochastic so that agents need to form expectations about future dividends to form their optimal portfolios. We assume that agents rationally use all available information to update their initial beliefs about the dividend growth rate but have different initial priors. When agents have different beliefs, they select different optimal portfolios and trading occur. The paper describes how incomplete information affects option prices and open interest in an incomplete market setting. Agents with a more pessimistic posterior estimate of the dividend growth rate demand state-contingent insurance protection from the optimist. This form of risk-sharing can be achieved by means of (out-the-money) put options. On the other hand, agents with more optimistic posterior estimates of the dividend growth rate demand (out-of-the-money) call options from the pessimists. Since the marginal utility of both agents is different in good and bad states of the world, differences in beliefs can generate a time-varying option smile and volatility skew.

An important by-product of the model is that although the instantaneous dividends volatility is deterministic, the endogenous equilibrium volatility of the underlying stock price is stochastic and driven by the level of difference in beliefs via the posterior mean of the dividend growth rate. The model therefore provides a structural underpinning for reduced-form stochastic volatility models.

The assumption of informational heterogeneity is motivated by a growing body of empirical evidence showing that options are used for informationally driven trades. Pan and Poteshman (2003) find that the put/call volume ratio of buyer-originated trades predicts future returns in the underlying asset. Amin and Lee (1997) show that good (bad) earnings news are positively correlated with an increase in call (put) open interest before the announcement. Cao, Chen, and Griffin (2000) show that take-over premia are predicted by the option volume in the pre-announcement period. Moreover, Easley, O'Hara, and Srinivas (1998) find a link between option volume and stock returns, even independent of take-over announcements.[1] These studies focus on individual stock options and

[1] Other important empirical contributions on the informational role of options include Manaster and Rendleman (1982), Stephan and Whaley (1990), Figlewski and Webb (1993), Mayhew, Sarin, and Shastri (1995), and Chakravarty, Gulen, and Mayhew (2002).

conjecture that some investors have private information on the underlying stock, which they try to exploit by trading in the option market. Our study, however, focuses on *index* options for which private information is a much less compelling trading motive. Therefore, we explore another source for informationally driven trades: heterogeneous beliefs of fully rational agents in a learning economy.

The model derives the optimal portfolio holdings in the underlying asset and in options as a function of the difference in beliefs. We then estimate the structural model using moment conditions on both option prices and open interest. Using the standard deviation of the beliefs distribution obtained from the Survey of Professional Forecasters and the Consumer Confidence Survey, we build a Difference in Beliefs Index (Ψ) on market fundamentals and show a strong link between heterogeneity in beliefs on market fundamentals and (a) open interest in index options, (b) future realized and implied volatility, and (c) the shape of the implied volatility smile. More specifically, we address the following questions:

First, what is the extent to which difference in beliefs on markets fundamentals can explain the dynamics of option trading and open interest? Unlike models with time-varying but deterministic volatility, the DB model generates testable restrictions on the dynamics of option volume and open interest. We use a panel data of options to run a Chi-square test to assess the extent to which changes in the difference in beliefs explain option trading. We fail to reject the overidentifying restrictions on the option open interest. Moreover, we find that the link between changes in the difference in beliefs and option open interest is economically significant: a one standard deviation increase in the Difference in Beliefs Index increases option open interest by 20%. We find the relationship to be non-linear.

Second, how does the model fare against traditional reduced form option pricing models in terms of hedging errors? We find that the Difference in Beliefs model generates lower hedging errors than both Heston (1993) and Black and Scholes (1973). We test the model by using a subset of options to estimate the structural parameters and then use the pricing errors on different subsets to build a Chi-square test statistic of the overidentifying restrictions. We fail to reject the model when using both out-of-the-money and in-the-money subsets of options.

Third, how do the differences in beliefs affect the shape of the volatility smile? We find that the implied volatility smile is sensitive to the Difference in Beliefs Index. For low Index levels, the smile level is about 15%. For high Index levels, it exceeds 20%. Moreover, we find that changes in the Index affect the steepness of the smile: the greater the Difference in Beliefs Index, the steeper the implied volatility smile.

Fourth, how much do differences in beliefs anticipate future realized and implied volatility? We find that current levels in the Difference in Beliefs Index have positive and statistically significant predictive power for the future realized volatility, even after controlling for the implied volatility.

Fifth, do differences in beliefs explain violations of text-book arbitrage bounds? Bakshi, Cao, and Chen (2000) find that the delta of a call option is often negative or larger than one. We run a Logit regression to assess the extent to which the Difference in Beliefs Index can explain these no-arbitrage violations. We find that the difference in beliefs slope coefficients are positive and statistically significant. An increase in the Index increases the probability that the Black and Scholes delta is negative or above one. This suggests that the heterogeneity in beliefs is an important pricing factor generating enough market incompleteness to generate deviations from arbitrage bounds based on one-factor models.

From these five sets of results we conclude that the information heterogeneity and belief structure of the economy has important option pricing and risk management implications.

Related Literature.

Our model draws from several contributions in the incomplete information and rational learning literature. David and Veronesi (2002) develop an incomplete information option pricing model in which investors' uncertainty about the drift of firms' dividends affects option prices through its impact on the stock's volatility. Stock return volatility is stochastic because of fluctuations in "uncertainty". They derive a Fourier Transform option pricing formula and show that the time-varying correlation between returns and volatility induced by uncertainty in dividends is related to higher order moments of the return distribution. A contribution of their approach is to circumvent the deterministic variance property of the estimation error of Gaussian models. Our approach is related to theirs: we assume that the drift of the dividends is not observable and stochastic and the dynamics of beliefs is implied by rational learning. The main differences in our approach are: (a) we introduce heterogeneity and derive equilibrium portfolio holdings and implications in terms of option open interest; (b) we model two forms of incomplete information: the first affecting the dividend drift, the second affecting a signal that is correlated with dividends. Both forms are described by a continuous time process as opposed to a switching regime model. Other option pricing models with uncertainty include Campbell and Li (1999), who study the implications of uncertainty in volatility regimes, and Garcia, Luger, and Renault (1999), who study a regime switching model. Similar to our approach, Guidolin and Timmermann (2000) model learning in an economy in which prices are endogenous. Their assumption of an i.i.d. drift process, however, makes option prices converge to Black-Scholes values in the continuous time limit. The original models of asset pricing in economies with incomplete information have been developed by Detemple (1986), Dothan and Feldman (1986), Gennotte (1986), Detemple and Murthy (1994a) and Detemple and Murthy (1994b). More recent contributions include Brennan (1998), Brennan and Xia (1998), David (1997), Veronesi (1999) and Veronesi (2000).

Our work is also related to the literature studying the role of options in incomplete markets. In the Black and Scholes (1973) and Merton (1973) models of derivative pricing, options are redundant securities as their payoff can be replicated using financial claims. To understand the empirical shortcomings of the model, several work have investigated the role of options in economies in which they are not redundant. Bates (2001) considers an economy in which crashes can occur and agents differ in their risk aversion. In his model the less crash-averse agents insure the more crash-averse agents through options that complete the market. Grossman and Zhou (1996) consider a model in which the demand for options is generated by the utility function of portfolio insurers. Related earlier work on this topic include Leland (1980) and Brennan and Solanki (1979). Franke, Stapleton, and Subrahmanyam (1998) consider a one-period economy in which agents are exposed to different non hedgeable background risk leading to market incompleteness. Options trading also arises in presence of additional risk factors such as stochastic volatility and jumps (Hull and White (1987), Wiggins (1987), Heston (1993), Liu and Pan (2003)) or because of informational reasons (Back (1993), Biais and Hillion (1994), Brennan and Cao (1996), John, Koticha, Narayanan, and Subramanyam (2000)). Buraschi and Jackwerth (2001) develop an empirical test of the dynamic spanning properties of option contracts and reject the null hypothesis that they are redundant. They find, moreover, that option returns embed a risk premium. Similarly, Coval and Shumway (2001) find that the options expected returns cannot be explained simply by the risk-return trade-off generated by the diffusion process of the underlying asset. Bakshi and Kapadia (2003) study delta-hedged gains of European options and find evidence of a negative market volatility risk premium.

Our work is also related to the literature of models with differences in beliefs. Detemple and Murthy (1994b) and Zapatero (1998) discuss the implications of differences in beliefs on financial innovations and interest rate volatility. Basak (2000) investigates the existence of a sunspot equilibrium in which an extraneous process that is uncorrelated with market fundamentals affect asset prices when agents have different beliefs. These models, however, do not investigate option pricing and trading implications of this form of heterogeneity.

The rest of the paper is organized as follows: Section 2 reviews the current literature on this topic and describes the contribution of this paper. Section 2 describes the model. The datasets are described in section 3. Section 4 presents the model's estimation and empirical evidence. The estimation differs from the literature as it uses joint information on both option prices and open interest. We discuss the results of a GMM test for the overidentifying restrictions of the model. The rest of the empirical results are articulated in the following four subsections: Section 4.1 discusses the model performance in terms of the fitting errors for option volume and open interest. Section 4.2 discusses the model performance in terms of hedging errors. Section 4.3 discusses the time series implications for the model-implied difference in beliefs and shows evidence that the Difference in Beliefs Index predicts future realized volatility even after controlling for the current implied volatility. Section 4.4 shows that the Difference in Beliefs Index can, at least partially, explain the Bakshi, Cao, and Chen (2000) no-arbitrage violation puzzle. Section 5 concludes. All proofs are contained in the Appendix.

2 The Model

We study a finite horizon general equilibrium economy in which two types of agents are endowed with shares in a production technology that generates a dividend flow. The agents are identical in terms of preferences and endowment but differ in their beliefs about the dividend growth rate. In our setup, which is related to Detemple and Murthy (1994b), options are not redundant and help hedging sources of risk.

Assumption 1. (Preferences).

The economy is populated by two sets of constant relative risk aversion (CRRA) agents who maximize *finite lifetime utility:*

$$\max E^n \left[\int_0^T \frac{c_n(t)^\gamma}{\gamma} dt \mid \mathcal{F}_t^n \right] \qquad n = 1, 2 \tag{1}$$

The two sets of agents differ in terms of their beliefs which affect their expectations.

The utility is maximized subject to a budget constraint. Let $\xi^n(t)$ be the equilibrium state price density of agent n. Following Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987) the budget constraint can be expressed as follows

$$E^n \left[\int_0^T \xi^n(t) c_n(t) dt \mid \mathcal{F}_t^n \right] = W_0 \tag{2}$$

The initial wealth $W_0 = e^n P_1(0)$, where e^n is the number of stock units with which agent n is initially endowed and $P_1(0)$ is the stock price at time zero. The martingale representation of the budget constraint in (2) says that in equilibrium the value of the future consumption stream is equal to the current value of the asset(s) of each individual. Since the individuals have different initial beliefs, the stochastic discount factor $\xi^n(t)$ is agent specific for (2) to be satisfied in equilibrium.

Assumption 2. (Dividend process).

The dividend process follows a geometric Brownian motion with stochastic drift:

$$\begin{aligned} d \ln \delta(t) &= \mu_\delta(t) dt + \sigma_\delta dW_\delta(t) \\ d\mu_\delta(t) &= (a_{0\delta} + a_{1\delta}\mu_\delta(t))dt + dW_{\mu_\delta}(t) \end{aligned} \tag{3}$$

Agents also observe a process $z(t)$ which contains a signal for the growth rate of dividends.

$$\begin{aligned} dz(t) &= (a\mu_\delta(t) + b\mu_z(t)) dt + \sigma_z dW_z(t) \\ d\mu_z(t) &= (a_{0z} + a_{1z}\mu_z(t))dt + n_z dW_{\mu_z}(t) \end{aligned}$$

Both agents are aware of $\delta(t)$ and $z(t)$ dynamics but do not know the current value of the stochastic drifts $\mu_\delta(t)$ and $\mu_z(t)$. Agents have different initial prior beliefs about the value of $\mu_\delta(0)$ and rationally update their estimate of $\mu_\delta(t)$ and $\mu_z(t)$ given the observed history of the dividend process $\delta(t)$ and the signal $z(t)$. With no loss of generality, we assume that $E(dW_\delta(t) dW_z(t)) = 0$.

The two stochastic drifts $\mu_\delta(t)$ and $\mu_z(t)$ are not observable. Agents observe the realizations of the dividend process $\delta(t)$ and a signal $z(t)$ and rationally computes posterior estimates of the drifts, given their initial priors $\mu_\delta^n(0)$ and $\mu_z^n(0)$ and all available information.[2] For convenience, let us rewrite our economy using vector notation: $Y_t = [\ln \delta_t, z_t]'$ and $\mu_t = [\mu_\delta(t), \mu_z(t)]'$, and $m_Y^n = E\left[\mu(t) | F_{Y_t}^n\right]$ so that

$$\begin{aligned} dY_t &= (A_0 + A_1 \mu_t)dt + B dW_Y(t) \\ d\mu_t &= (a_0 + a_1 \mu_t) dt + b dW_\mu(t) \end{aligned}$$

[2] For a discussion on how to endogenize the difference in beliefs see Morris (1995).

Given the different in priors, each agent observe different innovation processes dW_Y^n. Since $Y(t)$ is observable, however, the perceived innovations dW_Y^n and drifts $m^n(t)$ must be such that $Y_t^n = Y_t$. Thus:

$$\begin{aligned} dW_Y^n(t) &= B^{-1}\left[dY_t - (A_0 + A_1 m^n(t))dt\right] \\ &= dW_Y(t) + B^{-1}(\mu_t - m^n)dt \qquad n = 1, 2 \end{aligned} \tag{4}$$

Which implies that

$$dW_Y^2(t) = dW_Y^1(t) + \Psi_t dt, \qquad \Psi_t = B^{-1}(m_t^1 - m_t^2) \tag{5}$$

The processes Ψ_t is the disagreement between the two agents on the growth rates of the observable processes scaled by their volatility. It plays an important role in the characterization of the equilibrium since it parametrizes the difference in which the two agents process new information. Notice that agents may interpret the same innovation in the signal $z(t)$ differently: those with low estimates $m_z(t)$ may interpret a positive shock to $z(t)$ as good news for the dividend growth rate while those with high estimates for $m_z(t)$ may revise their estimates of the dividend growth downwards if the change in the signal is not large enough.

Market Completeness.

Uncertainty on the dividend growth rate and the correlation between the signal $z(t)$'s drift and the $\delta(t)$'s drift makes observed innovations in $z(t)$ affect the posterior estimate of the dividend process. Thus, the state space which affects security pricing is generated by a two-dimensional Brownian process dW_δ^n, dW_z^n. This implies that a market with continuous trading in the stock and a risk-free bond is incomplete. In this framework, uncertainty generates an equilibrium demand for options and changes in the difference in beliefs Ψ_t on either the dividend or the signal generate trading and changes in asset prices.

In what follows, we will assume that agents can trade in three assets: a riskless bond, a stock and an option. The bond price follows $dB(t) = B(t)\, r(t)\, dt$. The stock and option prices have the following general stochastic dynamics

$$\begin{aligned} dP(t) &= P(t)\left[\mu_p(t)\, dt + \sigma_p(t)'\, dW_Y\right] \\ dO(t) &= O(t)\left[\mu_o(t)\, dt + \sigma_o(t)'\, dW_Y\right] \end{aligned} \tag{6}$$

where the securities' expected return and volatility are endogenously determined in equilibrium. Due to uncertainty, agents have different perceived innovation processes dW_Y^n, thus they confront different *subjective* price processes with drifts μ_p^n and μ_o^n with Brownian innovation dW_Y^n. Since agents need to agree in equilibrium on security prices and the Brownian innovations dW_Y^n affect both dY_t and security prices, agents must disagree on security expected returns. The extent of the disagreement, however, is related to the difference in beliefs on fundamentals $\Psi(t)$. By substituting equation (4) into (6), it is easy to show that difference in the perceived rate of returns is:

$$\begin{aligned} \mu_p^1(t) - \mu_p^2(t) &= \sigma_p'\Psi(t) \\ \mu_o^1(t) - \mu_o^2(t) &= \sigma_o'\Psi(t) \end{aligned}$$

The difference in expected asset returns are equal to the difference in beliefs multiplied by their volatility. This clearly suggests that the difference in beliefs must also affect equilibrium risk premia. When option markets are open, the economy is dynamically complete under each perceived security price process. Thus, there exists a *unique* stochastic discount factor $\xi^n(t)$ for each of the two agents. Under no arbitrage, it follows that

$$\frac{d\xi^n(t)}{\xi^n(t)} = -r(t)\,dt - \theta^n(\delta(t), z(t))\,dW_Y^n$$

The market prices of the risk θ^n are individual specific and depends on both $\delta(t)$, as common in a Lucas tree economy, and $z(t)$. Since agents are uncertain about dividend growth rates they use the signal to make inferences. However, since the nature of the relationship between the signal and the dividend growth rate is stochastic, agents are uncertain about their interpretation of the signal. Thus, this uncertainty is priced in equilibrium: θ^n is a function of $z(t)$. This implies that asset prices can be volatile even if the dividend process is smooth. This is a desirable property given the well-known difficulty in reconciling the volatility of the stochastic discount factor with market fundamentals as discussed in the risk premium puzzle literature (see Hansen and Jagannathan (1997)).

Let us define $\pi_n(t) = (\pi_{n1}(t), \pi_{n2}(t))^\top$ as the number of consumption good units agent n invests in each security (stock and option respectively). We define the equilibrium concept as follows:

Definition 1 (Equilibrium)
An equilibrium is a collection of price processes ($P(t), O(t)$ and $r(t)$) along with individual consumption and portfolio choices ($c_n(t), \pi_n(t)$) such that:
(i) The pairs ($c_n(t), \pi_n(t)$) maximize each individual utility function subject to the budget constraints given the equilibrium prices ($P(t), O(t)$ and $r(t)$)
(ii) The goods, stock and option markets clear:

$$c_1(t) + c_2(t) = \delta(t), \quad \pi_{11}(t) + \pi_{21}(t) = 1, \qquad \pi_{21}(t) + \pi_{22}(t) = 0$$

where $c_n(t)$ are individual consumptions, $\pi_{n1}(t)$ are stock holdings and $\pi_{n2}(t)$ are option holdings for agent $n = 1, 2$.

To compute the equilibrium, we use the approach discussed in Detemple (1986). First, we solve for the optimal learning problem of each individual. Given this solution, each agent acts as a standard utility-maximizer in a world with complete information. Second, we use the individual-specific filtered dynamics of $\delta(t)$ and $z(t)$ to solve for the optimal portfolio problem. Finally, we aggregate the two optimal portfolio solutions and use the market clearing conditions to obtain equilibrium asset prices.

Since the drift of $\delta(t)$ and $z(t)$ are unobservable, individuals rationally compute their best estimate $m_\delta^n(t)$ and $m_z^n(t)$ using their priors and all available information. Using standard results in filtering theory, it is possible to prove the following results.

Proposition 1 (Learning)
(a) Let $m(t) = E\left[\mu(t)|\mathcal{F}_t^Y\right]$ and $\gamma(t) = E\left[(\mu_t - m_t)(\mu_t - m_t)'|\mathcal{F}_t^Y\right]$. Under some technical regularities, m_t and γ_t are unique continuous $\mathcal{F}_t^Y$-measurable for any t solutions of the system of equations

$$dm_t = [a_0 + a_1 m_t]dt + \gamma_t A_1'(BB')^{-1}\left[dY_t - (A_0 + A_1 m_t)dt\right]$$
$$\dot{\gamma}_t = a_1\gamma_t + \gamma_t a_1' + bb' - \gamma_t A_1'(BB')^{-1}A_1\gamma_t$$

with initial conditions $m_0 = E(\mu_0|\mathcal{F}_0^Y)$, $\gamma_0 = E\left[(\mu_0 - m_0)(\mu_0 - m_0)'|\mathcal{F}_t^Y\right]$. If the matrix γ_0 is positive definite, then the matrices γ_t, $0 \leq t \leq T$ will be positive definite as well.
(b) Moreover, if $\sum_{i=1}^2 E(\mu_i)^4 < \infty$ and if the initial belief $P(\mu_0 \leq a|Y_0)$ on the drift μ_t is conditionally Gaussian $N(m_0, \gamma_0)$ then for any t_j, $0 \leq t_0 < t_1 ... < t_n \leq t$ the conditional distribution $P(\mu_{t_0} \leq a_0,$ $... , \mu_{t_n} \leq a_n|\mathcal{F}_t^Y)$ is Gaussian.

The posterior belief on the dividend growth rate is stochastic and depends on future realizations of the observable variables Y_t. The sensitivity of the revision in the posterior belief depends on the "confidence" level γ_t. The lower γ_t, the higher the confidence in the initial prior and thus the lower the revision of the posterior given new observations on Y_t. The reverse is true for the less confident agent. Notice that when the two agents have different prior variances, i.e. $\gamma^1(0) \neq \gamma^2(0)$, the difference in beliefs $m^1(t) - m^2(t)$ is stochastic even if the process Y_t is commonly observed.

Remark 1. In the special case in which agents have heterogeneous initial beliefs $m^i(0) \neq m^j(0)$ but are homogeneous in terms of confidence, i.e. $\gamma^i(0) = \gamma^j(0)$, then the difference in beliefs $\Psi(t)$ follows an *o.d.e.* $\frac{d}{dt}\Psi = \tilde{A} + \tilde{B}\Psi$, even though each belief is itself stochastic:

$$\frac{d}{dt}\Psi_\delta = \tilde{A}_1 + \tilde{B}_{11}\Psi_\delta + \tilde{B}_{12}\Psi_z$$
$$\frac{d}{dt}\Psi_z = \tilde{A}_2 + \tilde{B}_{21}\Psi_\delta + \tilde{B}_{22}\Psi_z$$

The system admits solution $\Psi_\delta(t) = \varkappa_{\delta,o} + \varkappa_{\delta,1}\exp Q_1 t + \varkappa_{\delta,2}$ and $\Psi_z(t) = \varkappa_{z,o} + \varkappa_{z,1}\exp Q_1 t + \varkappa_{z,2}\exp Q_2 t$. The parameters $\varkappa$, Q_1 and Q_2 are discussed in the appendix. Convenient closed form solutions can be obtained using the stationary solution S of γ which satisfies $\dot{\gamma}_t = 0$ for $\gamma_t = S$.

