

# Exchange Rate Volatility and the Forward Premium Anomaly

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## Abstract

I ask whether the forward premium anomaly is consistent with an arbitrage-free model for the exchange rate and term structures of interest rates in two currencies. I use a two-currency term structure model to examine two sets of currency pairs: the U.S. Dollar and British Pound, and the U.S. Dollar and Euro. Previous papers in this literature have failed to match exchange rate volatility, which is an essential component of the risk premium in exchange rate returns. A distinguishing feature of this paper is that I use exchange rate option prices to estimate the model. Options provide valuable information because they are sensitive to the level of volatility and to the pricing of volatility risk. When options are used to estimate the model, it successfully captures both exchange rate volatility *and* the term structure of interest rates in both currencies. Using simulated data, I show that the model also replicates the empirical findings in Fama (1984) and is consistent with the forward premium anomaly.

# 1 Introduction

Changes in exchange rates are a significant determinant of returns on foreign investments, yet existing research has not identified a risk premium that reconciles the empirical properties of exchange rate returns with prices of other assets in financial markets. One such empirical property that existing pricing models have failed to match is the forward premium anomaly: the tendency for currencies with high interest rates to appreciate against currencies with lower interest rates, rather than depreciate as uncovered interest rate parity would suggest. I ask whether the forward premium anomaly is consistent with an empirical pricing model that captures exchange rate volatility *and* the term structures of interest rates in both currencies.

To address this question, I present and estimate a dynamic model for the exchange rate and term structure of interest rates in two currencies. A distinguishing feature of this paper is that I use exchange rate option prices and the term structures of interest rates in both currencies to estimate the model. I find that options provide valuable information about exchange rate volatility and the risk premium in exchange rate returns that is much harder to identify using only time-series data on exchange rates and interest rates in each currency.

It is logical to examine the risk premium in exchange rate returns jointly with the term structures of interest rates; the same factors that determine the risk premium in exchange rate returns can also affect risk premia in the term structure of interest rates in each currency. Moreover, exchange rates and nominal interest rates may both depend on expectations about the same macroeconomic variables such as inflation. Nielsen and Saá-Requejo (1993), Backus et al. (2001), Bansal (1997), and Hodrick and Vassalou (2002) first studied the forward premium anomaly in the context of two-currency term structure models. More recently, Ahn (2004), Dewachter and Maes (2001), and Inci and Lu (2004) have used more flexible

models. These previous papers fail to match, or neglect to examine, exchange rate volatility.<sup>1</sup> I use option prices to estimate the model in order to better match exchange rate volatility, which is a vital component of the risk premium in exchange rate returns.

I estimate a 4-factor version of the model with quasi-maximum likelihood using the joint time-series of swap rates in two currencies, exchange rate returns, and prices of at-the-money options on the exchange rate. Although the model nests previous two-currency affine term structure models that have been empirically studied in the literature, it does not accurately capture the variation in exchange rate volatility unless I include options in estimation. When I estimate the model with U.S. and U.K. data without using option prices, the 3-month option-implied volatility from the model has a correlation of -16% with the actual option-implied volatility in the data. When I use options to estimate the model it captures the variation in option-implied volatility and this correlation increases to 89%.

The results are similar when I estimate the model with U.S. and Euro data. When I don't use options to estimate the model, the 1-month option-implied volatility from the model has a correlation of 28% with the actual option-implied volatility in the data. When I use options to estimate the model, this correlation increases to 67%. Moreover, for both sets of currency pairs, when I estimate the model using options, it still maintains a very good fit to the term structures of interest rates in each currency.

I then examine whether the models that I estimate are consistent with the forward premium anomaly which is characterized by the following regression from Fama (1984),

$$\ln S_{t+\Delta t} - \ln S_t = \alpha + \beta (r_t - \tilde{r}_t) + \varepsilon_t .$$