Given the differences in beliefs $\Psi_\delta(t) = \frac{m_\delta^1(t) - m_\delta^2(t)}{\sigma_\delta(t)}$ and $\Psi_z(t) = \frac{m_z^1(t) - m_z^2(t)}{\sigma_z(t)}$ and the subjective process for dY perceived by each agent, we can solve for the equilibrium. Notice, however, that since agents have heterogeneous beliefs about the dynamics of the two underlying factors, they have different state price densities. Each state price density is a function of the individual specific beliefs. Thus, to solve for the equilibrium it is convenient to use the aggregation technique of Cuoco and He (1994) and Karatzas and Shreve (1998) in which the representative agent utility function is constructed by taking an average of individual utilities with state-dependent weights:

$$U(c, \eta) = \max_{c_1 + c_2 = c} c_1(t)^\gamma/\gamma + \lambda_t\left(c_2(t)^\gamma/\gamma\right) \tag{7}$$

The first order conditions for the optimal consumption plan of agent n are given by $c_n(t) = (y_n\xi^n(t))^{1/\gamma - 1}$, where y_n is the (constant) Lagrange multiplier associated to the budget constraint

(2). Thus, y_n must satisfy

$$E^n\left[\int_0^T \xi^n(t)\left(y_n\xi^n(t)\right)^{1/\gamma-1}dt \mid \mathcal{F}_t^n\right] = e^n P_1(0) \tag{8}$$

Imposing the market clearing condition $c_1(t) + c_2(t) = \delta(t)$, we can obtain the equilibrium consumption allocation of the two agents and the two stochastic discount factors. The results are summarized in the following Proposition:

Proposition 2 (Equilibrium)
In equilibrium, the relative weight of the second agent λ is state dependent and equal to $\lambda_t = \frac{y_1}{y_2}\eta_t$, with $\eta_t = \xi^1(t)/\xi^2(t)$ and the individual state price densities being equal to:

$$\xi^1(t) = [\delta(t)]^{\gamma-1}\frac{1}{y_1}\left(1+\left(\frac{y_1\eta(t)}{y_2}\right)^{\frac{1}{1-\gamma}}\right)^{1-\gamma} \tag{9}$$

y_n are constant satisfying the individual static budget constraint

$$E^n\left[\int_0^T \xi^n(t)\left(y_n\xi^n(t)\right)^{1/\gamma-1}dt \mid \mathcal{F}_{Y_0}^n\right] = e^n P_1(0)$$

Moreover, the relative marginal utilities evolve according to $\frac{d\lambda_t}{\lambda_t} = \frac{d\eta_t}{\eta_t}$

$$\frac{d\eta(t)}{\eta(t)} = -\Psi_\delta(t)\,dW_\delta^1(t) - \left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)dW_z^1(t) \tag{10}$$

The individual optimal consumption allocations are given by

$$c_1(t) = \delta(t)\frac{1}{\left(1+\left(y_1\eta(t)/y_2\right)^{\frac{1}{1-\gamma}}\right)};\quad c_2(t) = \delta(t)\frac{\left(y_1\eta(t)/y_2\right)^{\frac{1}{1-\gamma}}}{\left(1+\left(y_1\eta(t)/y_2\right)^{\frac{1}{1-\gamma}}\right)} \tag{11}$$

The main result is that since in equilibrium the relative weight λ_t of the second agent is proportional to the ratio of the two stochastic discount factors, then it must be a function of the difference in beliefs. Thus, the relative consumption share of the two agents is stochastic even if agents have identical CRRA preferences and the dividend volatility is deterministic. The stochastic process $\eta(t)$ plays a crucial role in the determination of equilibrium quantities and asset prices. The higher is $\eta(t)$, the higher is the relative share of the dividend allocation to the second agent. When $m_\delta^1(t) > m_\delta^2(t)$, the first agent is more optimistic about the dividend growth rate than the second agent, a negative dividend shock $d\hat{W}_\delta^n(t) < 0$ has two effects. First, the overall consumption level decreases. Second, $\eta(t)$ increases, thus increasing the second (pessimist) agent's weight and therefore his relative consumption share.

The difference in the two stochastic discount factors is driven by the two agents perceived innovations dW_Y. The following Proposition describe the market prices of risk for the two agents:

Proposition 3 *In equilibrium, agents are averse to both dividend and signal shocks. The two agent-specific prices of risk are equal to*

$$\theta_\delta^1(t) = (1-\gamma)\sigma_\delta + \frac{\Psi_\delta(t)(y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}}{\left(1+(y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}\right)}; \qquad \theta_\delta^2(t) = (1-\gamma)\sigma_\delta - \frac{\Psi_\delta(t)}{\left(1+(y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}\right)}$$

$$\theta_z^1(t) = \frac{(y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z}+b\Psi_z(t)\right)}{\left(1+(y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}\right)}, \qquad \theta_z^2(t) = -\frac{\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z}+b\Psi_z(t)\right)}{\left(1+(y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}\right)} \tag{12}$$

The price of dividend risk. The market price of dividend risk $\theta_\delta^1(t)$ is equal to the sum of two terms. The first term is standard and is equal to the product between the dividend volatility and the coefficient of relative risk aversion. The second term is new. When agents have different beliefs, the Pareto weight process is stochastic and its volatility is proportional to the difference in beliefs Ψ_δ. From equation (10), we see that when Ψ_δ is positive, the larger is Ψ_δ the more a negative shock to dividends, $dW_\delta(t)$, increases the Pareto weight of the second (pessimistic) agent. This means that the consumption of the first agent is reduced not only because aggregate dividends decline but also because his relative consumption allocation is reduced: the first (optimistic) agent ex-ante insures the second (pessimistic) agent. The extent of this effect is proportional to the difference in beliefs.

The price of the signal risk. In an economy with homogeneous agents, i.e. $\Psi_\delta = \Psi_z = 0$, the signal is used by each agent to update the estimate of the dividend growth rate. However, it is not priced. To reason is that in such an economy $Cov(dW_z, d\eta) = 0$: unexpected changes in the signal are not correlated with marginal utilities. When people have heterogeneous beliefs, however, the economy supports an equilibrium in which the signal itself has a positive market price of risk. This may at first seem strange since the signal does not separately affect consumption. Agents, however, are averse to innovations in the volatility since it changes the state-price density ratio and therefore their consumption share.

It is instructive to notice that the difference in the two prices of risk for each factor is proportional to the difference in beliefs:

$$\theta_\delta^1(t) - \theta_\delta^2(t) = \Psi_\delta(t) \qquad \theta_z^1(t) - \theta_z^2(t) = \left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)$$

When $\Psi_\delta = 0$ the dividend price of risk becomes agent independent; when both $\Psi_\delta = 0$ and $\Psi_z = 0$ the signal is no longer a priced source of risk and $\theta_z^1(t) = \theta_z^2(t) = 0$.

2.1 Asset Pricing

Given the stochastic discount factor $\xi^1(s)$, the stock price is equal to

$$P(t) = \frac{1}{\xi^1(t)} E_t^1\left[\int_t^T \xi^1(s)\,\delta(s)\,ds\right]$$

In equilibrium asset prices can be computed using the expectation operator and the stochastic discount factor of either agent. For simplicity we consider the pricing from the first agent's perspective. European stock options expiring at time H with final payoff equal to $C(H, \delta(H)) = \max(P(H) - K, 0)$ can be valued using the same stochastic discount factor, so that

$$C(t, H) = E_t^1 \left[\xi^1(H) \max(P(H) - K, 0)\right] / \xi^1(t).$$

The term structure of bond prices paying a unit of the numeráire at time H is given by $Q(t, H) = E_t^1 \left[\xi^1(H)\right] / \xi^1(t)$.

The pricing kernel depends on both the cash flow process $\delta(t)$ and the differences in beliefs $\Psi(t)$ that affect the stochastic weight process $\eta(t)$. Proposition A.1 (see Appendix) characterizes the joint distribution of $\delta(t)$ and $\eta(t)$, which is shown to be log-Normal. Using this result and Proposition 1, we obtain the following representation for stock, options and bonds prices:

Proposition 4 (Asset Pricing)
Let $N_2(z_\eta, z_\delta)$ be a standardized Bivariate Normal distribution. The equilibrium stock price is:

$$P(t) = \frac{1}{\xi^1(t)} \int_{-\infty}^{+\infty} \left[\int_t^T \frac{F_\delta(t, s)^\gamma}{(1 + F_\eta(t, s))^{\gamma - 1}} ds \right] dN_2(z_\eta, z_\delta)$$

European Call Option prices with time to maturity H are equal to

$$C(t, H) = \frac{1}{\xi^1(t)} \int_{-\infty}^{+\infty} \left[\frac{F_\delta(t, H)^{\gamma - 1}}{(1 + F_\eta(t, H))^{\gamma - 1}} \max(P(H) - K, 0) \right] dN_2(z_\eta, z_\delta)$$

The bond price is equal to

$$Q(t, H) = \frac{1}{\xi^1(t)} \int_{-\infty}^{+\infty} \left[\frac{F_\delta(t, H)^{\gamma - 1}}{(1 + F_\eta(t, H))^{\gamma - 1}} \right] dN_2(z_\eta, z_\delta)$$

where

$$\begin{aligned} F_\eta(t, s) &= F(\eta(t), y_1, y_2, s, t, z_\eta) = \left(\frac{y_1 \eta(t) e^{M_\eta(s,t) - \sqrt{V_\eta(s,t)} z_\eta}}{y_2} \right)^{\frac{1}{1-\gamma}} \\ F_\delta(t, s) &= F(\delta(t), y_1, y_2, s, t, z_\delta) = \left(\delta(t) e^{M_\delta(s,t) - \sqrt{V_\delta(s,t)} z_\delta} \right) \end{aligned} \tag{13}$$

and $M_\eta(s, t)$, $V_\eta(s, t)$, $M_\delta(s, t)$, $V_\delta(s, t)$, z_η and z_δ are as in Proposition A.1, equation (A.6).

Given the functional form of $F_\eta(s, t)$ and $F_\delta(s, t)$, asset prices are deterministic integrals and can be computed at any desired level of accuracy using standard numerical integration methods. To investigate the pricing implications of the model, we calibrate μ_δ and σ_δ to the growth rate and volatility of the $S\&P500$ dividend from 1950 to 2000, i.e. 3% and 5% respectively, and plot equilibrium asset pricing properties as a function of the difference in beliefs. We assume that the volatility of the signal equals that on dividends, i.e. $\sigma_z = 5\%$. Moreover, we calibrate the volatility of the stochastic drift of the dividend and signal processes to half the volatility of their levels, i.e. $n_\delta = n_z = 2.5\%$.[3] We assume a relatively small value for the risk aversion coefficient, i.e. $\gamma = -\frac{1}{2}$. In a log-utility economy, $\gamma = 0$ and differences in beliefs do not affect stock prices. When $\gamma < 0$ agents are more risk averse than a log-utility investor.

A. Stock Volatility and Equity Premiumy. The time variation in the difference in beliefs generates endogenous stochastic volatility. We obtain the formal relationship between the volatility, $\Psi_\delta(t)$ and $\Psi_z(t)$ by applying Ito's lemma to Proposition 4.[4] Figure 1 Panel B shows that when both differences in beliefs are zero, the stock volatility equals the dividend volatility, which is 5%. An increase in both differences in beliefs from zero to 0.5, however, increases the stock volatility from 5% to 13%, which is close to the average volatility of the S&P500 index. A further increase in both differences in beliefs to a value of 1 increases the stock volatility to 25%. The difference in beliefs about the dividend more strongly affects the stock volatility than that about the signal. The economic intuition linking the differences in beliefs and endogenous stochastic volatility is as follows: since differences in beliefs make each agent have their own specific stochastic discount factor $\xi^n(t)$ and the relative weights $\eta(t)$ of the two agents in the representative agent utility function is equal to the ratio of the two stochastic discount factors, the volatility of $\eta(t)$ is proportional to the difference in beliefs (see equation (10)). The consumption process and the investment decision in the stock is directly affected by the difference in beliefs even in the absence of dividend shocks (see equation (11)). This implies that the stock volatility is a function of the volatility of $\eta(t)$ and thus of the level of Ψ_δ and Ψ_z, i.e. $\sigma_p = \frac{\partial P}{\partial \eta}\sigma_\eta$.

The time variation in the difference in beliefs can reconcile relatively high stock volatility with low dividend volatility. Can the model, however, match the observed excess returns? The required excess return on a risky security is $\sigma_{i\delta}(t)\,\theta_\delta^n(t) + \sigma_{iz}(t)\,\theta_z^n(t)$. When both differences in beliefs are zero, the only priced source of risk is the dividend innovation. Further, the price of risk is linear in the coefficient of risk aversion $-\gamma$ as in the standard single-agent CRRA utility model. In this case, the model can generate an excess return of 8% with the low observed sample dividend and price volatilities only at the cost of assuming a coefficient of risk aversion of $\gamma = -24$. When the difference in beliefs increases from 0 to 1, however, the price of dividend risk θ_δ increases six-fold and the sample equity risk premium can be matched even with $\gamma = -2$. This occurs because a negative dividend realization not only reduces the cash flow stream, but it also affects the marginal valuation through the stochastic weight process $\eta(t)$. The higher the difference in beliefs, the higher the effect

[3] This parameter is calibrated from the values obtained by Xia (2001) using a VAR approach.

[4] $dP(t)/P(t) = \mu_p^1(t)\,dt + \sigma_{p\delta}(t)\,dW_\delta^1(t) + \sigma_{pz}(t)\,dW_z^1(t)$ where $\sigma_{p\delta}(t)$ equals $(\partial P/\partial\delta)\,\delta(t)\,\sigma_\delta - (\partial P/\partial\eta)\,\Psi_\delta(t)\,\eta(t)$ and $\sigma_{pz}(t)$ equals $(\partial P/\partial\eta)\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)\eta(t)$. Thus, the total stock volatility is equal to $\sigma^2\left(\frac{dP}{P}\right) = \left(\frac{\partial P}{\partial\delta}\delta(t)\,\sigma_\delta - \frac{\partial P}{\partial\eta}\Psi_\delta(t)\,\eta(t)\right)^2 + \left(\frac{\partial P}{\partial\eta}\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)\eta(t)\right)^2$.

of a dividend shock on $\eta(t)$ and the higher the price of risk.[5]

B. <u>Stock Prices</u>. Figure 1 Panel A shows that the difference in beliefs negatively affects the stock price level: the greater is the difference in beliefs (either about the dividend or about the signal) the smaller is the stock price. An increase in the difference in beliefs about the dividend (or signal) from 0 to 1 drives the value of the stock down by about 10%, from 33 to 30 consumption units. The intuition for this is that an increase in the difference in beliefs generates both a higher expected stock growth rate and a higher stock price volatility. When $\gamma < 0$, at the calibrated parameters, the higher dividend growth rate does not compensate for the increase in volatility. Further, the equilibrium stock price declines as the stock price becomes more heavily influenced by the (pessimistic) agent whose estimate of the dividend growth rate is lower than the actual value. The situation is reverse for positive values of γ (lower risk aversion than log-utility).

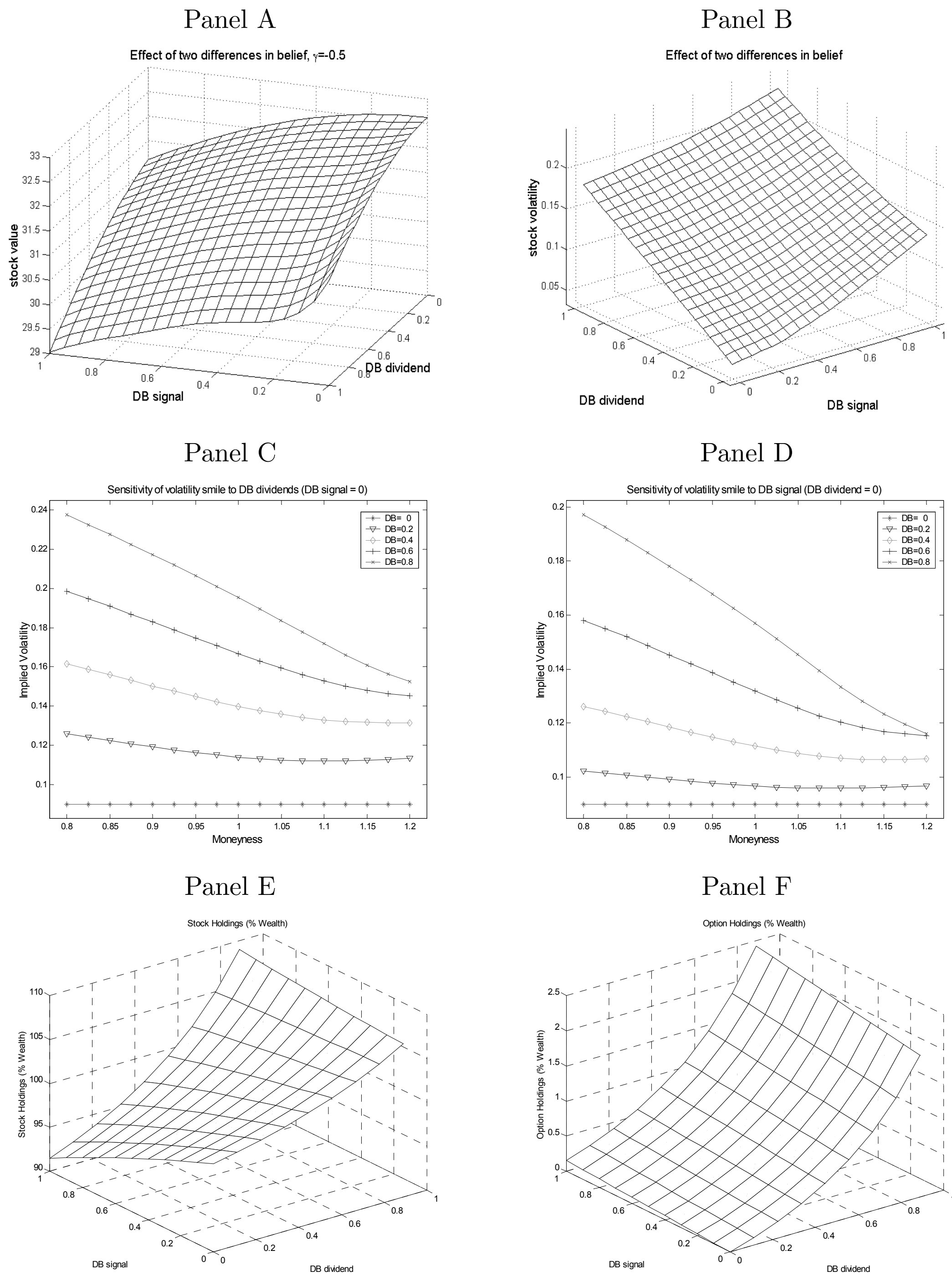
C. <u>Option Prices</u>. The difference in beliefs generates an option-implied smile. Figure 1, Panels C and D illustrate how the difference in beliefs impacts the option-implied volatility surface. Setting both differences in beliefs to zero yields a standard one-factor, Black-Scholes (BS) economy. The volatility smile is flat since in this case the stock price transition density is log-normal (see Appendix). When only one of the two differences in beliefs is non-zero, options become non-redundant. The effect of the differences in beliefs however is asymmetric. In-the-money calls (out-of-the-money puts) are more expensive than out-of-the-money calls (in-the-money puts). When the difference in beliefs about the dividend process increases from 0 to 0.6, the smile level (at-the-money volatility) increases from 7% to 17%. The slope of the volatility smile is significant and similar to the one commonly observed in the market: a 20% out-of-the-money put is priced at 20% implied volatility while the out-of-the-money call is priced at 15% implied volatility. The difference in beliefs affects the signal on the volatility smile similarly though with smaller magnitude. The smile level increases from 7% to 14%, when the difference in beliefs about the signal increases from 0 to 0.6. The corresponding smile slope is about 3.5%. The difference in beliefs creates a natural demand for insurance. When $\mu_\delta^1 > \mu_\delta^2$, so that $\Psi_\delta > 0$, trade occurs and the first (optimist) agent insures the consumption stream of the second (pessimist) agent writing OTM puts in exchange for OTM calls. The market value of the two options is, however, not symmetric. When $\Psi_\delta > 0$ a negative dividend shock $dW_\delta < 0$ increases $\eta(t)$ and thus the relative consumption share of the pessimist. This implies that negative dividend shocks increase the marginal utilities of the optimist more than that of the pessimist. Thus, in equilibrium, the optimist requires a higher price to write OTM puts and hedge the consumption

[5] In equilibrium, the required excess return is equal to:

$$\mu_P^1(t) - r(t) = (1-\gamma)\, cov\left(\frac{dP(t)}{P(t)}, \frac{d\delta(t)}{\delta(t)}\right) - \frac{(y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}}{\left(1 + (y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}\right)} cov\left(\frac{dP(t)}{P(t)}, \frac{d\eta(t)}{\eta(t)}\right) \tag{14}$$

The first term is the standard consumption CAPM where a risky security's risk premium is positively related to the covariance of its return with the return of the dividend (aggregate consumption). In the second term, the risk premium is decreasing in the covariance between this asset and the weight process. When $\eta(t)$ is high the representative agent puts less weight on the first agent. An asset negatively correlated with changes in $\eta(t)$ is more valuable to the first agent, therefore requiring smaller risk premium. For the second agent the situation is reverse.

stream of the pessimist than the pessimist requires to write OTM calls. This generates a (stochastic) smile whose slope depends on the extent of the difference in beliefs.