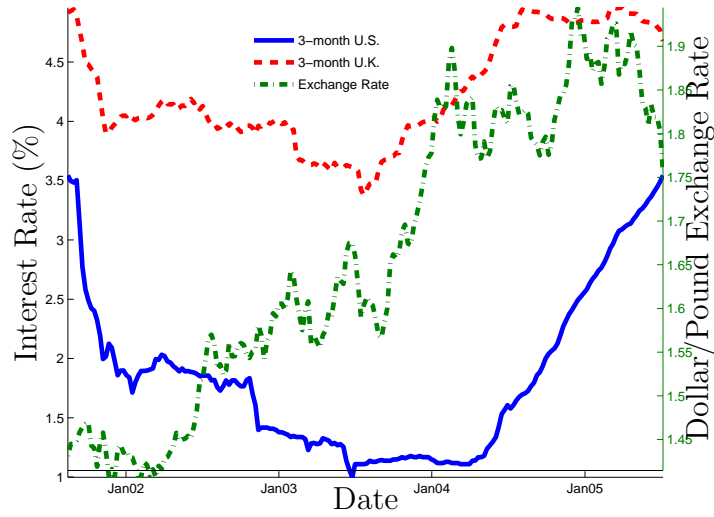
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<sup>1</sup>A notable exception is Brandt and Santa-Clara (2002) who use an additional risk factor to match exchange rate volatility. However, they do not examine the implications for the forward premium anomaly because they assume that the additional risk factor is not priced and therefore does not affect the risk premium in exchange rate returns.

Here,  $S_t$  is the nominal exchange rate expressed in units of domestic currency per unit of forward currency,  $r_t$  is the nominal domestic interest rate from  $t$  until  $t + \Delta t$ , and  $\tilde{r}_t$  is the nominal foreign interest rate from  $t$  until  $t + \Delta t$ . Uncovered interest rate parity predicts that  $\hat{\beta} = 1$ , but Fama (1984) and others find that  $\hat{\beta} < 0$  for most currency pairs.

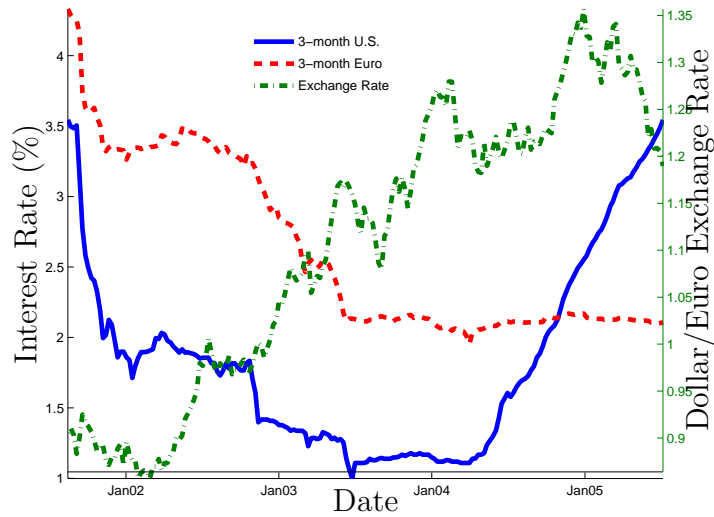
From August 2001 until July 2005 (the sample period over which I have option prices), 3-month Libor rates in the U.K. were about 2.41% higher on average than those in the U.S., but the Pound appreciated against the Dollar by 22.25%. Figure 1 plots this data. Fama's regression coefficient for this period is  $\hat{\beta} = -4.50$ . To test whether the model matches Fama's regression result, I simulate four years of weekly exchange rate and interest rate data 1,000 times using the historical estimates I obtained with exchange rate option prices. The mean regression coefficient,  $\hat{\beta}$ , from the simulations is -4.51 and the corresponding 95% confidence interval is  $[-23.31, 11.82]$ . Moreover, the mean expected return on the Pound is 2.59% and it varies between a low of -1.94% and a high of 5.95% (the in-sample standard deviation of the expected return is 1.28%). The model is also statistically consistent with Fama's regression results when I estimate it without using option prices. However, the model-implied expected return on the Pound varies between a low of -23.67% and a high of 23.78% (the in-sample standard deviation is 10.38%).

The results are similar when I estimate the model using U.S. and Euro data. As Figure 2 illustrates, 3-month Euro Libor rates were about 0.78% higher on average than those in the U.S., but the Euro appreciated against the Dollar by 26.78%. Fama's regression coefficient for this period is  $\hat{\beta} = -10.08$ . The mean regression coefficient using data simulated from the model when I estimate it with options is  $\hat{\beta} = -8.67$  and the corresponding 95% confidence interval is  $[-31.55, 6.72]$ . Again, the model-implied expected return on the Euro is also much less variable when I estimate the model with options than when I don't include options.