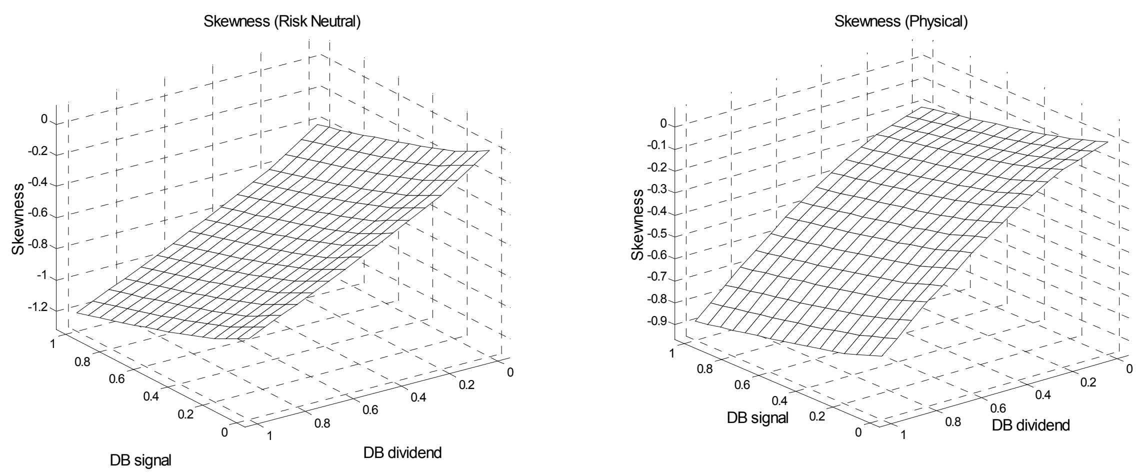


The figure presents how differences in beliefs affect the stock price (Panel A), stock volatility (Panel B), implied volatility smiles (Panels C and D) and optimal portfolio holdings (Panels E and F). On Panels C and D we fix one of the differences in beliefs at zero to illustrate the convergence to BS and change the other difference in beliefs between 0 and 1, solving for the price of options with different moneyness. Time to maturity is fixed at 1 month. On Panels E and F we change both differences in beliefs, re-solve for the equilibrium solution and present the portfolio holdings. Both holdings are calculated for the first agent (optimist) and are expressed in terms of wealth. Option holdings are expressed in terms of the option notional value multiplied by option delta.

FIGURE 1: **Effect of Difference in Beliefs**

D. Skewness. An important stream of the asset pricing literature analyzes the role of negative skewness in equity returns and the option smile. Ait-Sahalia and Lo (1998), Bakshi, Cao, and Chen (1997), Bates (2000), Duffie, Pan, and Singleton (2000), Madan, Carr, and Chang (1998), Pan (2002), and Rubinstein (1994) investigate the asymmetry in the underlying stock return risk neutral distribution. Bakshi, Kapadia, and Madan (2003) describe the link between risk-neutral skew, risk aversion and higher order moments of the physical distribution. Our model generates an endogenous volatility skew under both the risk neutral and physical measures. The results are summarized in Figure 2. The value of the risk-neutral skewness is close to -1.09, as reported by Bakshi, Kapadia, and Madan (2003) for the OEX index. Moreover, we find that the model-implied risk-neutral skewness is higher than the physical skewness, as discussed in Bakshi, Kapadia, and Madan (2003). The magnitude of the physical skewness ranges from 0, when both differences in beliefs are zero, to -0.8, when both differences in beliefs are unity. This compares to a sample historical skewness equal to -0.7 for $S\&P500$ monthly returns.

The intuition for the model-implied skewness is similar to that for the negative slope of the implied volatility smile. When negative stock returns follow a negative dividend shock, the first agent (optimist) receives an additional negative reallocation shock ($d\eta(t) > 0$). His consumption falls, his marginal utility increases and he becomes more risk averse, thus demanding a larger risk premium to hold the stock. This reduces the stock price by more than what would be caused simply by a dividend reduction. This effect generates an asymmetry in the risk-neutral distribution making the left tail extend more towards negative values. This is known as negative skewness.



These two figures present the skewness of the physical and risk-neutral distribution of stock returns as a function of the two differences in beliefs.

Figure 2: **Skewness**

2.2 Optimal Portfolio Holding and Volume

Optimal portfolio holding can be obtained by equating the dynamic budget constraint to its martingale representation (Cox and Huang (1989)). First we calculate the first agent's optimal position. Then, that of the second agent can be derived from the asset market clearing conditions (the stock net supply is one unit, while the option's is zero). The dynamic budget constraint satisfies the following stochastic differential equation:

$$dX^1(t) = -c_1(t)\,dt + X^1(t)\,r(t)\,dt + \pi_1(t)^\top \left(\mu_i^1(t) - r(t)\right) dt + \pi_1(t)^\top \sigma(t)\,dW_Y^1(t) \tag{15}$$

where $\pi_1(t) = (\pi_{11}(t), \pi_{12}(t))^\top$ is the number of units of the consumption good invested in each security (stock and option respectively) by the first agent and $dW_Y^1(t) = \left(dW_\delta^1(t), dW_z^1(t)\right)^\top$. Given the equilibrium stochastic discount factor, at any time $t < T$ the wealth level must also satisfy the static budget constraint:

$$X^1(t) = \frac{1}{\xi^1(t)} E^1\left[\int_t^T c_1(s)\,\xi^1(s)\,ds | \mathcal{F}_t^1\right] \tag{16}$$

Since $c^n(t)$ and $\xi^n(t)$ are functions of the state variables $\delta(t)$ and $\eta(t)$, Ito's Lemma applied to (16) yields:

$$\begin{aligned} dX^1(t) - E\left[dX^1(t)|\mathcal{F}_t^1\right] &= \frac{\partial X^1}{\partial \delta}\sigma_\delta \delta(t)\,dW_\delta^1(t) - \frac{\partial X^1}{\partial \eta}\Psi_\delta(t)\,\eta(t)\,dW_\delta^1(t) \\ &\quad - \frac{\partial X^1}{\partial \eta}\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)\eta(t)\,dW_z^1(t) \end{aligned} \tag{17}$$

The stochastic differential equations (15) and (17) are Ito representations of the same wealth process with respect to the same basis of Brownian motions. Thus, the coefficients of the Brownian motions must be identical in any state of the world. This yields the following system of two equations with two unknowns:

$$\pi_1(t)^\top \sigma(t) = \left[\frac{\partial X^1}{\partial \delta}\sigma_\delta \delta(t) - \frac{\partial X^1}{\partial \eta}\Psi_\delta(t)\,\eta(t), \quad -\frac{\partial X^1}{\partial \eta}\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)\eta(t)\right],$$

$$\text{where } \sigma(t) = \begin{pmatrix} \frac{\partial P(t)}{\partial \delta}\sigma_\delta \delta(t) - \frac{\partial P(t)}{\partial \eta}\Psi_\delta(t)\,\eta(t) & -\frac{\partial P(t)}{\partial \eta}\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)\eta(t) \\ \frac{\partial O(t)}{\partial \delta}\sigma_\delta \delta(t) - \frac{\partial O(t)}{\partial \eta}\Psi_\delta(t)\,\eta(t) & -\frac{\partial O(t)}{\partial \eta}\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)\eta(t) \end{pmatrix}$$

Solving for the portfolio holding, we obtain the following result for the first agent's optimal position:

Proposition 5 (Stock and Option Holding)
The optimal portfolio holding is

$$\pi_1(t) = \begin{bmatrix} \pi_{1,1}(t) \\ \pi_{1,2}(t) \end{bmatrix} = [\underset{(4\times 4)}{\sigma(t)^\intercal}]^{-1} \begin{bmatrix} \frac{\partial X^1}{\partial \delta}\sigma_\delta \delta(t) - \frac{\partial X^1}{\partial \eta}\Psi_\delta(t)\,\eta(t) \\ -\frac{\partial X^1}{\partial \eta}\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)\eta(t) \end{bmatrix} \tag{18}$$

The partial derivatives of the first agent's wealth, stock price and option price with respect to the dividend and stochastic weight are fully characterized in the Appendix.

The trading volume in each security can be calculated as the changes over time in the optimal portfolio holding. Equation (18) shows that the optimal portfolio holding is affected by the difference in beliefs about both the dividend and signal processes. Hence, positive trading volume can be observed even in the absence of dividend shocks.

The open interest is equal to

$$\widehat{OI}_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i\right) = \pi_{1,2}\left(t\right)/P\left(t\right) \quad (19)$$

where $\pi_{1,2}(t)$ is the portfolio holdings (in consumption units) invested in options as derived in equation (18). For convenience we standardize the open interest with respect to the underlying asset's value.

Figure 1, Panels E and F show how changes in the differences in beliefs impact the optimal portfolio holdings of at-the-money call options. When both differences in beliefs Ψ_δ and Ψ_z are zero, agents are identical and each holds his initial stock endowment. When the differences in beliefs increase, the first agent's growth rate estimate increases compared to that of the second agent. An increase in the difference in beliefs about the dividend from 0 to 1 increases the stock holding by 8% and options holding from 0% to 2% of wealth. This range of values compares to an actual market capitalization of S&P500 options which is about 3% of the S&P500 market capitalization. Change in the signal difference in beliefs from 0 to 1 increases the option holding by 0.5%. Options holding reacts less to signal DB than to dividend DB. The difference in beliefs creates a natural demand for insurance. When $\mu_\delta^1 > \mu_\delta^2$, the first agent is more optimistic about the dividend growth rate and acts as insurer of the second (pessimist) agent's consumption stream by writing OTM puts in exchange for OTM calls. The open interest level signals the extent of risk sharing between agents with different beliefs.

3 The Dataset

CBOE Options data

The dataset includes daily information on the $S\&P500$ index options from October 1986 to August 1996. It contains option trading prices, bid-ask quotes, volumes, open interest and the underlying index price. A key feature of the dataset is detailed open interest information.

S&P500 options are among the most actively traded derivative securities in the world. They are European and have no wildcard features (see Fleming and Whaley (1994)). They normally expire on the third Friday of the contract month. The expiration dates follow the March quarterly cycle (March, June, September, December). Originally, these options expired only at the market close and were denoted by the ticker SPX. In June 1987, the CBOE introduced a second set of options that expire at the market open, with ticker NSX. On August 24, 1992 the CBOE reversed the ticker symbols of the two option contracts. Our sample contains SPX options throughout: close-expiry options until August 24, 1992 and open-expiry thereafter. Following Dumas, Fleming, and Whaley (1998), during the first sub-period (close-expiry) we measure the option's time to expiration using the number of calendar days between the trade date and the expiration date. During the second subperiod (open expiry) we use the number of calendar days minus one. To avoid the problem of bid/ask bounce we follow Ait-Sahalia and Lo (1998) and use the midpoint of bid and ask quotes.

Three main issues affect any option dataset when used for asset pricing purposes: First, in-the-money options are much more illiquid than at- and out-of-the-money options. This reflects the demand by portfolio insurers for out-of-the-money puts. When the volume of in-the-money options is small, the recorded quoted prices are known to be noisy. Second, since the S&P500 index futures are traded on the Chicago Mercantile Exchange (CME) while options are traded at the CBOE, reported quotes may not be perfectly synchronized across the two markets. Third, the ex-ante future dividend rate is not known since the S&P500 only provides an ex-post series of realized dividends. We address these issues using the same approach as in Ait-Sahalia and Lo (1998). This consists of using the put-call parity and the spot-futures parity to infer the synchronous futures price and the implied future dividend rate respectively. Interest rates are obtained from the LIBOR market. We refer to Ait-Sahalia and Lo (1998) for specific details on these three filters. Finally, we delete any option that violates basic arbitrage restrictions:

$$Se^{-dT} \geq C_i \geq \max(0, Se^{-dT} - Ke^{-rt}) \quad \text{and} \quad e^{-rT}K \geq P \geq \max\left(e^{-rT}K - Se^{-dT}, 0\right)$$

These deleted options constitute less than one percent of the dataset. We then calculate implied volatilities and eliminate all options with implied volatility over 100%.

Table 1 shows the summary statistics for option volume and open interest. Most of the option open interest is concentrated in short maturity/near-the-money contracts beyond which the volume declines quickly. For instance, about 75 percent of option trading volume is in options with moneyness between 0.98 and 1.02. Table 2 describes the annual dynamics of the option trading volume for different moneyness levels. Since the trading volume contains a deterministic time trend, we follow Gallant, Rossi, and Tauchen (1992) and detrend the volume time series assuming a quadratic deterministic component.

The Difference in Beliefs Index.

We use both the Survey of Professional Forecasters[6] and the Consumer Confidence Survey to construct a proxy for the difference in beliefs about fundamentals.

> **Survey of Professional Forecasters.** Every three months, the Federal Reserve Bank of Philadelphia conducts a survey of economic variable forecasts (including output, inflation, and interest rates), which are prepared by private sector economists. On average, there are 30 forecasters participating in each survey and the composition is relatively stable. The participants to the survey are asked to forecast approximately 27 economic variables over the subsequent five quarters. We focus on GDP, GDP implicit price deflator, corporate profits after tax, civilian unemployment, industrial production and the start of new housing units. These are the variables most related to our definition of the real valued economic fundamentals.

[6] David and Veronesi (2001) use the Survey of Professional Forecasters data to approximate the estimation error of the representative agent and uncertainty in their model.

Consumer Confidence Index. This survey is conducted for The Conference Board by the NFO Research Company. Questionnaires are mailed to a nationwide representative sample of 5,000 households. The data has been available since June 1977 and the questions have not changed. The Index is based on responses to 5 questions included in the survey: (1) Appraisal of current business conditions; (2) Expectations regarding business conditions six months hence; (3) Appraisal of the current employment conditions; (4) Expectations regarding employment conditions six months hence; (5) Expectations regarding their total family income six months thereafter. For each of these 5 questions there are three possible answers: positive, negative and neutral. The response proportions to each question are seasonally adjusted. The relative value is defined as the percentage of the positive answer with respect to the sum of positive and negative and is initialized to the beginning of 1985. The Consumer Confidence Index is the average of all 5 Indices. The Present Situation Index averages the indexes for questions 1 and 3, while the Expectations Index averages the indices for questions 2, 4, and 5.

We construct a Dispersion in Beliefs Index (Ψ_δ) about market fundamentals based on both surveys. Each serves different goals. The Survey of Professional Forecasters is based on a restricted pool of market professionals, while the Consumer Confidence Index is available at monthly frequency. First, we compute the time series of the cross-sectional standard deviations of the beliefs from both surveys. Then, since the sampling frequency of the two surveys is different, we aggregate information from the two time-series by computing the first principal component. We find that this explains about 80 percent of the original time series variations. Figure 3 (left panel) shows the dynamics of the difference in beliefs implied from the Consumer Confidence Survey and by the Survey of Professional Forecasters. Figure 3 (right panel) shows the Difference in Beliefs Index.

4 Empirical Results

The empirical analysis is organized as follows: We estimate the DB model using its key insight: option prices and open interest are simultaneously influenced by differences in beliefs. Thus, we first characterize the joint moment conditions on option prices and open interests and use a panel data on the CBOE to estimate the model. Second, we assess the model's performance using out-of-sample (cross-sectionally) option pricing errors and open interest. Third, we compare these results to an alternative reduced-form model with exogenous stochastic volatility derived from Heston (1993) and Liu and Pan (2003). Fourth, we describe the performance of the model in terms of hedging errors at different (out-of-sample) horizons. Fifth, we study how current changes in the difference in beliefs affect future stock volatility and the shape of the option-implied volatility smile.

Let $C_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right)$ and $OI_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right)$ be the model-implied call option prices and open interest respectively, with Θ being the structural parameters of the model and K_i the strike prices. Let the empirical values of the option prices and open interest be $C_t\left(K_i/S, T_i\right)$ and $OI_t\left(K_i/S, T_i\right)$ respectively. We estimate the structural model by minimizing a GMM quadratic criterion defined in terms of the estimation errors for the option prices, open interest, and the first two moments of the dividend process. For easier interpretation and better numerical properties, we standardize the estimation errors by the observed level of the endogenous variables. Thus, the estimation errors are percentage deviations, rather than absolute dollar deviations.

$$\min \sum_{t=2}^{T} h_t\left(\hat{\Theta}\right)' \sum^{-1} h_t\left(\hat{\Theta}\right)$$

where

$$h_t\left(\hat{\Theta}\right) = \left[\begin{array}{l} \frac{C_t\left(\Theta,\Psi_{z,t},\Psi_{\delta,t},K_i/S,T_i\right)}{C_t(K_i/S,T_i)} - 1 \\ \frac{OI_t\left(\Theta,\Psi_{z,t},\Psi_{\delta,t},K_i/S,T_i\right)}{OI_t(K_i/S,T_i)} - 1 \\ \left[\ln\left(\frac{\delta_{t+1}}{\delta_t}\right)\right] - \mu_\delta \\ \left[\ln\left(\frac{\delta_{t+1}}{\delta_t}\right)\right]^2 - \left[\sigma^2\left(\frac{d\delta}{\delta}\right) + \mu_\delta^2\right] \end{array}\right]$$

The last two entries of the vector $h_t\left(\hat{\Theta}\right)$ are specification errors for the first two moments of the dividend process. This enforces the structural parameters to be empirically directly related to the dynamics of the actual dividend process. Notice that since the dividend drift is stochastic, the second moment of the dividends is also affected by the drift volatility.[7] $\widehat{\Psi}_{\delta,t}$ is the Differences in Beliefs Index directly obtained from survey data as earlier described. At each time step t, $\widehat{\Psi}_{z,t}$ is treated as a latent variable and obtained by minimizing $h_t\left(\hat{\Theta}\right)' \Sigma^{-1} h_t\left(\hat{\Theta}\right)$.[8] Clearly, this reduces the degrees of freedom of the GMM function by one.[9] The weighting matrix Σ is the Newey and West (1987) covariance matrix of the estimation errors.

A. PARAMETER ESTIMATES. The parameter estimates are summarized in Table 3.

The values of a and b are 0.65 and 0.35, respectively, with a standard deviation of 0.11 and 0.13 suggesting that both differences in beliefs are important to explain the joint dynamics of option prices and open interests. The estimated values of the signal and dividend volatility are 6.1% and 3.8% respectively. The estimated dividend volatility is slightly below its sample counterpart which is 4%. The difference is however not statistically significant with a p-value of 0.24. Since in the model the stochastic dividend growth rate affects the overall dividend volatility, the estimate of the instantaneous volatility σ_δ is slightly below the overall sample dividend volatility. The estimated dividend growth rate is 5.9%, higher than the empirical dividend growth rate, which is 4.5%. The difference is however not statistically significant, with a p-value of 0.35. Finally, the volatility of the stochastic drift of the dividend and signal are 1.3% and 2.5%, respectively.

The risk aversion coefficient γ is -0.95. This level of risk aversion is higher than the logarithmic utility case but lower than the value of $\gamma = -1$ used by Campbell and Cochrane (1999) to study the implications of introducing habit persistence in a model with low (local) risk aversion.

[7] See Appendix for a formal derivation of the second moments: $\sigma^2\left(\frac{d\delta}{\delta}\right) = n_\delta^2/2 + \sigma_\delta$.

[8] Alternatively, one could select $\Psi_z(t)$ to match the time series of one particular option open interest or price. While this alternative approach gives an implied time series that is independent of the weighting matrix Σ, it has the disadvantage of being more sensitive to the observation errors of the selected option.

[9] The Lagrange multipliers y_1 and y_2 are obtained from the budget constraint (2) by substituting the equilibrium consumption process obtained in Proposition 2. Option prices are then obtained numerically from Proposition 3, noting that $\xi^1(t) = \delta(t)^{\gamma-1}\left(1 + (y_1\eta(t)/y_2)^{1/(1-\gamma)}\right)^{1-\gamma}$.

The average level of the signal difference in beliefs $\Psi_z(t)$ is 0.61. This compares to an observed average level of dividend difference in beliefs $\Psi_\delta(t)$ equal to 0.64. The standard deviations of the dividend difference in beliefs is 0.1. The dynamics of the two differences in beliefs is presented in Figure 4. The difference in beliefs about the dividend process is quite persistent. It increases substantially during the recession in the early 1990s from 0.20 to about 0.80. After 1992, a period characterized by a strong rally in the stock market, the dispersion in beliefs decreased to historically low levels. The time variation of the difference in beliefs on the signal is less persistent. To gain insight on the dynamics of the dispersion in beliefs on $z(t)$, we show in Figure 5 a non-parametric scatter plot of the difference in beliefs on $z(t)$ with respect to the daily S&P500 index return (panel C) and the intra-day volatility of the S&P500 index (panel D). We find that differences in beliefs on the signal are negatively correlated with the stock market. Large dispersions in beliefs on the signal take place during large market declines. When the difference in beliefs is one standard deviation above its long term mean, the average annualized stock market return has been -13%. The relationship is asymmetric: when the difference in beliefs is smaller than its long term mean, the average daily change in the stock market is not particularly sensitive to changes in beliefs. When the difference in beliefs is large, the average stock market sensitivity is high. Moreover, we find that large dispersions in beliefs about the signal are correlated with high intraday volatility of the S&P500 index (panel D). The relationship is both statistically and economically significant. When the difference in beliefs is one standard deviation below the long term mean, the intraday volatility is about 70 basis points. For values one standard deviation above the long term mean the intraday volatility is on average 85 basis points.

We test the overidentifying restrictions of the structural model using a GMM test that takes advantage of the large cross-sectional information contained in the dataset. To do this, we first estimate the structural parameters using a subset of options with moneyness $K_i/S = \{0.97, 1.03\}$ and maturities $T_i = \{45, 135\}$ days. Then we compute the pricing errors on the remaining set of options with moneyness levels $K_i/S = \{0.92, 1.00, 1.08\}$ and maturities $T_i = \{30, 180\}$ days. We use these pricing errors to construct an out-of-sample GMM test statistic.[10] Using different sets of options for estimation and testing allows us to increase the test's power. More specifically, let $C_t(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i)$ and $OI_t(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i)$ be the model-implied call option price and open interest conditional on the vector of structural parameters Θ, the differences in beliefs $\Psi_{z,t}$, and $\Psi_{\delta,t}$, the strike price K_i and the time to maturity T_i. Let the sample counterparts of the previous variables be $C(t, K_i/S, T_i)$ and $OI(t, K_i/S, T_i)$ respectively.