**Figure 1: U.S. and U.K. Libor Rates and Exchange Rate**

This plot shows the Dollar/Pound exchange rate and the 3-month Libor rates in the U.S. and U.K. from August 2001 until July 2005.



**Figure 2: U.S. and Euro Libor Rates and Exchange Rate**

This plot shows the Dollar/Euro exchange rate and the 3-month Libor rates in the U.S. and Euro Zone from August 2001 until July 2005.

I examine how each of the risk factors in the model affects the exchange rate risk premium and interest rate risk premiums in each currency. For each model that I estimate, there is a risk factor that primarily affects the exchange rate risk premium and has little effect on the risk premium for interest rates in either currency. Brandt and Santa-Clara (2002) and Dewachter and Maes (2001) also estimate two-currency term structure models with risk factors that can affect the exchange rate but not interest rates in either currency. However, these papers impose the restriction that risk factors that do not affect interest rates are not priced and therefore do not affect the risk premium in exchange rate returns. Instead, I find that these risk factors are priced, and exchange rate options are useful for identifying their prices. Bakshi et al. (2005) also use option prices to estimate a dynamic exchange rate model with jumps, but do not examine the forward premium anomaly because they assume that the term structure of interest rates in both currencies is constant and flat.

My results also complement recent research by Lustig and Verdelhan (2007). They find that standard consumption-based models that are frequently used for pricing the cross-section of equity returns are also consistent with the cross-section of currency returns. I extend the standard single-currency dynamic term structure models to a two-currency setting and find that they are consistent with the forward premium anomaly when I use option prices to estimate the model.

The remainder of the paper is organized as follows. Section 2 describes dynamic international asset pricing models and the forward premium anomaly. Section 3 presents the specific class of two-currency affine term structure models that I study in the paper and discusses valuation of zero coupon bonds and exchange rate options. Section 4 describes the data and estimation procedure I use. Section 5 discusses the empirical results and Section 6 concludes. Most technical details are relegated to the appendix.

## 2 Dynamic International Asset Pricing Models

In the absence of arbitrage, it is well known that there exists a unique minimum variance pricing kernel (stochastic discount factor),  $M$ , such that the price  $P_t$  at time  $t$ , of a payoff  $P_T$  at time  $T$  is

$$P_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} P_T \right]. \quad (1)$$

In an international asset pricing model, if financial markets are open and integrated then the minimum variance pricing kernel must also price payoffs in a foreign currency when they are exchanged to the domestic currency. Let  $\tilde{P}_t$  be the price in foreign currency of a payoff  $\tilde{P}_T$  in foreign currency. If  $S$  is the exchange rate (expressed in units of domestic currency per unit of foreign currency) and  $M$  is an international pricing kernel, then

$$S_t \tilde{P}_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} \left( S_T \tilde{P}_T \right) \right].^2 \quad (2)$$

I assume that the minimum variance international pricing kernel follows a diffusion process of the form

$$dM_t = -M_t r_t dt - M_t \Lambda_t \cdot dW_t. \quad (3)$$

The drift,  $r$ , of the pricing kernel is commonly referred to as the short interest rate and the

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<sup>2</sup>Previous papers in this literature have characterized international asset pricing models using domestic and foreign pricing kernels,  $M$  and  $\tilde{M}$ . From equations (1) and (2), one can verify that

$$\frac{\tilde{M}_T}{\tilde{M}_t} := \frac{M_T}{M_t} \frac{S_T}{S_t},$$

is a pricing kernel that prices assets in the foreign currency when payoffs are denominated in either the domestic or foreign currency. Section B shows that these two approaches are equivalent, and if  $M$  is the minimum variance international pricing kernel denominated in the domestic currency, then  $\tilde{M}$  is in fact the minimum variance international pricing kernel denominated in the foreign currency.

volatility,  $\Lambda$ , is referred to as the market price of risk. A dynamic international asset pricing model also requires the dynamics of the exchange rate. Let  $\tilde{r}$  be the short interest rate in the foreign currency and suppose that the exchange rate also follows a diffusion process that depends on the Brownian motion,  $W$ , with volatility,  $\sigma$ . Then

$$dS_t = S_t [r_t - \tilde{r}_t + \sigma_t \Lambda_t] dt + S_t \sigma_t dW_t. \quad (4)$$

From equation (4), the expected change in the exchange rate is equal to the difference between the domestic and foreign short interest rates,  $r_t - \tilde{r}_t$ , plus a risk premium,  $\sigma_t \Lambda_t$ .