B. Pricing Errors. The out-of-sample *pricing errors* for the remaining subset of options expressed in percentage terms are defined as:

$$\varepsilon_t(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i) = \frac{C_t(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i)}{C(t, K_i/S, T_i)} - 1$$

If the expected value of the pricing errors is zero then, under some regularity conditions, its sample

[10] The same out-of-sample test is used in David and Veronesi (2002).

counter-part converges to zero. Thus, we consider the following test statistic

$$J = \left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t\right)' \Sigma^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t\right)$$

The variance-covariance matrix is estimated using a Newey and West (1987) correction with 10 lags to adjust for potential heteroskedasticity and autocorrelation in the pricing errors. Since these options are not used in the estimation, under the null hypothesis of zero mean errors, this test statistic is asymptotically distributed as a Chi-square with 6 degrees of freedom.

Table 4 summarizes the GMM test results for different option moneyness and maturity parameters. The joint test on all options, independent of moneyness and maturity, gives a J-statistic of 7.02 with a p-value of 32%. The overidentifying restrictions of the model are not rejected. We therefore explore how the model performance depends on the option characteristics by stratifying the sample with respect to the moneyness and maturity parameters. We find that the price dynamics of out-of-the-money options, i.e. $K/S = 1.08$, is the most difficult to fit. However, for these options, the p-values still range between 10% and 14%. The aggregate p-value is 10%. At-the-money options, i.e. $K/S = 1$, have the highest p-values ranging between 44% and 54% while in-the-money calls, i.e. $K/S = 0.92$, have p-values ranging between 33% and 48%. Looking at the option maturity, we find that the p-value is 36% for short term options and 41% for long term options. The model is never rejected.

C. Open Interest. We now turn to the *open interest* assessment of the structural model. The spirit of the test is similar to that for option prices. We first estimate the structural parameters of the model using prices of a subset of options. Then we compute the errors for the open interest on the remaining set of options with moneyness levels $K_i/S = \{0.97, 1.03\}$ and maturities $T_i = \{45, 135\}$ days. The out-of-sample open interest fitting errors for the remaining subset of options are expressed in percentage terms as follows:

$$\varepsilon_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right) = \frac{OI_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right)}{OI\left(t, K_i/S, T_i\right)} - 1$$

We construct the J_T-statistic $J_T = \left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t\right)' \Sigma^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t\right)$ in the same fashion as for the pricing errors. Table 5 summarizes the results of the GMM test performed on open interest data. The joint test on all options, independent of moneyness and maturity, have a J-statistic of 6.09, with a p-value of 19%. The overidentifying restrictions of the model are not rejected. We stratify the sample for different maturity and moneyness levels to study how the model performance depends on these characteristics. Similar to the pricing errors GMM tests, we find that the dynamics of the out-of-the-money option open interest is the most difficult to fit. However, for these options the p-value is still between 12% and 17%, with an aggregate p-value equal to 13%. The overidentifying restrictions of the model are not rejected. In-the-money options have the highest p-values ranging between 22% and 30%. We also compare the model performance with respect to the option maturity and find that the p-value is 23% for short term options and 20% for long term options.

4.1 Option Volume and Open Interest

The DB model generates implications for the dynamics of the option open interest as a function of the Difference in Beliefs Index.[11] This contrasts with generalized deterministic models which can fit by construction any cross-section of option prices at any desired level of accuracy (in sample) but leaves the dynamics of the option open interest indeterminate. In this section we quantify the extent to which the DB model explains the option open interest.

The Difference in Beliefs Index is positively correlated with option open interests. During the 1987 market crash, the Difference in Beliefs Index drops from 0.40 to 0.15. At the same time, the open interest to S&P500 capitalization ratio declines to reach the lowest value in 1987: in March 1987 the open interest to market capitalization ratio is 2.5 times lower than in March 1985. During the high growth expansionary period of the nineties, we again find a positive correlation between the two time series. The open interest to S&P500 capitalization ratio peaks during the 1991-92 recession, then it steadily declines. In April 1992 the open interest is about twice than in January 1995. At the same time, in April 1992 the Dispersion in Beliefs Index reaches 0.80 and then declines to 0.25 in January 1995.

Table 6 summarizes the average open interest fitting error for different maturities and moneyness levels. For the sake of comparison, we also estimate and compute fitting errors based on the Liu and Pan (2003) model. They generalize Heston (1993) in the context of a partial equilibrium model in which options are held to hedge unexpected changes in volatility and jumps and solve the dynamic portfolio problem. We refer to this model as SV-J and use it as a benchmark, since in models with generalized deterministic volatility, such as Derman and Kani (1994), options are redundant and open interest is undetermined. We find that for all option classes, the DB model produces smaller open interest fitting errors than the SV-J model. The largest differences are for short-term ITM and long term ATM options. In the first case, the fitting errors are 33.27% for the SV-J model and 18.94% for the DB model. In the second case, the fitting errors are 15.86% for the SV-J model and 10.13% for the DB model. For OTM call options, the pricing errors of the two models are not significantly different (30% for short-term options and 20% for long-term options).

To study the economic significance in the link between open interest and the difference in beliefs, we plot the logarithm of the option trading volume as a function of the difference in beliefs (see Figure 5, panel A).[12] We find that the option trading volume is positively correlated with the difference in

[11] Other important related works that assume a positive link between trading volume and difference in beliefs include Varian (1989), Bessembinder, Chan, and Seguin (1996), Harris and Raviv (1993) and Shalen (1993).

[12] We use standard kernel regression to find the exact non-parametric relationship between the two variables. Let $\Psi_z(t)$ be the option-implied difference in beliefs and y be an observed financial variable. For each level of $\Psi_z(t)$ the non-parametric function $y_t = y(\Psi_z(t))$ is estimated as

$$y(x_0) = \frac{\sum_{t=1}^{T}\left[\frac{1}{h}K\left(\frac{x_0-x_t}{h}\right)y_t\right]}{\sum_{t=1}^{T}\left[\frac{1}{h}K\left(\frac{x_0-x_t}{h}\right)\right]};\ x_t = \Psi_z(t)$$

The function $K(\cdot)$ is a kernel function, the scalar h is the bandwidth and x_j are the grid-points. As a kernel function we select Epanechnikov kernel function $K(x) = \frac{3}{4\sqrt{5}}\left(1-\frac{1}{5}x^2\right)I_{|x|\leq\sqrt{5}}$. See Silverman (1986) for a discussion of its accuracy with respect to the more traditional Gaussian kernel. The bandwidth is selected via standard cross-validation procedures.

beliefs. For instance, a one standard deviation change in the difference in beliefs from 0.45 to 0.65 results in a 20% increase in the option trading volume. The relationship is nonlinear and steeper at both higher and lower levels of the difference in beliefs. For instance, an increase in the difference in beliefs from 0.65 to 0.85 results in a 60% increase in the level of the option trading volume. We find a similar relationship for the total NYSE stock trading volume. An increase in the difference in beliefs from 0.45 to 0.65 results in a 30% increase of the NYSE trading volume.

The time variation in the DB generates endogenous stochastic volatility. As such, it nests some features of the Heston (1993) and Liu and Pan (2003) models. The DB model, therefore, should not be interpreted as an alternative to these reduced-form specifications. Rather, it offers a structural explanation for the stochastic and time varying endogenous volatility. To study in further detail the marginal contribution of changes in the difference in beliefs with respect to the stochastic volatility, we regress the option trading volume on the difference in beliefs for both the non-dividend and dividend processes, controlling for both the stochastic volatility implied by the DB model. Moreover, since in the DB model volatility is stochastic and endogenously driven by changes in $\Psi_{z,t}$ and $\Psi_{\delta,t}$, we decompose the total observable volatility at time t, V_t, into two components: (a) the part that is predicted by the DB model, i.e. $\widehat{V}_t(\Psi)$ and (b) the volatility that is unexplained by changes in the difference in beliefs, i.e. $V_t - \widehat{V}_t(\Psi)$. This second component captures the portfolio reallocation decision driven by volatility changes due to non-informational reasons. We consider the following regression:

$$\log\left(OptVlm_t\right) = \beta_0 + \beta_1 \log\left(\widehat{V}_t(\Psi)\right) + \beta_2\left[V_t - \widehat{V}_t(\Psi)\right] + \beta_3 \log\left(\Psi_{z,t}\right) + \beta_4 \log\left(\Psi_{\delta,t}\right) + e_t$$

where $OptVlm_t$ is the option trading volume at time t. This regression allows us to assess the relative importance of the informational component for the trading activity, with respect to other exogenous components affecting changes in volatility. We run the equivalent regression for the stock trading volume. Trading volumes are detrended using the method described in the data section, to ensure stationarity; moreover, the standard errors are corrected for heteroskedasticity and autocorrelation using Newey and West (1987).

The results are given in Table 7. Consistent with the literature, we find that changes in volatility are positively correlated with the option open interest. A 10% increase in the model-implied volatility $\hat{V}_t(\Psi)$ increases the option volume by 3.2%. The residual change in volatility, unexplained by the model, i.e. $V_t - \hat{V}_t(\Psi)$ is statistically significant. The size of its slope coefficient, however, is about one third of the slope coefficient for the model-implied volatility. This suggests that the model is effective in capturing a large component of the realized volatility. The main result is that even after controlling for the two measures of volatility, the two Difference in Beliefs Indices are still positive and statistically significant. Of these two indices, the largest coefficient is for the difference in beliefs on the signal $\Psi_{z,t}$. A 10% increase in the difference in belief increases the option volume by 9.6%. The effect is almost as strong as that to the model-implied volatility. The R^2 of the regression is 22%. We conclude that even after controlling for portfolio reallocation decisions driven by changes in the volatility, a large component of the trading activity is directly driven by informationally related reasons.

4.2 Hedging Errors

We follow Bakshi, Cao, and Chen (1997) and David and Veronesi (2002) by using the hedging errors to assess the model's pricing performance. We use this metric, as opposed to in-sample pricing errors, because Dumas, Fleming, and Whaley (1998) show that several of the option pricing models that perform well in terms of in-sample pricing errors display large out-of-sample pricing errors. They interpret this result as evidence of model overfitting in the context of generalized deterministic volatility function models. Thus, we consider the time t problem of hedging a short position in a call option with τ periods to maturity and strike price K. In both Heston and the DB models, the volatility of the underlying asset is stochastic. In the DB model this is due to changes in the difference in beliefs. Thus, the replication portfolio returns require a third security to hedge the risk of changes in the difference in beliefs, which has asset pricing effects. Let $x_S(t)$ be the model-implied quantity of the underlying assets, $x_0(t)$ the bond position, and $x_c(t)$ the quantity invested in another call option with the same maturity but different strike price K_1. For the DB model, the option with strike price K_1 is used to hedge the risk of unexpected changes in the difference in beliefs. The optimal hedge ratio can be obtained from the derivative of the option price with respect to the state variables S and η:

$$x_c(t) = \frac{\partial C(t,K)/\partial \eta}{\partial C(t,K_1)/\partial \eta}$$
$$x_s(t) = \frac{\partial C(t,K)}{\partial S_t} - \frac{\partial C(t,K_1)}{\partial S_t} x_c(t)$$

The initial bond amount $x_0(t)$ is selected so that the value at time t of the portfolio is equal to the value of the option to be hedged. For comparison, we also present results based on the Heston (1993) model. In this case, the optimal hedging portfolio weights are given by:

$$x_c(t) = \frac{\partial C(t,K)/\partial \sigma}{\partial C(t,K_1)/\partial \sigma}$$
$$x_s(t) = \frac{\partial C(t,K)}{\partial S_t} - \frac{\partial C(t,K_1)}{\partial S_t} x_c(t)$$

For completeness, we consider a delta-vega hedging strategy based on the Black and Scholes model. This strategy is clearly inconsistent since, in the Black and Scholes model, the volatility is not time-varying. Nonetheless, it is one of the most widely adopted hedging strategies by risk managers.

The hedge requires continuous rebalancing to replicate the option return. We select the rebalancing period to be one day. We calculate the hedging errors at time $t + \Delta t$ as

$$H(t+\Delta t) = x_s(t)\, S(t+\Delta t) + x_0(t)\, e^{r(t)\Delta t} + x_C(t)\, C(t+\Delta t, \tau-\Delta t, K_1) - C(t+\Delta t, \tau-\Delta t, K)$$

We calculate the hedging errors at a one week (5 trading days) horizon. The average hedging errors are reported in Table 8.

Across all maturities and moneyness levels, the hedging error of the DB model is \$0.305, versus \$0.32 for the Heston (1993) model and \$0.353 for a Black-Scholes delta-vega hedging strategy with continuous recalibration. For short dated options, i.e. less than 45 days, the difference in hedging performance is even larger: \$0.30 for the DB model, \$0.325 for the Heston (1993) model and \$0.35 for the Black-Scholes delta-vega strategy. The percentage difference is remarkable. For short dated options, the DB model generates hedging errors which are 10% lower than the Heston (1993) model and almost 20% lower than the Black-Scholes delta-vega strategy.

To study the performance difference with respect to other option characteristics, we stratify the sample according to moneyness level. We find that, independent of the moneyness level, the DB model always produces lower hedging errors than both the Heston (1993) model and the Black-Scholes delta-vega strategy. The biggest difference is for short-term ITM options. In this case, the hedging errors of the DB model are \$0.316, 15% lower than the \$0.365 of the Heston (1993) model and 14% lower than the \$0.361 of the Black-Scholes delta-vega strategy. For OTM options, the hedging errors of the DB model are however larger than Heston (1993), but still 16% better than the BS delta-vega hedging strategy. The previous results suggest that the explicit use of open interest information is an effective forward looking measure for the dynamics of the priced risk factors affecting option prices. As such, it helps to reduce the hedging errors from replication strategies.

4.3 Difference in Beliefs, the Smile and the Future Volatility

Differences in beliefs are statistically significant to help explain both the cross-section and the time-series of option prices. We now examine in more detail why this model is successful. In particular, we examine the relation between difference in beliefs and the shape of the contemporaneous implied volatility smile. Then, we examine whether differences in beliefs predict future realized price volatility.

We find that the signal difference in beliefs affects option prices in a non-linear way. For low difference in beliefs levels, the volatility smile is 15%. Larger levels of heterogeneity in beliefs induce a substantial shift in the option-implied volatility smile. For high levels the average level of the smile is above 20%. The difference is economically large and statistically significant. Figure 5 Panel E shows the effect of the signal difference in beliefs on the slope of the volatility smile. The correlation is positive for values of the signal difference in beliefs above 0.70.

Figure 6 shows the shape of the smile for various levels of the dividend difference in beliefs. In this figure, we keep the signal difference in beliefs equal to zero and change only the dividend difference in beliefs. The level of the differences in beliefs is positively related to the slope of the implied volatility smile. For small amounts of dispersion the smile is almost negligible. For large amounts the volatility smile is quite steep. For example, when the level of the difference in beliefs is 0.80 the corresponding slope of the smile is approximately 7%. This corresponds to an implied volatility of 23% for in-the-money calls and 16% for out-of-the-money calls. When the difference in beliefs is 0.20, the slope of the smile is less than 1%.

Does the current level of the difference in beliefs predict future volatility? To address this we regress the future realized volatility τ-periods ahead $V_{t+\tau}$ onto the current level of the option implied difference in beliefs. The empirical literature has shown substantial evidence that the current level of the implied volatility is a significant, albeit imperfect, predictor of the future realized volatility (Christensen and Prabhala (1998)). Thus, in order to correctly assess the marginal contribution of the difference in beliefs we control for the implied volatility. Moreover, since volatility is persistent, we also control for the current realized volatility. The forecasting regression is as follows:

$$V_{t+\tau} = \alpha_0 + \alpha_1 IV_t + \alpha_2 V_t + \alpha_3 \Psi_{z,t} + \alpha_4 \Psi_{\delta,t}$$

where $V_{t+\tau}$ is the future realized volatility τ periods ahead and IV_t is the option implied volatility at time t. We select the forecasting horizon to range between one week and 6 months. The results are given in Table 9. We find that the null hypothesis that the option-implied difference in beliefs on the signal predicts the future realized volatility is not rejected, with t-statistics ranging between 1.6 and 3.4, depending on the horizon. The slope coefficient is positive: the higher the current level of the signal difference in beliefs, the higher the future realized volatility. Consistent with the literature, for short horizons lagged volatility is statistically significant. At horizons longer than one week, however, the lagged volatility is not significant. Moreover, we find that differences in beliefs about the drift of the dividend process have large and positive slope coefficients. The t-statistics are however small. The R^2 of the regressions ranges between 23% at a 1 week horizon and 43% at a 3 month horizon. Consistent with the literature, we find that the current implied volatility has a positive and significant slope coefficient. We also find that the relationship between differences in beliefs and future volatility is *economically* significant. A change of one standard deviation in the difference in beliefs, i.e from 0.61 to 0.80, increases the expected future volatility in 2 weeks time by 0.7%, for example from 15% to 15.7%.

The difference in beliefs improves the forecast of the future realized volatility. This may explain why the DB model performs better than the Heston model in terms of hedging errors. The DB model is fit to information on both the open interest and the implied volatility smile. Andersen (1996) shows evidence that the joint dynamics of the volatility and volume is likely driven by a common factor, which he labels "information flow". Hence, a model that links the structural relation between information flow to both prices and volumes more accurately accounts for the future dynamics of volatility and produces a better hedging performance.

4.4 Violations of One Factor Option Pricing Models

We now examine the model's performance with respect to a different dimension. Several studies document that the dynamics of option prices are not consistent with one-factor models. Bakshi, Cao, and Chen (2000) explore simple static arbitrage violations of option pricing models that assume a one-factor structure such as Black and Scholes. For instance, in these models an increase in the underlying asset value implies an increase (decrease) in the value of a call (put) option. The value of a call option is monotonically increasing in the value of the underlying asset.

Definition 2 *(*Bakshi, Cao, and Chen (2000)*) In a frictionless one-factor economy the delta of a call (put) must be positive (negative) and lower (greater) than one (minus one) in order for prices* *to be consistent with the Principle of No-Arbitrage*[13]

- VIOLATION 1*:* $\Delta S \Delta C < 0$ *for call options and* $\Delta S \Delta P > 0$ *for put options.*

- VIOLATION 2*:* $\Delta S \neq 0$ *and* $\Delta C/\Delta S > 1$ *and* $\Delta P/\Delta S < -1$ *for put options.*

where S *is the stock (underlying) price,* C *is the call option price and* P *is the put option price.*

Violations of these restrictions can be interpreted either as evidence of frictions or of a second (priced) stochastic factor. In what follows, we first explore the empirical violation frequency of the one-factor no-arbitrage restrictions in our sample. We find (see Table 10) that violations of type-1 occur with frequency between 17% and 24% for call options and between 15% and 22% for put options. Violations of type-2 occur with frequency between 4% and 11% for call options and between 2% and 4% for put options. The dynamics of the violation frequencies is relatively stable across years, and are not specific to a particular subperiod. The magnitude of violations is close to those reported in Bakshi, Cao, and Chen (2000). They interpret these violations as evidence against one-factor models of option pricing. We explore whether the difference in beliefs can be the additional second stochastic factor that explains these violations.

In Table 11 we compare the empirical (Panel A) violation frequency with those obtained by simulating (Panel B) the model at the estimated parameter values. We find that type-1 violation frequencies are relatively stable across moneyness and maturity varying from 12% for short maturity out-of-the-money calls to 17% percent for long in-the-money calls. The type-2 violations frequencies show strong dependence on moneyness, varying from 26% for ITM short calls to 3% for OTM short calls.

The model generates frequencies of type-1 violations which are very close to the empirical ones, across all moneyness and maturities. For instance, for short-term OTM calls the model-implied type-1 violation frequency is 12.01%, compared to an observed empirical frequency of 12.65%. For long-term OTM options the difference is still small, i.e. less than 2%. The biggest discrepancy is for the long ITM calls for which the model-implied violation frequency is 9.45% with respect to an observed empirical frequency of 16.50%. The type-2 violation frequencies are harder to replicate. The model is able to generate a frequency of about 2%, while the empirical frequency varies from 3% to 25%. However, as observed in Bakshi, Cao, and Chen (2000), violations of type-2 are more likely due to the tick size, as opposed to the effect of a second stochastic factor.

An alternative explanation for the observed large violations of the one factor no-arbitrage restrictions is that volatility is stochastic. We therefore directly examine the marginal contribution of the difference in beliefs dynamics with respect to the dynamics of the volatility to explain the observed pattern of violations. Since the violations have an inherent binary structure, i.e. it is equal to one if the violation has occurred and zero if the violation has not occurred, we use logit techniques[14] to

[13] Bakshi, Cao, and Chen (2000) consider two other violations related to the fact that the option delta should be different from zero. We do not explore these issues.

[14] The probit results are similar and are not reported to save space. They are available on request.

regress violation events onto the explanatory variables. Let $y_{it}^{(j)}$ be the occurrence of a violation of type-j for option i at time t. The probability that a violation event occurs can be specified as

$$\Pr\left(y_{it}^{(j)} = 1\right) = F\left(\beta_0 + \beta_1 \log\left(\widehat{V}_{ti}(\Psi)\right) + \beta_2(V_{ti} - \widehat{V}_{ti}(\Psi)) + \beta_3 \log\left(\Psi_{z,t}\right) + \beta_4 \log\left(\Psi_{\delta,t}\right)\right)$$

where $\widehat{V}_{ti}$ is the implied-volatility from the model at time t for option i, which is stochastic and a function of Ψ, while V_{ti} is the actual observed implied volatility. On the right hand side, we also include option specific characteristics, such as moneyness and time to maturity, to ensure that $\Psi_{z,t}$ is not capturing any spurious effects. The model estimation is done by maximum likelihood and the results are summarized in Table 12. The t-statistics of the coefficients are given in parenthesis.

The probability of violations of type-1 and 2 is positively related to the level of $\Psi_{z,t}$ for both puts and calls. In Panel B we calculate the average marginal impact on the violation probability given a one standard deviation change in the explanatory variables. The marginal effect is computed at the average value of the explanatory variables.

The results show that a 0.1 change in moneyness (from 1 to 1.10) increases the probability of type-1 violations by 2.0% for call options, keeping the other variables constant. A one standard deviation change in $\Psi_{z,t}$ (from the average value of 0.72 to 0.97) increases the probability of type-1 violations by 1.56% for call options and 1.55% for put option. An equivalent change in the volatility increases the probability of type-1 violation by 1.27% and 0.41% for call and put options respectively. Thus, we find that the change in the difference of beliefs is at least as important as the change in the volatility. The size of the effect is significant if one considers that the average violation frequency ranges over the years between 17% and 24%. The positive coefficient means that a high level in the difference in beliefs makes options behave more as non-redundant securities.

For type-2 violations, the coefficient of $\Psi_{z,t}$ is positive and statistically significant, although it is smaller than that of type-1. A one standard deviation change in $\Psi_{z,t}$ yields a 0.57% increase in the probability of type-2 violations (Calls), while that of the stochastic volatility yields a 0.17% increase. The difference in beliefs about the dividend process $\Psi_{\delta,t}$ does not have a significant effect on the probability of type-2 violations for call options.

The marginal effect of changes in the implied volatility that is not due to changes in the differences in beliefs, $V_{ti} - \widehat{V}_{ti}$, is substantially smaller than both the direct and indirect effect of changes in Ψ. For instance, in the case of type-1 violations for call options, the marginal effect of changes in $\widehat{V}_{ti}(\Psi)$ is 1.27%, the direct effect of $\Psi_{z,t}$ is 1.56%, while the effect of $(V_{ti} - \widehat{V}_{ti})$ is only 0.33%.

To summarize, we find that the two Difference in Beliefs Indices are significant explanatory variables when describing the probability of violations to the no-arbitrage restrictions, which are consistent with one-factor models of option pricing. The significance of the difference in beliefs persists even after controlling for changes in the stochastic volatility.

5 Conclusion

In this paper we study both theoretically and empirically an economy in which options are not redundant. Agents are uncertain about the stochastic drifts of the risky factors and they have heterogeneous beliefs. We characterize the way in which heterogeneity in beliefs have option pricing and volume implications. We model two forms of beliefs: (a) those about the drift of the dividend process, (b) those about a factor (signal) that is correlated with the drift of the dividend process. Then we estimate and test the structural model. We use survey data to construct a Difference in Beliefs Index that serves as a direct proxy for the difference in beliefs about market fundamentals. We find the following:

First, we test the structural overidentifying pricing restrictions of the model by using an out-of-sample GMM test. We fail to reject the model at a p-value of 31.9% when using restrictions on option prices and of 19.3% when using restrictions on open interest. The results are robust to different subsamples. We also fail to reject the model when we consider only ITM, OTM, short-term or long-term options.

Second, for all option classes the Difference in Beliefs model produces open interest fitting errors which are smaller than a stochastic volatility model that abstracts from the heterogeneity in beliefs, as in Liu and Pan (2003). The largest differences between the two models are for short-term ITM and long term ATM options. In the first case, the fitting errors difference is 14.3%; in the second case it is 5.7%.

Third, the Difference in Beliefs model generates lower hedging errors at a one-week horizon than both Heston (1993) and Black and Scholes (1973). Across all maturities and moneyness levels, the hedging error of the DB model is 5% lower than the Heston (1993) model and about 17% lower than a Black-Scholes delta-vega hedging strategy. For short dated options, i.e. less than 45 days, the difference in hedging performance is even larger, with hedging errors that are 8% lower than the Heston (1993) model and 16% lower than the Black-Scholes delta-vega strategy.

Fourth, current levels of the Index of dispersion in beliefs have positive and statistically significant predictive power for the future realized volatility, even after controlling for the current implied volatility. The R^2 of the regression ranges between 23%, for a one-week horizon, to 43%, for three months horizon. Both the current and future implied volatility smile is very sensitive to our Difference in Beliefs Index: the greater the dispersion of beliefs, the steeper the implied volatility smile.

Fifth, we use the DB model to address the puzzle highlighted by Bakshi, Cao, and Chen (2000). They document evidence of violations of basic arbitrage bounds implied by one factor option pricing models. For instance, in most of these models an increase in the value of the underlying asset implies an increase (decrease) in the value of a call (put) option: the option delta is restricted to values between 0 and $+1(-1)$. Thus, we run a Logit regression to assess the extent to which the Index of dispersion in beliefs can explain these no-arbitrage violations. We find that an increase in the Index substantially increases the probability that the Black and Scholes call delta is negative or above $+1$. The results are both economically and statistically significant. We find that most arbitrage violations, and the extent of these violations, are correlated with abnormal changes in the Dispersion in Beliefs.

The empirical evidence gives strong support to the null hypothesis that the information heterogeneity and belief structure of the economy has important option pricing and risk management implications.

References

Ait-Sahalia, Y., and A. W. Lo (1998): "Estimation of State-Price Densities Implicit In Financial Asset Prices," *Journal of Finance*, 53, 499–547.

Amin, K., and C. M. Lee (1997): "Option Trading, Price Discovery and Earnings News Dissemination," *Contemporary Accounting Research*, 14, 153–192.

Andersen, T. G. (1996): "Return Volatility and Trading Volume: An Information Flow Interpretation of Stochastic Volatility," *Journal of Finance*, 51(1), 169–204.

Back, K. (1993): "Asymmetric Information and Options," *Review of Financial Studies*, 6(3), 435–472.

Bakshi, G., C. Cao, and Z. Chen (1997): "Empirical Performance of Alternative Option Pricing Models," *Journal of Finance*, 52(5), 2003–2049.

——— (2000): "Do Call Prices and the Underlying Stock Always Move in the Same Direction?," *The Review of Financial Studies*, 13, 549–584.

Bakshi, G., and N. Kapadia (2003): "Delta-Hedged Gains and the Pricing of Volatility Risk," *The Review of Financial Studies, forthcoming.*

Bakshi, G., N. Kapadia, and D. Madan (2003): "Stock Return Characteristics, Skew Laws, and Differential Pricing of Individual Equity Options," *Review of Financial Studies*, Winter.

Basak, S. (2000): "A Model of Dynamic Equilibrium Asset Pricing with Heterogeneous Beliefs and Extraneous Risk," *Journal of Economic Dynamics and Control*, 24, 63–95.

Bates, D. (2000): "Post '87 Crash Fears in the SP500 Futures Option Market," *Journal of Econometrics*, 94, 181–238.

Bates, D. (2001): "The Market for Crash Risk," NBER working paper 8557.

Bessembinder, H., K. Chan, and P. J. Seguin (1996): "An Empirical Examination of Information, Differences in Opinions, and Trading Activity," *Journal of Financial Economics*, 40, 105–134.

Biais, B., and P. Hillion (1994): "Insider and Liquidity Trading in Stock and Options Markets," *Review of Financial Studies*, 7(4), 743–780.

Black, F., and M. Scholes (1973): "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637–659.

Brennan, M. J. (1998): "The Role of Learning in Dynamic Portfolio Decisions," *European Finance Review*, 1, 295–306.

Brennan, M. J., and H. H. Cao (1996): "Information, Trade and Derivative Securities," *Review of Financial Studies*, 9(1), 165–208.

Brennan, M. J., and R. Solanki (1979): "Optimal Portfolio Insurance," working paper, University of British Columbia.

Brennan, M. J., and Y. Xia (1998): "Resolution of an Asset Allocation Puzzle," Working Paper, Anderson School of Management, UCLA.

Buraschi, A., and J. C. Jackwerth (2001): "The Price of a Smile: Hedging and Spanning in Option Markets," *The Review of Financial Studies*, 14, 495–527.

Campbell, S. D., and C. Li (1999): "Option Prices with Unobserved and Regime Switching Volatility," working paper, Department of Economics, University of Pennsylvania.

Cao, C., Z. Chen, and J. Griffin (2000): "The Informational Content of Option Volume Prior to Takeovers," working paper, Pennsylvania State University.

Chakravarty, S., H. Gulen, and S. Mayhew (2002): "Informed Trading in Stock and Option Markets," working paper, University of Georgia.

Christensen, B., and N. Prabhala (1998): "The Relation between Implied and Realized Volatility," *Journal of Financial Economics*, 3, 125–150.

Coval, J. D., and T. G. Shumway (2001): "Expected Option Returns," *Journal of Finance*, 56, 983–1009.

Cox, J., and C.-F. Huang (1989): "Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, 49, 33–83.

Cuoco, D., and H. He (1994): "Dynamic Equilibrium in Infinite-Dimensional Economies with Incomplete Financial Markets," Working paper. Wharton School, University of Pennsylvania.

David, A. (1997): "Fluctuating Confidence in Stock Markets: Implications for Returns of Volatility," *Journal of Financial And Quantitative Analysis*, pp. 427–462.

David, A., and P. Veronesi (2001): "Inflation and Earnings Uncertainty and the Volatility of Asset Prices: An Empirical Investigation.," working paper, University of Chicago.

——— (2002): "Option Prices with Uncertain Fundamentals," working paper, University of Chicago.

Derman, E., and I. Kani (1994): "Riding on a Smile," *Risk*, 7(2).

Detemple, J., and S. Murthy (1994a): "Intertemporal Asset Pricing with Heterogeneous Beliefs," *Journal of Economic Theory*, 62, 294–320.

——— (1994b): "Intertemporal Asset Pricing with Heterogeneous Beliefs," *Journal of Economic Theory*, 62, 294–320.

Detemple, J. B. (1986): "Asset Pricing in a Production Economy with Incomplete Information," *Journal of Finance*, 41, 383–391.

Dothan, M. U., and D. Feldman (1986): "Equilibrium Interest Rates and Multiperiod Bonds in a Partially Observable Economy," *Journal of Finance*, pp. 369–382.

Duffie, D., J. Pan, and K. Singleton (2000): "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica*, 68, 1343–1376.

Dumas, B., J. Fleming, and R. E. Whaley (1998): "Implied Volatility Functions: Empirical Tests," *Journal of Finance*, 53, 2059–2106.

Easley, D., M. O'Hara, and P. S. Srinivas (1998): "Option Volume and Stock Prices: Evidence on Where Informed Traders Trade," *Journal of Finance*, 53(2), 431–465.

Figlewski, S., and G. P. Webb (1993): "Options, Short Sales, and Market Completeness," *Journal of Finance*, 48, 761–777.

Fleming, J., and R. E. Whaley (1994): "The Value of Wildcard Options," *The Journal of Finance*, 49, 215–236.

Franke, G., R. Stapleton, and M. Subrahmanyam (1998): "Who Buys and Who Sells Options? The Role of Options in an Economy with Background Risk," *Journal of Economic Theory*, 82, 89–109.

Gallant, R., P. Rossi, and G. Tauchen (1992): "Stock Prices and Volume," *The Review of Financial Studies*, pp. 199–242.

Garcia, R., R. Luger, and É. Renault (1999): "Asymmetric Smiles, Leverage Effects and Structural Parameters," working paper, CIRANO.

Gennotte, G. (1986): "Optimal Portfolio Choice Under Incomplete Information," *Journal of Finance*, 41, 733–746.

Grossman, S. J., and Z. Zhou (1996): "Equilibrium Analysis of Portfolio Insurance," *Journal of Finance*, 51, 1379–1403.

Guidolin, M., and A. Timmermann (2000): "Option Prices and Implied Volatility Dynamics under Bayesian Learning," working paper, University of California at San Diego.

Hansen, L. P., and R. Jagannathan (1997): "Assessing Specification Errors in Stochastic Discount Factor Models," *Journal of Finance*, 52(2), 557–590.

Harris, M., and A. Raviv (1993): "Differences in Opinion Make a Horse Race," *Review of Financial Studies*, 6, 473–506.

Heston, S. L. (1993): "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *The Review of Financial Studies*, 6(2), 327–343.

Hull, J. C., and A. White (1987): "The Pricing of Options with Stochastic Volatilities," *Journal of Finance*, 42, 281–300.

John, K., A. Koticha, R. Narayanan, and M. Subramanyam (2000): "Margin Rules, Informed Trading in Derivatives and Price Dynamics," working paper, New York University.

Karatzas, I., J. P. Lehoczky, and S. E. Shreve (1987): "Optimal Portfolio and Consumption Decisions for a "small Investor" on a Finite Horizon.," *SIAM Journal of Control and Optimization*, 25, 1557–1586.

Karatzas, I., and S. E. Shreve (1998): *Methods of Mathematical Finance*. Springer-Verlag.

Leland, H. E. (1980): "Who Should Buy Portfolio Insurance?," *Journal of Finance*, 35, 581–594.

Liu, J. (2001): "Dynamic Portfolio Choice and Risk Aversion," working paper, UCLA.

Liu, J., and J. Pan (2003): "Dynamic Derivative Strategies," *Journal of Financial Economics, forthcoming.*

Lucas, R. E. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429–1446.

Lui, J., F. Longstaff, and J. Pan (2003): "Dynamics Asset Allocation with Event Risk," *Journal of Finance*, 58.

Madan, D., P. Carr, and E. Chang (1998): "The Variance Gamma Process and Option Pricing," *European Journal of Finance*, 1, 39–55.

Manaster, S., and R. J. Rendleman (1982): "Option Prices as Predictors of Equilibrium Stock Prices," *Journal of Finance*, 37, 1043–1057.

Mayhew, S., A. Sarin, and K. Shastri (1995): "The Allocation of Informed Trading Across Related Markets: An Analysis of the Impact of Changes in Equity-Option Margin Requirements," *Journal of Finance*, 50, 1635–1653.

Merton, R. C. (1973): "Theory of Rational Option Pricing," *Bell Journal of Economics*, 4(1), 141–83.

Morris, S. (1995): "The Common Prior Assumption in Economic Theory," *Economics and Philosophy*, 11, 227–253.

Newey, W. K., and K. D. West (1987): "A Simple Positive Semi-Definite, Heteroscedasticity and Autocorrelation Consistent Covariance Matrix," *Econometrica*, 55, 703–708.

Pan, J. (2002): "The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study," *Journal of Financial Economics*, 63, 3–50.

Pan, J., and A. M. Poteshman (2003): "The Information in Option Volume for Stock Prices," working paper, Sloan School and UIUC.

Rubinstein, M. (1994): "Implied Binomial Trees," *Journal of Finance*, 49(3), 771–818.

Shalen, C. T. (1993): "Volume, Volatility and the Dispersion of Beliefs," *Review of Financial Studies*, 6, 405–434.

Silverman, B. (1986): *Density Estimation*. London: Chapman and Hall.

Stephan, J. A., and R. E. Whaley (1990): "Intraday Price Change and Trading Volume Relations in the Stock and Stock Option Markets," *Journal of Finance*, 45, 191–220.

Varian, H. R. (1989): "Differences of Opinion in Financial Markets," in *Financial Risk: Theory, Evidence and Implications: Proceedings of the 11th Annual Economic Policy Conference of the Federal Reserve Bank of St. Louis*, pp. 3–37.

Veronesi, P. (1999): "Stock Market Overreaction to Bad News in Good Times: A Rational Expectations Equilibrium Model," *The Review of Financial Studies*, 12, 975–1007.

——— (2000): "How Does Information Quality Affect Stock Returns," *Journal of Finance*, 55, 807–837.

Wiggins, J. B. (1987): "Option Values Under Stochastic Volatility: Theory and Empirical Estimates," *Journal of Financial Economics*, 19, 351–372.

Xia, Y. (2001): "Learning About Predictability: The Effects of Parameter Uncertainty on Dynamic Asset Allocation," *Journal of Finance*, 56(1), 205–246.

Zapatero, F. (1998): "Effects of Financial Innovations on Market Volatility When Beliefs are Heterogeneous," *Journal of Economic Dynamics and Control*, 22, 597–626.

Proofs

Proposition 1.

It is easy to verify that the parameters of the system of equations satisfy the conditions of Theorem 12.6 and 12.7 in Liptser and Shiryaev (1974). The result follows.

Remark 1. Using standard techniques it is possible to show that the solution of the following two-dimensional linear *o.d.e.*

$$\frac{d}{dt}m_\delta(t) = \tilde{A}_1 + \tilde{B}_{11}x + \tilde{B}_{12}y$$
$$\frac{d}{dt}m_z(t) = \tilde{A}_2 + \tilde{B}_{21}x + \tilde{B}_{22}y$$

is equal to

$$m_\delta(t) = \varkappa_{\delta,o} + \varkappa_{\delta,1}\exp Q_1t + \varkappa_{\delta,2}\exp Q_2t$$
$$m_z(t) = \varkappa_{z,o} + \varkappa_{z,1}\exp Q_1t + \varkappa_{z,2}\exp Q_2t$$

where $Q_1t = \left(\frac{1}{2}t\tilde{B}_{11} + \frac{1}{2}t\tilde{B}_{22} - \frac{1}{2}tP\right)$, $Q_2t = \left(\frac{1}{2}t\tilde{B}_{11} + \frac{1}{2}t\tilde{B}_{22} + \frac{1}{2}tP\right)$ and $P = \sqrt{4\tilde{B}_{12}\tilde{B}_{21} - 2\tilde{B}_{11}\tilde{B}_{22} + \tilde{B}_{11}^2 + \tilde{B}_{22}^2}$.
Let $V_1 = P - \tilde{B}_{22} - \tilde{B}_{11}$, and $V_2 = \tilde{B}_{11} + \tilde{B}_{22} + P$, the parameters $\varkappa$ are equal to:

$$\varkappa_{\delta,o} = \frac{\tilde{A}_1}{V_1} - \frac{\tilde{A}_1}{V_2} - \tilde{A}_1\frac{\tilde{B}_{11}}{P(V_2)} + \tilde{A}_1\frac{\tilde{B}_{22}}{P(V_2)} - \frac{1}{2}\frac{\tilde{A}_2}{\tilde{B}_{21}}\frac{P}{V_2} - \tilde{A}_2\frac{\tilde{B}_{11}}{\tilde{B}_{21}}\frac{\tilde{B}_{22}}{P(V_2)} - \tilde{A}_1\frac{\tilde{B}_{11}}{P(V_1)} + \tilde{A}_1\frac{\tilde{B}_{22}}{P(V_1)} + \frac{1}{2}\tilde{A}_2\frac{\tilde{B}_{11}^2}{\tilde{B}_{21}P(V_2)}$$
$$+\frac{1}{2}\frac{\tilde{A}_2}{\tilde{B}_{21}}\frac{\tilde{B}_{22}^2}{P(V_2)} - \frac{1}{2}\frac{\tilde{A}_2}{\tilde{B}_{21}}\frac{P}{V_1} - \tilde{A}_2\frac{\tilde{B}_{11}}{\tilde{B}_{21}}\frac{\tilde{B}_{22}}{P(V_1)} + \frac{1}{2}\tilde{A}_2\frac{\tilde{B}_{11}^2}{\tilde{B}_{21}P(V_1)} + \frac{1}{2}\frac{\tilde{A}_2}{\tilde{B}_{21}}\frac{\tilde{B}_{22}^2}{P(V_1)}$$
$$\varkappa_{\delta,1} = \frac{1}{2}K_2\frac{\tilde{B}_{11}}{\tilde{B}_{21}} - \frac{1}{2}\frac{K_2}{\tilde{B}_{21}}\tilde{B}_{22} - \frac{1}{2}\frac{K_2}{\tilde{B}_{21}}P$$
$$\varkappa_{\delta,2} = \frac{1}{2}K_1\frac{\tilde{B}_{11}}{\tilde{B}_{21}} - \frac{1}{2}\frac{K_1}{\tilde{B}_{21}}\tilde{B}_{22} + \frac{1}{2}\frac{K_1}{\tilde{B}_{21}}P$$

$$\varkappa_{z,o} = \frac{\tilde{A}_2}{V_1} - \frac{\tilde{A}_2}{V_2} - 2\tilde{A}_1\frac{\tilde{B}_{21}}{P(V_2)} + \tilde{A}_2\frac{\tilde{B}_{11}}{P(V_2)} - \tilde{A}_2\frac{\tilde{B}_{22}}{P(V_2)} - 2\tilde{A}_1\frac{\tilde{B}_{21}}{P(V_1)} + \tilde{A}_2\frac{\tilde{B}_{11}}{P(V_1)} - \tilde{A}_2\frac{\tilde{B}_{22}}{P(V_1)}$$
$$\varkappa_{z,1} = K_2$$
$$\varkappa_{z,1} = K_1$$

with K_1 and K_2 being the two constant of integration.

Proposition 2 (The Equilibrium)

We outline the proof for the equilibrium proposition in three steps: (a) We characterize how individuals update their beliefs about the dynamics of the dividend and signal processes, $\delta(t)$ and $z(t)$, respectively. (b) We compute the price of risk as a function of these beliefs. (c) We derive the aggregation properties of the economy and characterize the market prices of risk. (d) Finally, we derive the equilibrium consumption allocation. To solve for the equilibrium in the economy with heterogeneous agents we use the representative agent technique with stochastic weights discussed by Cuoco and He (1994).

By simple substitution, the agent-specific Brownian motions are equal to:

$$dW_\delta(t) = \frac{m_\delta^n(t) - \mu_\delta(t)}{\sigma_\delta}dt + dW_\delta^n(t)$$
$$dW_z(t) = \left[a\frac{m_\delta^n(t) - \mu_\delta(t)}{\sigma_z} + b\frac{m_z^n(t) - \mu_z(t)}{\sigma_z}\right]dt + W_z^n(t)$$

Let asset prices follow the diffusion processes:

$$dP_i(t) = P_i(t)\left[\mu_i(t)\,dt + \sigma_{i\delta}(t)\,dW_\delta(t) + \sigma_{iz}(t)\,dW_z(t)\right]$$

Substituting the agent-perceived Brownian motions:

$$dP_i(t) = P_i(t)\left[\mu_i^n(t)\,dt + \sigma_{i\delta}(t)\,dW_\delta^n(t) + \sigma_{iz}(t)\,dW_z^n(t)\right]$$
$$\mu_i^n(t) = \mu_i(t) + \sigma_{i\delta}(t)\frac{\mu_\delta^n(t) - \mu_\delta(t)}{\sigma_\delta} + \sigma_{iz}(t)\left[a\frac{\mu_\delta^n(t) - \mu_\delta(t)}{\sigma_\delta} + b\frac{\mu_z^n(t) - \mu_z(t)}{\sigma_z}\right]$$

where $\mu_i^n(t)$ is the expected instantaneous return for the asset i from the prospective of agent n and $W_z^n(t)$ is agent n's individual Brownian motion. Thus, the difference in expected returns from the prospective of two different agents is given by

$$\mu_i^1(t) - \mu_i^2(t) = \sigma_{i\delta}(t)\frac{\mu_\delta^1(t) - \mu_\delta^2(t)}{\sigma_\delta(t)} + \sigma_{iz}(t)\left[a\frac{\mu_\delta^1(t) - \mu_\delta^2(t)}{\sigma_z} + b\frac{\mu_z^1(t) - \mu_z^2(t)}{\sigma_z}\right] \tag{A.1}$$

$$= \sigma_{i\delta}(t)\Psi_\delta(t) + \sigma_{iz}(t)\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right) \tag{A.2}$$

Let us define $\theta_\delta^n(t)$ to be agent n's price of dividend risk and $\theta_z^n(t)$ to be his price of signal risk. Since by no-arbitrage excess returns need to satisfy:

$$\mu_i^n(t) - r(t) = \sigma_{i\delta}(t)\theta_\delta^n(t) + \sigma_{iz}(t)\theta_z^n(t) \tag{A.3}$$

From the previous two equations we have

$$\sigma_{i\delta}(t)\left[\theta_\delta^1(t) - \theta_\delta^2(t)\right] + \sigma_{iz}(t)\left[\theta_z^1(t) - \theta_z^2(t)\right] = \sigma_{i\delta}(t)\Psi_\delta(t) + \sigma_{iz}(t)\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)$$

Since the last equation has to hold for any $\sigma_{i\delta}(t)$ and $\sigma_{iz}(t)$:

$$\theta_\delta^1(t) - \theta_\delta^2(t) = \Psi_\delta(t)$$

$$\theta_z^1(t) - \theta_z^2(t) = \left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)$$

Representative Agent with stochastic weight. Since the econonmy is dynamically complete with trading in a stock, option and riskless security, each agent stochastic discount factor is unique. Thus, to characterize the equilibrium it is convenient to think in terms of a representative agent. Clearly, due to the different filtrations perceived by each agent, the relative Pareto weights of the two agents are stochastic (see Karatzas and Shreve (1998) and Cuoco and He (1994)). Let the utility function of the representative agent be:

$$V(\delta;\lambda) = \max_{\substack{c_1,c_2 \\ s.t\ \ c_1+c_2=\delta}} \frac{c_1^\gamma}{\gamma} + \lambda\frac{c_2^\gamma}{\gamma} \tag{A.4}$$

From the first order condition, $u_1'(c_1) = \lambda u_2'(c_2)$. Solving for the optimal allocation of the aggregate consumption yields:

$$V(\delta;\lambda) = \frac{\delta^\gamma}{\gamma}\left(1 + \lambda^{\frac{1}{1-\gamma}}\right)^{1-\gamma}$$

Thus, the marginal utility is $V_\delta' = \delta^{\gamma-1}\left(1 + \lambda^{\frac{1}{1-\gamma}}\right)^{1-\gamma}$. Since the f.o.c. of each agent require $u_i'(c_i) = y_i\xi^i$ and $V_\delta'(\delta;\lambda_2) = u'(c_1) = \lambda u'(c_2)$, then

$$\xi_t^1 = \frac{1}{y_1}\delta_t^{\gamma-1}\left(1 + \lambda_t^{\frac{1}{1-\gamma}}\right)^{1-\gamma} \quad \text{and} \quad \xi_t^2 = \frac{1}{y_1}\delta_t^{\gamma-1}\frac{1}{\lambda_t}\left(1 + \lambda_t^{\frac{1}{1-\gamma}}\right)^{1-\gamma}$$

with

$$\lambda_t = \frac{y_1}{y_2}\xi_t^1/\xi_t^2 = \frac{y_1}{y_2}\eta_t$$

The constants $\lambda(0)$, y_1 and y_2 solve the static individual first order conditions and budget constraints: $E^n\left[\int V_\delta' \cdot (y_n\xi_t^n)^{\gamma-1}dt\right] = e_nP(0)$, which imply

$$E^1\left[\int (1 + \lambda_t{}^{1/1-\gamma})^{-\gamma}dt\right] = e_1P(0)$$

$$E^2\left[\int (1 + \lambda_t{}^{1/1-\gamma})^{-\gamma}\lambda_t{}^{1/1-\gamma}dt\right] = e_2P(0)$$

Using Ito's rule, λ_t satisfies the differential equation $\frac{d\lambda_t}{\lambda_t} = \frac{d\eta_t}{\eta_t} = \frac{d(\xi_1/\xi_2)}{\xi_1/\xi_2} = -(\theta_\delta^1 - \theta_\delta^2)dW_\delta^1 - (\theta_z^1 - \theta_z^2)dW_z^1$. Thus:

$$\frac{d\lambda_t}{\lambda_t} = -\Psi_\delta dW_\delta^1 - \left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)dW_z^1$$

The first order conditions for the optimal consumption choice for agent n is given by $c_n(t) = (y_n\xi^n(t))^{1/(\gamma-1)}$. Hence the individual consumption allocations can be calculated as

$$c_1(t) = \delta(t)\frac{1}{\left(1 + (y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}\right)}; \quad c_2(t) = \delta(t)\frac{(y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}}{\left(1 + (y_1\eta(t)/y_2)^{\frac{1}{1-\gamma}}\right)} \tag{A.5}$$

Proposition 2 (The Market Prices of Risk)

Applying Ito's lemma to the individual consumption process we obtain:

$$dc_n(t) = \mu^n_{c_n}(t)\,dt - \frac{c_n(t)}{\gamma-1}\theta^n_\delta(t)\,dW^n_\delta(t) - \frac{c_n(t)}{\gamma-1}\theta^n_z(t)\,dW^z_\delta(t)$$

The market clearing condition for consumption requires that $c_1(t) + c_2(t) = \delta(t)$, thus the diffusion process for the sum of individual consumptions should be identical to the diffusion process for the dividend. Hence, the following restriction follows:

$$\frac{c_1(t)}{1-\gamma}\theta^1_\delta(t) + \frac{c_2(t)}{1-\gamma}\theta^2_\delta(t) = \sigma_\delta\delta(t)$$

From $\theta^1_\delta(t) - \theta^2_\delta(t) = \Psi_\delta(t)$, the price of dividend risk is equal to $\theta^1_\delta(t) = \left[\frac{c_1(t)}{1-\gamma} + \frac{c_2(t)}{1-\gamma}\right]^{-1}\left[\sigma_\delta\delta(t) + \frac{\Psi_\delta(t)}{1-\gamma}\right]$. Simplifying the terms and using the market clearing condition we obtain the price of the dividend risk for both agent

$$\theta^1_\delta(t) = (1-\gamma)\sigma_\delta + \Psi_\delta(t)\frac{c_2(t)}{\delta(t)}; \quad \theta^2_\delta(t) = (1-\gamma)\sigma_\delta - \Psi_\delta(t)\frac{c_1(t)}{\delta(t)}$$

Substituting the solution (A.5) for the endogenously determined individual consumptions into the equation above we obtain the solution for the dividend price of risk

$$\theta^1_\delta(t) = (1-\gamma)\sigma_\delta + \frac{\Psi_\delta(t)\left(y_1\eta(t)/y_2\right)^{\frac{1}{1-\gamma}}}{\left(1+\left(y_1\eta(t)/y_2\right)^{\frac{1}{1-\gamma}}\right)}; \quad \theta^2_\delta(t) = (1-\gamma)\sigma_\delta - \frac{\Psi_\delta(t)}{\left(1+\left(y_1\eta(t)/y_2\right)^{\frac{1}{1-\gamma}}\right)}$$

For the individual price of signal risk the restriction is:

$$\frac{c_1(t)}{1-\gamma}\theta^1_z(t) + \frac{c_2(t)}{1-\gamma}\theta^2_z(t) = 0$$

Since the difference of the individual prices of signal risk is equal to $\theta^1_z(t) - \theta^2_z(t) = \left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)$, then the prices of signal risk are equal to:

$$\theta^1_z(t) = \frac{c_2(t)}{\delta(t)}\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right), \quad \theta^2_z(t) = -\frac{c_1(t)}{\delta(t)}\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)$$

Substituting the solution (A.5) for the endogenously determined individual consumptions into the equation above we obtain the solution for the signal price of risk

$$\theta^1_z(t) = \frac{\left(y_1\eta(t)/y_2\right)^{\frac{1}{1-\gamma}}\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)}{\left(1+\left(y_1\eta(t)/y_2\right)^{\frac{1}{1-\gamma}}\right)}, \quad \theta^2_z(t) = -\frac{\left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)}{\left(1+\left(y_1\eta(t)/y_2\right)^{\frac{1}{1-\gamma}}\right)}$$

Proposition A.1 (Joint distribution of dividend and SPD ratio)
The joint distribution of the stochastic weight process $\eta(t)$ and dividend process $\delta(t)$ at time s, conditional on the value of these processes at time $t < s$, $(\eta(t), \delta(t))$ is bivariate Lognormal:

$$\xi^1(t) = \eta(t)\cdot\xi^2(t) \tag{A.6}$$

$$\eta(s) = \eta(t)\exp\left(M_\eta(t,0) - \sqrt{V_\eta(s,t)}Z_\eta\right) \tag{A.7}$$

$$\delta(s) = \delta(t)\exp\left(M_\delta(t,0) - \sqrt{V_\delta(s,t)}Z_\delta\right)$$

where

$$\begin{pmatrix} Z_\eta \\ Z_\delta \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho(s,t) \\ \rho(s,t) & 1 \end{pmatrix}\right)$$

$$\rho(s,t) = Cov(s,t)/\left(\sqrt{V_\delta(s,t)V_\eta(s,t)}\right); \qquad Cov(s,t) = \sigma_\delta\left[\sum_{i=1}^{2}\frac{C_i\alpha_{\delta,i}}{\phi_i}\left(e^{\phi_i s} - e^{\phi_i t}\right)\right]$$

$$V_\delta(s,t) = \sigma_\delta(s-t) + \left[\left(\frac{s_{11}+a\,s_{12}}{\sigma_\delta}\right)^2 + \left(\frac{b\,s_{12}}{\sigma_z}\right)^2\right]\left[\frac{(s-t)^2}{2}\right] \qquad V_\eta(s,t) = V_{\eta,\delta}(s,t) + V_{\eta,\delta}(s,t)$$

$$M_\delta(s,t) = \mu^1_\delta(t)(s-t) \qquad M_\eta(s,t) = -\tfrac{1}{2}M_{\eta,z}(s,t) - \tfrac{1}{2}M_{\eta,\delta}(s,t)$$

Where $V_{\eta,\delta}(s,t)$, $V_{\eta,\delta}(s,t)$, $M_{\eta,\delta}(s,t)$ and $M_{\eta,z}(s,t)$ are defined in (A.8) and (A.9).

(1) Marginal Distribution of $\eta(s)$ conditional on time t

The dynamics of the stochastic weight process $\eta(t)$ is given by

$$\frac{d\eta(t)}{\eta(t)} = -\Psi_\delta(t)\,dW_\delta^1(t) - \left(a\Psi_\delta(t)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(t)\right)dW_z^1(t)$$

So that

$$\eta(s) = \eta(t)\exp\left\{-\int_t^s \Psi_\delta(u)\,dW_\delta^1(u) - \int_t^s \left(a\Psi_\delta(u)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(u)\right)dW_z^1(u)\right\}$$
$$\times\exp\left\{\int_t^s\left[-\frac{1}{2}\left[\Psi_\delta(u)\right]^2 - \frac{1}{2}\left[a\Psi_\delta(u)\frac{\sigma_\delta}{\sigma_z} + b\Psi_\delta(u)\right]^2\right]du\right\}$$

The Ito integrals are Normally distributed because the subintegral functions $\Psi_\delta(u)$ and $\Psi_z(u)$ are deterministic. Thus, at time s, the value of the stochastic weights process $\eta(s)$ conditional on time t, is given by

$$\eta(s) = \eta(t)\exp\left(M_\eta(s,t) - \sqrt{V_\eta(s,t)}Z_\eta\right),\ \ Z_\eta \sim N(0,1)$$

with the expected drift and variance given by

$$M_\eta(s,t) = -\frac{1}{2}M_{\eta,z}(s,t) - \frac{1}{2}M_{\eta,\delta}(s,t)$$
$$V_\eta(s,t) = V_{\eta,\delta}(s,t) + V_{\eta,\delta}(s,t)$$

$$V_{\eta,\delta}(s,t) = E\left(\int_t^s \Psi_\delta(u)\,du\right)^2 = \left[\sum_{i=1}^{2}\frac{C_i\alpha_{\delta,i}}{\phi_i}\left(e^{\phi_i s} - e^{\phi_i t}\right)\right]^2 \tag{A.8}$$
$$V_{\eta,z}(s,t) = \left[\sum_{i=1}^{2} a\frac{\sigma_\delta}{\sigma_z}\frac{C_i\alpha_{\delta,i}}{\phi_i}\left(e^{\phi_i s} - e^{\phi_i t}\right) + b\frac{C_i\alpha_{z,i}}{\phi_i}\left(e^{\phi_i s} - e^{\phi_i t}\right)\right]^2$$

Let us define the function characterizing the integrated squared difference in beliefs as

$$V_\delta = \int_t^s \Psi_\delta^2(u)\,du = \sum_{i=1}^{2}\frac{C_i\alpha_{\delta,i}^2}{2\phi_i}\left(e^{2\phi_i s} - e^{2\phi_i t}\right) + 2\frac{C_1\alpha_{\delta,1}C_2\alpha_{\delta,2}\left(e^{(\phi_1+\phi_2)s} - e^{(\phi_1+\phi_2)t}\right)^2}{\phi_1+\phi_2}$$

The functional form V_z is defined in the same fashion as V_δ with dividend related parameters replaced with signal parameters. Then we can define the components of the expected growth rate of stochastic weights $\eta(t)$ as

$$M_{\eta,\delta}(s,t) = E\left(\int_t^s \Psi_\delta^2(u)\,du\right) = V_\delta \tag{A.9}$$
$$M_{\eta,z}(s,t) = a^2\frac{\sigma_\delta^2}{\sigma_z^2}V_\delta + b^2V_z + 2\frac{ab\sigma_\delta}{\sigma_z}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\pi_{\delta,i}\pi_{z,j}\left(e^{\left(\phi_i+\phi_j\right)s} - e^{\left(\phi_i+\phi_j\right)t}\right)}{\phi_i+\phi_j}$$

The generic integral calculation used in all expressions above is $\int_t^s C\alpha e^{\phi u}du = \frac{C\alpha}{\phi}\left(e^{\phi s} - e^{\phi t}\right)$

(2) Marginal Distribution of $\delta(s)$ conditional on time t:

The dynamics of the log-dividend process $\delta(s)$, from the perspective of the first agent is:

$$d\ln\delta(t) = \mu_{l,\delta}^1(t)\,dt + \sigma_\delta dW_\delta^1(t)$$

The estimated value $\mu_{l,\delta}^1(t)$ follows the stochastic differential equation defined in (**??**). Hence the value of $\mu_{l,\delta}^1(t)$ can be presented as

$$\mu_{l,\delta}(t) = \mu_{l,\delta}(0) + \frac{s_{11} + a\,s_{12}}{\sigma_\delta}W_\delta^n(t) + \frac{b\,s_{12}}{\sigma_z}W_z^n(t)$$

Therefore, the dynamics of the log-dividend process becomes

$$\log\delta(s) = \log\delta(t) + \left[\mu_{l,\delta}^1(0)\right](s-t) + \sigma_\delta W_\delta^1(s-t) + \int_t^s\left[\frac{s_{11} + a\,s_{12}}{\sigma_\delta}W_\delta^n(u) + \frac{b\,s_{12}}{\sigma_z}W_z^n(u)\right]du$$

Therefore, the process followed by the log-dividend is that of a standard Brownian motion plus additional terms. Both terms are normal random variables as they are the sum of Normal variables $W_\delta^1(u)$ and $W_z^n(u)$. The mean and variance of the additional terms can be calculated as

$$E\left[\int_t^s \left[\frac{s_{11}+a\,s_{12}}{\sigma_\delta} W_\delta^n(u) + \frac{b\,s_{12}}{\sigma_z} W_z^n(u)\right] du\right] = 0$$
$$E\left[\int_t^s \left(\frac{s_{11}+a\,s_{12}}{\sigma_\delta} W_\delta^1(u) + \frac{b\,s_{12}}{\sigma_z} W_z^n(u)\right) du\right]^2 = \left[\left(\frac{s_{11}+a\,s_{12}}{\sigma_\delta}\right)^2 + \left(\frac{b\,s_{12}}{\sigma_z}\right)^2\right]\left[\frac{(s-t)^2}{2}\right]$$

Hence the distribution of $\log(\delta(t))$ is Normal with

$$M_\delta(s,t) = \log\delta(0) + \left[\mu_\delta^1(0) - \frac{1}{2}\sigma_\delta^2\right](s-t)$$
$$V_\delta(s,t) = \sigma_\delta(s-t) + \left[\left(\frac{s_{11}+a\,s_{12}}{\sigma_\delta}\right)^2 + \left(\frac{b\,s_{12}}{\sigma_z}\right)^2\right]\left[\frac{(s-t)^2}{2}\right]$$

(3) Covariance and Joint Distribution of $\eta(s)$ and $\delta(s)$:
Both $\log(\delta(s))$ and $\log(\eta(s))$ are conditionally Normal. Their conditional covariance can be computed as follows: since

$$\log\frac{\delta(s)}{\delta(t)} = \left[\mu_{l,\delta}^1(0)\right](s-t) + \sigma_\delta\left[W_\delta^1(s) - W_\delta^1(t)\right] + \int_t^s \frac{s_{11}+a\,s_{12}}{\sigma_\delta} W_\delta^n(u) + \frac{b\,s_{12}}{\sigma_z} W_z^n(u)\,du$$
$$\log\frac{\eta(s)}{\eta(t)} = M_\eta(s,t) - \int_t^s \Psi_\delta(u)\,dW_\delta^1(u) - \int_t^s \left(a\Psi_\delta(u)\frac{\sigma_\delta}{\sigma_z} + b\Psi_z(u)\right) dW_z^1(u)$$

their covariance is equal to the covariance between $\sigma_\delta\left[W_\delta^1(s) - W_\delta^1(t)\right]$ and $\int_t^s \Psi_\delta(u)\,dW_\delta^1(u)$. Since $\sigma_\delta\left[W_\delta^1(s) - W_\delta^1(t)\right]$ can be represented as $\int_t^s \sigma_\delta dW_\delta^1(u)$ we can calculate the covariance as a deterministic intergral

$$Cov_t(\log(\delta(s)), \log(\eta(s))) = \int_t^s \sigma_\delta\Psi_\delta(u)\,du = \sigma_\delta\left[\sum_{i=1}^2 \frac{C_i\alpha_{\delta,i}}{\phi_i}\left(e^{\phi_i s} - e^{\phi_i t}\right)\right]$$

(4) Marginal Distribution of $\delta(s)$ under true probability measure:
We characterized the transition densities from the first agent's perspective because it is convenient from asset pricing purposes. However, for econometric estimation it is important to know the transition densities under the true probability measure. The dynamics of the dividend is given by

$$d\ln\delta(t) = \mu_{l,\delta}(t)\,dt + \sigma_\delta dW_\delta(t)$$
$$d\mu_{l,\delta}(t) = n_\delta dW_{\mu_\delta}(t) \quad \text{(A.10)}$$

The integral forms of the above stochastic differential equations are given by:

$$\ln\delta(s) = \ln\delta(t) + \int_s^t \mu_{l,\delta}(u)\,du + \sigma_\delta W_\delta(t)$$
$$\mu_{l,\delta}(u) = \mu_{l,\delta}(t) + n_\delta W_{\mu_\delta}(u)$$

Following the same arguments as in section (2) it is easy to see that the transition densities are normal. The following first two moments completely define the distribution

$$E_t[\ln\delta(s)] = \ln\delta(t) + \mu_{l,\delta}(t)(s-t)$$
$$Var_t[\ln\delta(s)] = E_t\left(\left[\int_t^s n_\delta W_{\mu_\delta}(u)\,du\right]^2 + \sigma_\delta W_\delta(s)\right) = n_\delta^2\frac{(s-t)^2}{2} + \sigma_\delta(s-t)$$

■

Proposition 3 (Asset Pricing)
Stock Price:

From the Euler equation, the stock price $P(t) = \frac{1}{\xi^1(t)} E^1 \left[\int_t^T \xi^1(s) \delta(s) ds \right]$. Thus,

$$P(t) = \frac{E_t^1 \left[\int_t^T [\delta(s)]^{\gamma-1} \frac{1}{y_1} \left(1 + \left(\frac{y_1 \eta(s)}{y_2} \right)^{\frac{1}{1-\gamma}} \right)^{1-\gamma} \delta(s) ds \right]}{[\delta(t)]^{\gamma-1} \frac{1}{y_1} \left(1 + \left(\frac{y_1 \eta(s)}{y_2} \right)^{\frac{1}{1-\gamma}} \right)^{1-\gamma}}$$

where $\xi^1(t)$ is the stochastic discount factor for the first agent and $E^1[\cdot]$ is the expectation operator conditional on the information set of agent 1. From Proposition A.1, $\delta(s)$ and $\eta(s)$ are conditionally Normally distributed. Hence, the numerator can be written as

$$E_t^1 \left[\int_t^T \frac{F_\delta(s)^\gamma}{(1+F_\eta(s))^{\gamma-1}} ds \right] = \int_t^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{F_\delta(t,s,Z_\delta)^\gamma}{(1+F_\eta(t,s,Z_\eta))^{\gamma-1}} N(Z_\delta, Z_\eta, \rho(s)) \, dZ_\delta dZ_\eta ds$$

where $N(Z_\delta, Z_\eta, \rho(s))$ is bivariate Normal probability density function with correlation $\rho(s) = \frac{Cov(s,t)}{\sqrt{V_\delta(s,t)}\sqrt{V_\eta(s,t)}}$ and

$$F_\eta(t,s) = F(t,s,\eta(t), y_1, y_2, Z) = \left(\frac{y_1 \eta(t) e^{M_\eta(s,t) - \sqrt{V_\eta(s,t)} Z_\eta}}{y_2} \right)^{\frac{1}{1-\gamma}} \tag{A.11}$$
$$F_\delta(t,s) = F(t,s,\delta(t), y_1, y_2, Z) = \left(\delta(t) e^{M_\delta(s,t) - \sqrt{V_\delta(s,t)} Z_\delta} \right)$$

The calculation of the remaining part is trivial. Therefore, the stock prices are representable in terms of a deterministic integral.

<u>Option Price</u>:
From the Euler conditions, the option price $C(t,H) = \frac{1}{\xi^1(t)} E^1 \left[\xi^1(H) \max(P(H) - K, 0) \right]$. Thus,

$$C(t,H) = \frac{E^1 \left[\frac{F_\delta(t,H)^{\gamma-1}}{(1+F_\eta(t,H))^{\gamma-1}} \max(P(H, \delta(H), \eta(H)) - K, 0) \right]}{\frac{F_\delta(t,t)^{\gamma-1}}{(1+F_\eta(t,t))^{\gamma-1}}}$$

The functional form $F_\eta(s)$ and $F_\delta(s)$ are given in Equation (A.11). Since $\delta(t)$ and $\eta(t)$ are jointly Normal, as derived in Proposition A.1, then $E^1 \left[\frac{F_\delta(t,H)^{\gamma-1}}{(1+F_\eta(t,H))^{\gamma-1}} \max(P(H, \delta(H), \eta(H)) - K, 0) \right]$ is given by

$$C(t,H) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{F_\delta(t,H,Z_\delta)^{\gamma-1}}{(1+F_\eta(t,H,Z_\eta))^{\gamma-1}} \max(P(H) - K, 0) \, N(Z_\delta, Z_\eta, \rho(H,t)) \, dZ_\delta dZ_\eta$$

where $N(Z_\delta, Z_\eta, \rho(s))$ is a bivariate Normal probability density function with correlation $\rho(H,t) = \frac{Cov(H,t)}{\sqrt{V_\delta(H,t)}\sqrt{V_\eta(H,t)}}$

<u>Bond Price</u>:
Similarly, the bond price is given by $B(t,H) = \frac{1}{\xi^1(t)} E_t^1 \left[\xi^1(t,H) \right]$. Thus,

$$B(t,H) = \frac{F_\delta(t)^{1-\gamma}}{(1+F_\eta(t))^{1-\gamma}} E_t^1 \left[\frac{F_\delta(t,H)^{\gamma-1}}{(1+F_\eta(t,H))^{\gamma-1}} \right]$$

The expectation part of the product can be calculated using the joint Normality of $\delta(t)$ and $\eta(t)$ derived in Proposition A.1. Therefore,

$$E_t^1 \left[\frac{F_\delta(t,T)^\gamma}{(1+F_\eta(t,T))^{\gamma-1}} \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{F_\delta(t,s,Z_\delta)^\gamma}{(1+F_\eta(t,s,Z_\eta))^{\gamma-1}} N(Z_\delta, Z_\eta, \rho(s)) \, dZ_\delta dZ_\eta$$

Given the functional form of $F_\eta(s,t)$ and $F_\delta(s,t)$, the prices of the stock, option and bond are deterministic integrals and can be computed at any desired level of accuracy using standard numerical integration methods.

Proposition 4 (Stock and Option Holdings)
From $F_\delta(s)$ and $F_\eta(s)$, we obtain

$$\frac{\partial F_\delta(s)}{\partial \delta(t)} = \frac{F_\delta(s)}{\delta(t)}; \quad \frac{\partial F_\eta(s)}{\partial \eta(t)} = \frac{1}{1-\gamma} \frac{F_\eta(s)}{\eta(t)}; \quad \frac{\partial}{\partial \eta(t)} \left[\frac{1}{(1+F_\eta(t))^\beta} \right] = -\frac{\beta}{1-\gamma} \frac{F_\eta(s)}{\eta(t)(1+F_\eta(t))^{\beta+1}} \tag{A.12}$$

Wealth Dynamics and Sensitivity to $\delta(t)$ and $\eta(t)$:

The wealth process of the first agent is $X_1(t) = \frac{1}{\xi^1(t)} E^1 \left[\int_t^T \xi^1(s) c_1(s) ds \right]$. The consumption allocation is given by

$$[\delta(t)]^{\gamma-1} \frac{1}{y_1} \left(1 + \left(\frac{y_1 \eta(t)}{y_2}\right)^{\frac{1}{1-\gamma}}\right)^{1-\gamma}; \qquad c_1(t) = [\delta(t)] \left(1 + \left(\frac{y_1 \eta(t)}{y_2}\right)^{\frac{1}{1-\gamma}}\right)^{-1}$$

Therefore, using the results in (A.11), the wealth process is given by $X_1(t) = \frac{F_\delta(t)^{-\gamma}}{\left(1+F_\eta(t)\right)^{1-\gamma}} E^1 \left[\int_t^T \frac{F_\delta(s, Z_\delta)^\gamma}{\left(1+F_\eta\left(s, Z_\eta\right)\right)^\gamma} ds \right]$. From Fubini's theorem, the derivative of the wealth process $X(t)$, with respect to $\delta(t)$ is

$$\frac{\partial X(t)}{\partial \delta(t)} = \frac{F_\delta(t)^{-\gamma}}{(1+F_\eta(t))^{1-\gamma}} \left\{ \frac{-\gamma}{\delta(t)} E^1 \left[\int_t^T \frac{F_\delta(s, Z_\delta)^\gamma}{(1+F_\eta(s, Z_\eta))^\gamma} ds \right] + E^1 \left[\int_t^T \frac{\gamma F_\delta(s, Z_\delta)^{\gamma-1}}{\delta(t)(1+F_\eta(s, Z_\eta))^\gamma} ds \right] \right\}$$

Similarly, the derivative of the wealth process with respect to $\eta(t)$ is given by

$$\frac{\partial X(t)}{\partial \eta(t)} = \frac{F_\delta(t)^{-\gamma}}{(1+F_\eta(t))^{1-\gamma}} \left\{ -\frac{F_\eta(s)(1+F_\eta(t))}{\eta(t)} E^1 \left[\int_t^T \frac{F_\delta(s, Z_\delta)^\gamma}{(1+F_\eta(s, Z_\eta))^\gamma} ds \right] + E^1 \left[\int_t^T \frac{-\gamma}{(1-\gamma)} \frac{F_\delta(s, Z_\delta)^\gamma F_\eta(s)}{\eta(t)(1+F_\eta(t))^{\gamma-1}} ds \right] \right\}$$

Asset price Sensitivity to $\delta(t)$ and $\eta(t)$:

The stock and option price sensitivities with respect to $\delta(t)$ and $\eta(t)$ are computed from the results of Proposition 4 and equation (A.12). Since $P(t) = \frac{F_\delta(t)^{-\gamma}}{\left(1+F_\eta(t)\right)^{1-\gamma}} E_t^1 \left[\int_t^T \frac{F_\delta(s, Z_\delta)^\gamma}{\left(1+F_\eta\left(s, Z_\eta\right)\right)^{\gamma-1}} ds \right]$, from Fubini's theorem, the derivative of the wealth process $X(t)$ with respect to $\delta(t)$ is

$$\frac{\partial P(t)}{\partial \delta(t)} = \frac{F_\delta(t)^{-\gamma}}{(1+F_\eta(t))^{1-\gamma}} \left\{ \frac{-\gamma}{\delta(t)} E_t^1 \left[\int_t^T \frac{F_\delta(s, Z_\delta)^\gamma}{(1+F_\eta(s, Z_\eta))^{\gamma-1}} ds \right] + E_t^1 \left[\int_t^T \frac{\gamma F_\delta(s, Z_\delta)^{\gamma-1}}{\delta(t)(1+F_\eta(s, Z_\eta))^{\gamma-1}} ds \right] \right\}$$

Similarly, the derivative of the wealth process with respect to $\eta(t)$ is

$$\frac{\partial P(t)}{\partial \eta(t)} = \frac{F_\delta(t)^{-\gamma}}{(1+F_\eta(t))^{1-\gamma}} \left\{ -\frac{F_\eta(s)(1+F_\eta(t))}{\eta(t)} E_t^1 \left[\int_t^T \frac{F_\delta(s, Z_\delta)^\gamma}{(1+F_\eta(s, Z_\eta))^{\gamma-1}} ds \right] + E_t^1 \left[\int_t^T \frac{F_\delta(s, Z_\delta)^\gamma F_\eta(s)}{\eta(t)(1+F_\eta(t))^{\gamma-2}} ds \right] \right\}$$

Since the option price is $C(t) = \frac{F_\delta(t)^{-\gamma}}{\left(1+F_\eta(t)\right)^{1-\gamma}} E_t^1[\cdot]$, with

$$E_t^1[\cdot] = \int \int_{-\infty}^\infty \frac{F_\delta(H, Z_\delta)^{\gamma-1}}{(1+F_\eta(H, Z_\eta))^{\gamma-1}} \max(P(H) - K, 0) N(Z_\delta, Z_\eta, \rho(H,t)) dZ_\delta dZ_\eta$$

the derivative with respect to $\delta(t)$ and $\eta(t)$ are given by

$$\frac{\partial C(t)}{\partial \delta(t)} = \frac{F_\delta(t)^{-\gamma}}{(1+F_\eta(t))^{1-\gamma}} \left[\frac{-\gamma}{\delta(t)} E_t^1[\cdot] + \frac{\partial\left(E_t^1[\cdot]\right)}{\partial \delta(t)} \right]$$

$$\frac{\partial C(t)}{\partial \eta(t)} = \frac{F_\delta(t)^{-\gamma}}{(1+F_\eta(t))^{1-\gamma}} \left[-\frac{F_\eta(s)(1+F_\eta(t))}{\eta(t)} E_t^1[\cdot] + \frac{\partial\left(E_t^1[\cdot]\right)}{\partial \eta(t)} \right]$$

where the derivatives of the expectation part can be calculated as

$$\frac{\partial\left(E_t^1[\cdot]\right)}{\partial \delta(t)} = \int \int_{P(H)>K} \frac{F_\delta(H, Z_\delta)^{\gamma-1}}{\left(1+F_\eta\left(H, Z_\eta\right)\right)^{\gamma-1}} \left\{ \frac{-\gamma}{\delta(t)} (P(H) - K) + \frac{\partial P(H)}{\partial \delta(t)} \right\} N(Z_\delta, Z_\eta, \rho(H,t)) dZ_\delta dZ_\eta$$

$$\frac{\partial\left(E_t^1[\cdot]\right)}{\partial \eta(t)} = \int \int_{P(H)>K} \frac{F_\delta(H, Z_\delta)^{\gamma-1}}{\left(1+F_\eta\left(H, Z_\eta\right)\right)^{\gamma-1}} \left\{ -\frac{F_\eta(s)\left(1+F_\eta(t)\right)}{\eta(t)} (P(H) - K) + \frac{\partial P(H)}{\partial \eta(t)} \right\} N(Z_\delta, Z_\eta, \rho(H,t)) dZ_\delta dZ_\eta$$

the derivative of the future stock prices with respect to changes in $\delta(t)$ and $\eta(t)$ can be calculated using the chain rules $\frac{\partial P(H)}{\partial \delta(t)} = \frac{\partial P(H)}{\partial \delta(H)} \frac{\partial \delta(H)}{\partial \delta(t)}$ and $\frac{\partial P(H)}{\partial \eta(t)} = \frac{\partial P(H)}{\partial \eta(H)} \frac{\partial \eta(H)}{\partial \eta(t)}$.

Portfolio Choice in the Heston Model

The Heston (1993) model assumes that under the risk-neutral measure the processes for stock returns and volatility are given by

$$\frac{dS_t}{S_t} = (r - \delta)\, dt + \sqrt{V_t} dW_S\,(t)$$
$$dV_t = k_v\,(\theta_v - V_t)\, dt + \sigma_v \sqrt{V_t} dW_V\,(t)$$

The correlation coefficient between stock prices and volatility is $Cov\,(dW_S\,(t)\,, dW_V\,(t)) = \rho dt$. We consider an investor who maximizes the expected utility of his terminal wealth

$$\max_{\pi(t)} E\left(\frac{W^\gamma}{\gamma}\right)$$

As Liu and Pan (2003) show, the solution of this problem is

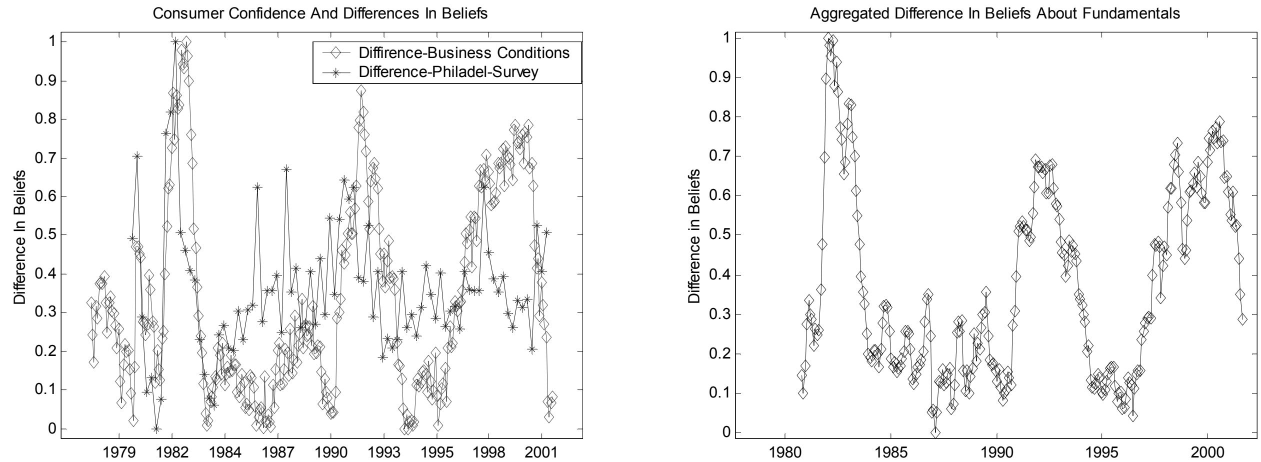
$$\pi_1\,(t) = \frac{\lambda_1}{1-\gamma} - \frac{\lambda_2 \rho}{(1-\gamma)\sqrt{1-\rho^2}} - \pi_2\,(t)\,\frac{(\partial C_t / \partial S_t)\, S_t}{C_t}$$
$$\pi_2\,(t) = \left[\frac{\lambda_2}{(1-\gamma)\,\sigma_v \sqrt{1-\rho^2}} + H\,(T-t)\right]\frac{C_t}{(\partial C_t / \partial V_t)}$$

where C_t is the price of an option calculated as in Heston, $(\partial C_t / \partial S_t)$ and $(\partial C_t / \partial V_t)$ are the sensitivities of the option price to changes in the stock price and volatility, λ_1 and λ_2 are the prices of risk of two diffusions, stock diffusion and volatility diffusion, and $H\,(T-t)$ is the following functional form

$$H\,(\tau) = \frac{\exp(k_2 \tau)}{2k_2 + (k_1 + k_2)\,(\exp\,(k_2 \tau) - 1)}\,\frac{\gamma\,(\lambda_1 + \lambda_2)}{(1-\gamma)^2}, \qquad k = k_v - \lambda_2$$
$$k_1 = k - \frac{\gamma}{1-\gamma}\left(\lambda_1 \rho + \lambda_2 \sqrt{1-\rho^2}\right)\sigma, \qquad k_2 = \sqrt{k_1^2 - \frac{\gamma\,(\lambda_1 + \lambda_2)}{(1-\gamma)^2}\sigma_v^2}$$

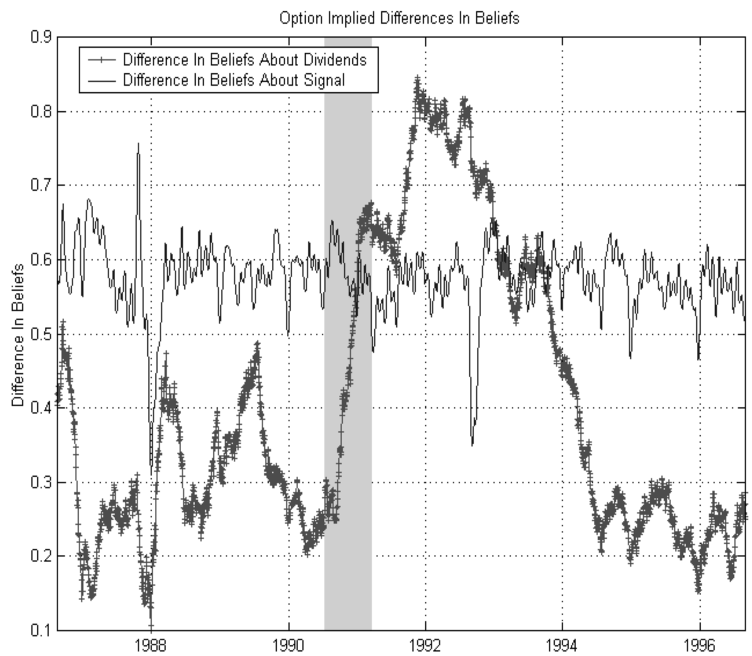
See Liu and Pan (2003), Liu (2001) and Lui, Longstaff, and Pan (2003) for a detailed derivation of the formulae.

Difference in Beliefs



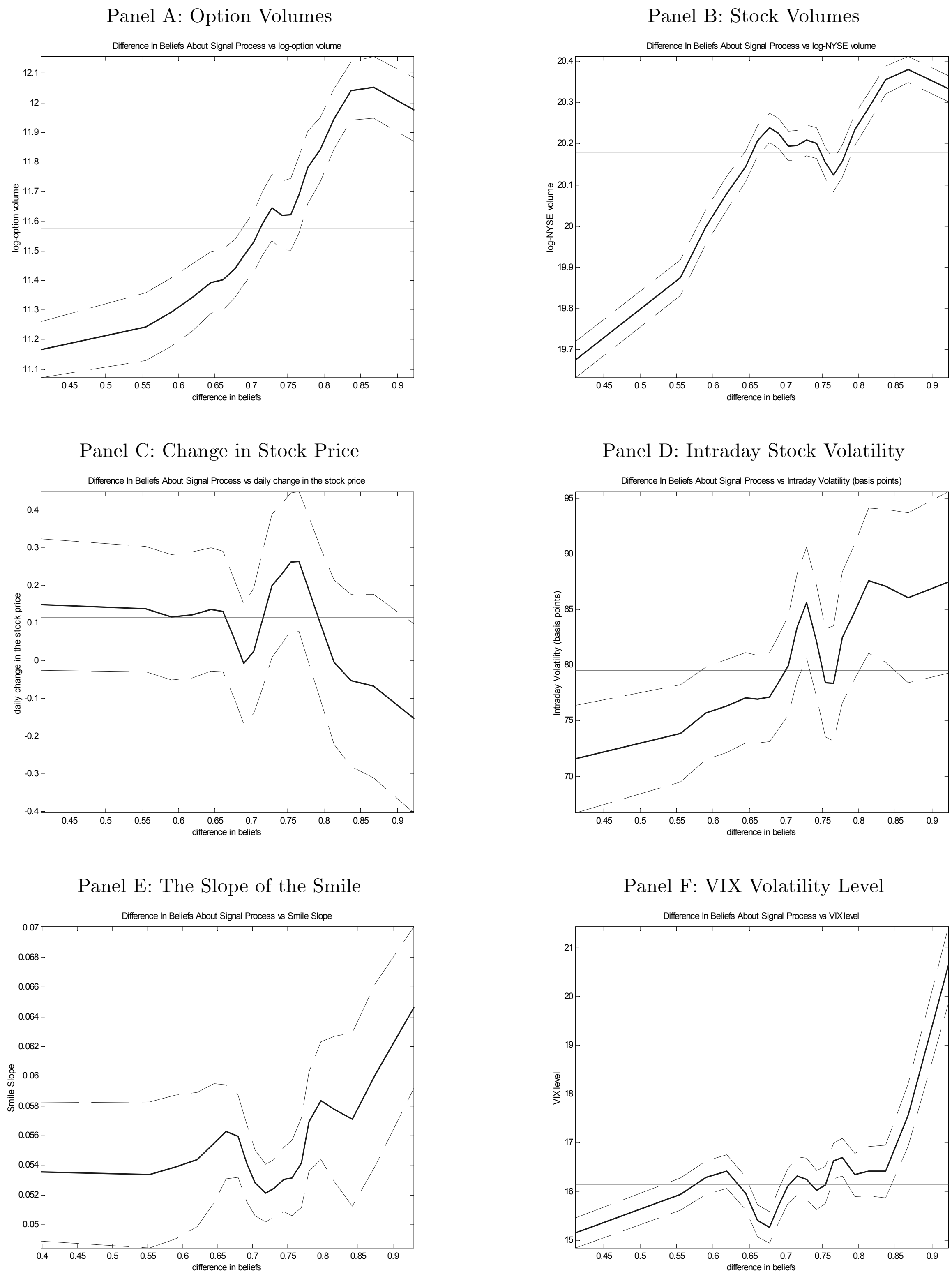
The left figure shows the dynamics of the difference in beliefs implied by the consumer confidence survey and the survey of Professional forecasters data. The right figure shows the Difference in Beliefs Index about the dividend process. This is obtained by aggregating information from the two surveys.

Figure 3: **Differences In Beliefs**



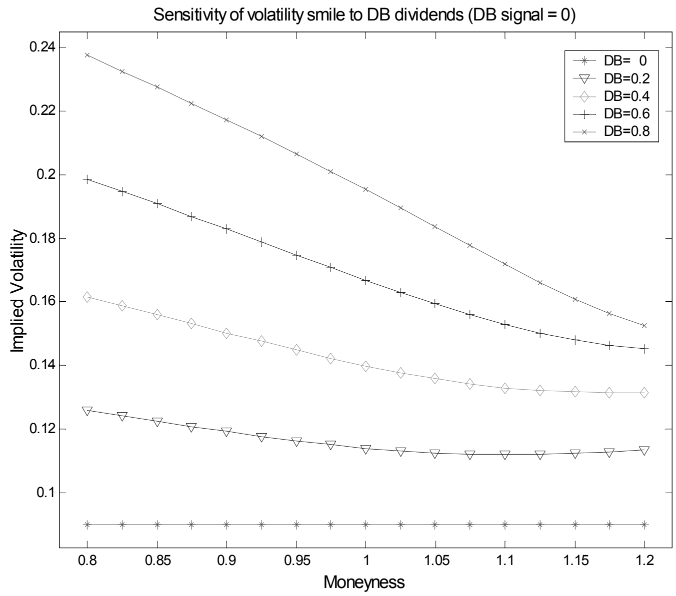
This figure shows the dynamics of the difference in beliefs about the signal and dividend processes implied by the option market. The difference in beliefs about the dividend risk is estimated directly from survey data and is model independent. The signal difference in beliefs is estimated as a latent variable using the model pricing restrictions.

Figure 4: **Option-Implied Difference In Beliefs**



Panel A shows the relation between the logarithm of option volume and the difference in beliefs $\Psi_{z,t}$. Panel B shows the relation between the logarithm of the NYSE stock volume and $\Psi_{z,t}$. Panel C shows the relation between changes in the stock price and $\Psi_{z,t}$. Panel D shows the relation between intraday volatility and $\Psi_{z,t}$. Panel E shows the relation between the slope of the smile and $\Psi_{z,t}$. Panel F shows the relation between the VIX Volatility index and $\Psi_{z,t}$. The solid line is the non-parametric estimate, while the dotted lines show the 95% confidence interval.

FIGURE 5: **The Impact of Differences In Beliefs: A Non-Parametric View**



This figure presents the shape of the model-implied volatility smile for different levels of the signal difference in beliefs $\Psi_{z,t}$.

FIGURE 6: **Implied Volatilty Smile and Differences in Beliefs**

Table 1: Open Interest Summary Statistics

This table summarizes the sample statistics for the open interest of SPX call (put) options. The open interest of Put options is in parenthesis. The open interest is measured as the average number of outstanding contracts. The sample period is between January 1986 and August 1996.

SPX Open Interest: Calls (Puts)

	$0.00 < \tau < 45.00$	$45.00 < \tau < 90.00$	$90.00 < \tau < 369.00$
$0.00 < K/S < 0.78$	4165	4844	730
	(7408)	(7896)	(3556)
$0.78 < K/S < 0.92$	2878	2419	1356
	(7845)	(7137)	(5470)
$0.92 < K/S < 0.98$	6354	3390	2501
	(12208)	(6829)	(5293)
$0.98 < K/S < 1.00$	11926	5329	3565
	(12623)	(6446)	(4568)
$1.00 < K/S < 1.02$	12377	5703	3916
	(7216)	(4124)	(3394)
$1.02 < K/S < 1.08$	8538	5295	3499
	(2494)	(2015)	(1766)
$1.08 < K/S < 1.20$	4140	3589	2388
	(1886)	(2668)	(895)
$1.20 < K/S < 1.59$	2557	2170	702
	(2088)	(1874)	(599)

Table 2: Open Interest Summary Statistics

This table presents the summary statistics for the put and call option volume and open interest on an annual basis. The average daily volume is measured in terms of the number of contracts traded per day. The average open interest is the average number of contracts outstanding per day. Put option volume and open interest is given in parentheses.

SPX Trading Volume: Calls (Puts)

	1986	1987	1988	1989	1990	1991	1992	1993	1994	1995
$0.78 < K/S < 0.92$	328	899	329	877	495	1269	784	236	397	1262
	(432)	(2054)	(543)	(2129)	(3953)	(4314)	(3929)	(1900)	(2205)	(5528)
$0.92 < K/S < 0.98$	828	2342	936	2287	3391	5041	3154	2982	2930	6027
	(1466)	(3907)	(1148)	(3111)	(6165)	(7898)	(8934)	(14364)	(21218)	(24492)
$0.98 < K/S < 1.00$	609	2233	1494	2019	3397	4434	4045	5507	9044	9769
	(677)	(2338)	(1128)	(2074)	(3919)	(5053)	(5475)	(9309)	(18407)	(16401)
$1.00 < K/S < 1.02$	696	2235	1636	2564	3942	3924	4241	8422	13342	13366
	(364)	(1371)	(973)	(1732)	(2550)	(2424)	(2612)	(4787)	(10437)	(5986)
$1.02 < K/S < 1.08$	1180	2822	2112	2810	6226	5295	5511	8639	17749	11512
	(271)	(843)	(447)	(666)	(3282)	(1390)	(1134)	(1695)	(4180)	(1241)
$1.08 < K/S < 1.20$	553	1474	814	835	2476	1165	715	258	1531	1219
	(63)	(759)	(1270)	(335)	(1047)	(241)	(169)	(61)	(239)	(268)

SPX Open Interest: Calls (Puts)

	1986	1987	1988	1989	1990	1991	1992	1993	1994	1995
$0.78 < K/S < 0.92$	1111	1891	1026	2257	1036	2864	1559	865	996	4565
	(1243)	(2429)	(1819)	(4823)	(5782)	(9461)	(8206)	(7375)	(7597)	(12336)
$0.92 < K/S < 0.98$	1581	2224	2467	3293	3061	5304	3505	3610	3539	7115
	(1436)	(2349)	(2435)	(4409)	(5386)	(8018)	(6684)	(9677)	(13323)	(12595)
$0.98 < K/S < 1.00$	1290	2249	4081	4520	5315	7649	6174	8152	9522	11035
	(1112)	(1806)	(3023)	(4171)	(6168)	(7928)	(6234)	(9481)	(15166)	(10004)
$1.00 < K/S < 1.02$	1209	2094	3680	4875	5887	6472	6267	9387	12623	10807
	(909)	(1415)	(2285)	(3453)	(5083)	(4601)	(3589)	(5613)	(10817)	(3982)
$1.02 < K/S < 1.08$	890	1667	2605	3368	5346	5837	5396	6674	11569	6731
	(440)	(936)	(1252)	(1433)	(4113)	(2096)	(1318)	(1698)	(4210)	(1207)
$1.08 < K/S < 1.20$	821	1877	1782	3124	4531	5346	4124	1069	4183	3038
	(201)	(1263)	(3002)	(1863)	(3003)	(1640)	(1003)	(306)	(655)	(542)

Table 3: Estimated Parameters of The Models

This table reports the parameter estimates for the three option pricing models: the Black and Scholes model, the Heston stochastic volatility model and the Difference in Beliefs (DB) model. The Black and Scholes and the Heston models are estimated cross-sectionally so that the parameter values reported are the sample averages of these cross-sections. The standard deviations are given in the parenthesis.

Parameter Estimates

	Heston	DB-model
σ_z		0.061
		(0.02)
σ_δ		0.038
		(0.01)
μ_δ		0.059
		(0.04)
γ		-0.95
		(0.45)
a		0.65
		(0.11)
b		0.35
		(0.13)
n_z		0.025
		(0.05)
n_δ		0.037
		(0.02)
Implied Average Ψ_z		0.61
		(0.19)
Implied Average Ψ_δ		0.64
		(0.1)
θ_v	0.042	
	(0.023)	
k_v	1.294	
	(1.402)	
σ_v	0.431	
	(0.252)	
ρ	-0.634	
	(0.383)	

Table 4: Specification Test for Option Prices

This table presents the results for the model specification test. Let the estimation error be

$$\varepsilon_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right) = \frac{C_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right)}{C\left(t, K_i/S, T_i\right)} - 1$$

The structural parameters of the model are estimated using option prices and open interest for options with $K_i/S = \{0.97, 1.03\}$ and $T_i = \{45, 90\}$ days. Then, we use the pricing errors of six options with moneyness levels K_i/S ={0.92, 1.00 and 1.08} and maturity T_i ={30, 135} to compute a Chi-square test. Let $C_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right)$ and $OI_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right)$ be the model-implied call option price and open interest, conditional on the vector of structural parameters Θ, the differences in beliefs $\Psi_{z,t}$, and $\Psi_{\delta,t}$, the strike price K_i and the time to maturity T_i. The sample counterparts of the previous variables are $C\left(t, K_i/S, T_i\right)$ and $OI\left(t\right)$ respectively. Consider the following J_T statistics

$$J_T = \left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t\right)' \Sigma^{-1} \left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t\right)$$

The variance-covariance matrix Σ is estimated using Newey and West (1987) with 10 lags to adjust for potential heteroscedasticity and autocorrelation in the pricing errors. The asymptotic distribution of the test-statistics is a Chi-square with 6 degrees of freedom. The table reports the level of the J_T statistics with the associated p-values in parenthesis.

	Maturity		
Moneyness (K/S)	30 days	135 days	All Maturities
$K/S = 0.92$	0.874	0.950	1.468
	(0.35)	(0.33)	(0.48)
$K/S = 1.00$	0.609	0.656	1.227
	(0.435)	(0.418)	(0.542)
$K/S = 1.08$	2.176	2.340	4.587
	(0.14)	(0.126)	(0.101)
All Strikes	3.209	2.856	7.015
	(0.36)	(0.414)	(0.319)

Table 5: Specification Test for Open Interests

This table presents the results for the specification test based on option interest. We first estimate the structural parameters for the model using (a) prices of the subset of options with moneyness $K_i/S = \{0.97, 1.03\}$ and maturity $T_i = \{45, 135\}$ days and (b) the open interest of at-the-money option with $K_i/S = 1$ and maturity $T_i = 90$ days. Then we compute the fitting error on the open interest of the remaining set of options with moneyness levels $K_i/S = \{0.97, 1.03\}$ and maturity $T_i = \{45, 135\}$. The out-of-sample fitting errors for the remaining subset of options expressed in percentage terms are defined as:

$$\varepsilon_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right) = \frac{OI_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right)}{OI\left(t, K_i/S, T_i\right)} - 1$$

where $OI_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right)$ is the model-implied call open interest, conditional on the vector of structural parameters Θ, the differences in beliefs $\Psi_{z,t}$, and $\Psi_{\delta,t}$, the strike price K_i, and the time to maturity T_i. We construct the following J_T statistics

$$J_T = \left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t\right)' \Sigma^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t\right)$$

The variance-covariance matrix Σ is estimated using Newey and West (1987) with 10 lags to adjust for potential heteroscedasticity and autocorrelation in the pricing errors. The asymptotic distribution of the test-statistics is a Chi-square with 4 degrees of freedom. The table reports the level of the J_T statistics with the associated p-values in parenthesis.

	Maturity		
Moneyness (K/S)	30 days	180 days	All Maturities
$K/S = 0.97$	1.065	1.470	3.384
	(0.302)	(0.225)	(0.184)
$K/S = 1.03$	1.904	2.413	4.129
	(0.168)	(0.12)	(0.127)
All Strikes	2.960	3.266	6.088
	(0.228)	(0.195)	(0.193)

Table 6: The Open Interest

This table presents the mean absolute percentage error for the open interest. We consider the open interest implication for two models: the general equilibrium DB model and the partial equilibrium model with stochastic volatility and jumps of Liu and Pan (2003) and Heston (1993). We fit both models to all option prices and to one open interest $K_i/S = \{0.97, 1.00, 1.03\}$ and maturity $T_i = \{45, 135\}$ days. The open interest fitting errors are defined as:

$$\varepsilon_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right) = abs\left[\frac{OI_t\left(\Theta, \Psi_{z,t}, \Psi_{\delta,t}, K_i/S, T_i\right)}{OI\left(t, K_i/S, T_i\right)} - 1\right]$$

		Maturity	
Moneyness (K/S)	Model	45 days	135 days
$K/S = 0.97$	SVJ-Open Interest	33.27%	14.93%
	DB-Open Interest	18.94%	13.11%
$K/S = 1$	SVJ-Open Interest	24.75%	15.86%
	DB-Open Interest	25.12%	10.13%
$K/S = 1.03$	SVJ-Open Interest	31.90%	21.22%
	DB-Open Interest	27.30%	19.11%

Table 7: Option-Implied Difference In Beliefs: Regressions

This table reports the results of the regression of S&P 500 Index options and NYSE spot trading volume on information flow proxies and differences in beliefs. We run the following regressions:

$$\log(OptVlm_t) = \beta_0 + \beta_1 \log(V_t) + \beta_2(V_t - \widehat{V}_t) + \beta_3 \log(\Psi_{z,t}) + \beta_4 \log(\Psi_{\delta,t}) + e_t$$

where $OptVlm_t$ stands for option volume at time t, V_t stands for implied volatility, $\widehat{V}_t$ is the volatility implied by the model, $\Psi_{z,t}$ is the difference in beliefs about signal and $\Psi_{\delta,t}$ is the difference in beliefs about dividends. We also run the equivalent regression for the stock trading volume.

$$\log(StockVlm_t) = \beta_0 + \beta_1 \log(V_t) + \beta_2(V_t - \widehat{V}_t) + \beta_3 \log(\Psi_{z,t}) + \beta_4 \log(\Psi_{\delta,t}) + e_t$$

The Newey-West heteroscedasticity and autocorrelation adjusted t-statistics are reported in parenthesis.

	log-Option Volume	log-spotVolume
Intercept	10.16	18.14
t-stats	(3.14)	(4.57)
$V_t - \widehat{V}_t(\Psi)$	0.25	0.53
t-stats	(2.49)	(2.44)
$\widehat{V}_t(\Psi)$	0.32	0.51
t-stats	(3.92)	(2.67)
$\Psi_{z,t}$	0.96	0.38
t-stats	(1.95)	(2.57)
$\Psi_{\delta,t}$	0.14	-0.16
t-stats	(1.45)	(0.34)
R^2	22%	13%

Table 8: Hedging Errors

This table reports the hedging performance of the model. For comparison, we report the equivalent hedging errors generated by the BS and Heston models. We consider the time t problem of hedging a short position in a call option with τ periods to maturity and strike price K. The hedging errors are defined as:

$$H(t+\Delta t) = x_s(t)\, S(t+\Delta t) + x_0(t)\, e^{r(t)\Delta t} + x_C(t)\, C(t+\Delta t, \tau-\Delta t, K_1) - C(t+\Delta t, \tau-\Delta t, K)$$

where $x_S(t)$, $x_0(t)$ and $x_C(t)$ are the model-implied portfolio weights for the underlying asset, the cash account and a call option, respectively, with the same maturity but different strike prices. In the case of Black-Scholes, we consider delta-vega hedging. We construct the desired hedge and calculate the hedging errors at time $t+\Delta t$. The estimation is updated at each time step. We calculate the average pricing errors for each moneyness and maturity.

		1 week ahead hedging		
Moneyness (K/S)	Model	<45 days	45 to 180 days	All Maturities
$K/S < 0.95$	Black ScholesDelta-Vega	0.361	0.382	0.367
	Heston	0.365	0.348	0.359
	DB	0.316	0.406	0.333
$0.95 < K/S < 1.05$	Black ScholesDelta-Vega	0.313	0.306	0.311
	Heston	0.296	0.288	0.293
	DB	0.271	0.298	0.280
$K/S > 1.05$	Black ScholesDelta-Vega	0.361	0.397	0.373
	Heston	0.313	0.289	0.304
	DB	0.331	0.351	0.321
Average	Black ScholesDelta-Vega	0.348	0.364	0.353
	Heston	0.325	0.309	0.320
	DB	0.301	0.339	0.305

Table 9: Predicting Future Volatility

This table shows the results of the volatility regressions. We run the following regressions

$$V_{t+\tau} = \alpha_0 + \alpha_1 IV_t + \alpha_2 V_t + \alpha_3 \Psi_{z,t} + \alpha_4 \Psi_{\delta,t}$$

where $V_{t+\tau}$ is the future realized volatility τ periods ahead, IV_t is the option-implied volatility,$\Psi_{z,t}$ is the the difference in beliefs about signal and $\Psi_{\delta,t}$ is the difference in beliefs about dividend. We analyze the forecasting power for horizons from 1 week to 6 month. The standard errors are adjusted for overlapping errors and the t-statistics are shown in parenthesis.

	Forecast Horizon					
	5 days	2 weeks	1 month	2months	3 monthts	6 months
Intercept	0.018	0.034	0.035	0.036	0.029	0.057
t-stats	(1.08)	(3.14)	(3.07)	(3.96)	(5.65)	(4.39)
IV_t	0.650	0.780	0.752	0.573	0.446	0.385
t-stats	(5.54)	(5.21)	(4.32)	(2.39)	(3.05)	(2.11)
V_t	0.345	0.270	0.245	0.294	0.190	0.167
t-stats	(2.43)	(1.34)	(1.54)	(1.21)	(0.83)	(0.74)
$\Psi_{z,t}$	0.018	0.037	0.031	0.044	0.035	0.021
t-stats	(1.56)	(1.71)	(2.16)	(3.36)	(3.38)	(1.86)
$\Psi_{\delta,t}$	0.039	0.032	0.060	0.131	0.190	0.134
t-stats	(0.98)	(0.17)	(0.24)	(0.89)	(1.33)	(1.25)
R^2	24%	36%	36%	41%	43%	39%

Table 10: Violation Frequencies for One Factor Models

The table shows the empirical frequency of type-1 and type-2 violations (Definition 2). The violation frequency is reported on an annual basis.

CBOE-CALLS

Year	Number of Obs	Violation 1	Violation 2
1986	9984	18.90%	7.97%
1987	17283	20.97%	8.25%
1988	13760	24.06%	4.16%
1989	15529	17.78%	7.31%
1990	19720	20.25%	4.92%
1991	17260	21.29%	5.04%
1992	18939	23.10%	6.05%
1993	19711	16.93%	5.80%
1994	21118	18.68%	4.08%
1995	21796	20.37%	10.87%
1996	14346	21.24%	5.50%

CBOE-PUTS

Year	Number of Obs	Violation 1	Violation 2
1986	9984	19.16%	1.51%
1987	17283	15.45%	2.00%
1988	13760	20.80%	4.30%
1989	15529	15.41%	2.71%
1990	19720	19.30%	2.07%
1991	17260	20.84%	1.63%
1992	18939	22.07%	2.73%
1993	19711	19.30%	2.02%
1994	21118	17.54%	3.46%
1995	21796	18.61%	2.66%
1996	14346	21.26%	3.58%

Table 11: Empirical and Model-Implied Violation Frequency

The table summarizes the violation frequency of one factor no-arbitrage restrictions. The violations are classified as in Definition 2. Panel A shows the observed empirical frequencies. Panel B shows the violation frequencies implied by the model using Monte-Carlo simulations. For each set of parameters (moneyness, time to maturity) we calculate the option price. Then, we step ahead one day and simulate 1000 realizations for the price of the underlying asset. For each realization, we calculate the corresponding option price. Finally, we compute the violation frequency.

Panel A: Empirical Frequency of Violations

Moneyness		<45 days	45 to 180 days
$K/S < 0.95$	Violation 1	16.06%	16.50%
	Violation 2	25.96%	14.74%
$0.95 < K/S < 1.05$	Violation 1	14.30%	16.61%
	Violation 2	9.35%	11.22%
$K/S > 1.05$	Violation 1	12.65%	15.07%
	Violation 2	3.41%	5.17%

Panel B: Model-Implied Frequency of Violations

Moneyness-K/S		<45 days	45 to 180 days
$K/S < 0.95$	Violation 1	13.10%	8.53%
	Violation 2	2.19%	2.12%
$0.95 < K/S < 1.05$	Violation 1	8.98%	9.74%
	Violation 2	1.95%	1.32%
$K/S > 1.05$	Violation 1	12.01%	14.30%
	Violation 2	1.67%	1.84%

TABLE 12: LOGIT ANALYSIS OF VIOLATIONS

This table summarizes the Logit regression results for the violation frequency according to Definition 2.

$$\Pr\left(y_{it}^{(j)}=1\right)=F\left(\beta_0+\beta_1\log\left(\widehat{V}_{ti}(\Psi)\right)+\beta_2(V_{ti}-\widehat{V}_{ti}(\Psi))+\beta_3\log\left(\Psi_{z,t}\right)+\beta_4\log\left(\Psi_{\delta,t}\right)\right)$$

Panel A reports the parameter estimates. Panel B reports the marginal impact of a one standard deviation change in $\Psi_{z,t}$, $\Psi_{\delta,t}$ and the Implied Volatility onto the probability of violation. For moneyness and maturity, the marginal impact is computed given a 1% change in the exogeneous variables. V_t is the implied volatility, $\hat{V}_t(\Psi)$ is the stochastic model implied volatility.

PANEL A: Logit Parameter Estimates

	Violation 1 Calls	Violation 2 Calls	Violation 1 Puts	Violation 2 Puts
Constant	-0.42	-0.66	-1.51	-1.48
	(3.45)	(6.77)	(11.22)	(4.07)
Moneyness (K/S)	1.02	-0.99	-0.56	-0.42
	(2.85)	(6.09)	(0.83)	(5.13)
Maturity	0.30	-0.25	0.20	-0.11
	(7.29)	(5.95)	(5.22)	(2.41)
$\Psi_{z,t}$	0.32	0.14	0.42	0.11
	(10.39)	(2.58)	(5.36)	(1.43)
$\Psi_{\delta,t}$	0.36	-0.16	0.32	0.52
	(1.62)	(1.12)	(0.92)	(0.98)
$\widehat{V}_{ti}(\Psi)$	1.30	0.21	0.55	0.16
	(9.15)	(3.02)	(5.35)	(1.43)
$V_{ti}-\widehat{V}_{ti}(\Psi)$	0.85	0.10	0.20	0.10
	(1.64)	(1.25)	(1.18)	(1.09)
R^2	16%	13%	11%	9%

PANEL B: Marginal Impact of each Variable

		Effect on the Probability of Violations			
	Average Change	Violation 1	Violation 2	Violation 1	Violation 2
Variable Name		Calls	Calls	Puts	Puts
Moneyness (K/S)	0.1	2.00%	-1.61%	-0.83%	-0.50%
Maturity	0.02	0.12%	-0.08%	0.06%	-0.03%
$\Psi_{z,t}$	0.25	1.56%	0.57%	1.55%	0.35%
$\Psi_{\delta,t}$	0.1	0.69%	-0.26%	0.47%	0.63%
$\widehat{V}_{ti}(\Psi)$	0.05	1.27%	0.17%	0.41%	0.10%
$V_{ti}-\widehat{V}_{ti}(\Psi)$	0.02	0.33%	0.03%	0.06%	0.02%