Exchange rate volatility affects both the volatility and expected excess return of foreign asset prices when they are exchanged to the domestic currency. If a foreign asset's price,  $\tilde{P}_t$ , in the foreign currency depends on the Brownian motion  $W$  with volatility  $\tilde{\sigma}_t$ , then the dynamics of its price,  $P_t^f := S_t \tilde{P}_t$ , in the domestic currency are given by

$$dP_t^f = P_t^f [r_t + (\tilde{\sigma}_t + \sigma_t) \Lambda_t] dt + P_t^f (\tilde{\sigma}_t + \sigma_t) dW_t.$$

In a risk-neutral world with  $\Lambda_t = 0$ , the uncovered interest rate parity hypothesis states that the expected change in the exchange rate is equal to the difference between the domestic and foreign short interest rates,  $r_t - \tilde{r}_t$ . Empirically, exchange rates do not tend to change by the difference in interest rates, as uncovered interest rate parity would suggest. There is strong evidence that investors are not risk-neutral (i.e.  $\Lambda_t \neq 0$ ). We also know that exchange rates are volatile (i.e.  $\sigma_t \neq 0$ ). Hence, it is not surprising that exchange rate returns contain

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<sup>3</sup>This characterization of the exchange rate in terms of the foreign short interest rate,  $\tilde{r}_t$ , is without loss of generality. If the exchange rate follows a diffusion process of the form

$$dS_t = S_t \mathcal{K}_t dt + S_t \sigma_t dW_t,$$

then it can be shown that the short interest rate in the foreign currency must be  $\tilde{r}_t := r_t + \sigma_t \Lambda_t - \mathcal{K}_t$ .

a risk premium (i.e.  $\sigma_t \Lambda_t \neq 0$ ).

What *has* proven puzzling is that currencies with high interest rates actually tend to appreciate, rather than depreciate, against currencies with lower interest rates. This puzzle is often referred to as the forward premium anomaly. More formally, Fama (1984) performs the following regression

$$\ln S_{t+\Delta t} - \ln S_t = \alpha + \beta (r_t - \tilde{r}_t) + \varepsilon_t, \quad (5)$$

and finds that  $\hat{\beta} < 0$  for most currency pairs. Using the exchange rate dynamics in equation (4), the population value of  $\beta$  in equation (5) is

$$\beta = \frac{\text{cov}(\ln S_{t+\Delta t} - \ln S_t, r_t - \tilde{r}_t)}{\text{var}(r_t - \tilde{r}_t)} \approx \frac{\text{cov}(r_t - \tilde{r}_t + \sigma_t \Lambda_t - \frac{1}{2} \sigma_t \sigma_t^\top, r_t - \tilde{r}_t)}{\text{var}(r_t - \tilde{r}_t)} .^4$$

Fama shows that  $\beta < 0$  implies that

$$\text{var}(r_t - \tilde{r}_t) < -\text{cov}\left(\sigma_t \Lambda_t - \frac{1}{2} \sigma_t \sigma_t^\top, r_t - \tilde{r}_t\right) < \text{var}\left(\sigma_t \Lambda_t - \frac{1}{2} \sigma_t \sigma_t^\top\right). \quad (6)$$

That is, if  $\beta < 0$  then the implied risk premium in exchange rate returns is negatively correlated with, and more variable than, the difference in interest rates.

As equation (6) illustrates, the forward premium anomaly depends on the relationship

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<sup>4</sup>There is a convexity adjustment to the drift when the dynamics of the exchange rate are expressed in logs, so that from equation (4),

$$d \ln S_t = \left[ r_t - \tilde{r}_t + \sigma_t \Lambda_t - \frac{1}{2} \sigma_t \sigma_t^\top \right] dt + \sigma_t dW_t,$$

which implies that for small  $\Delta t$ ,

$$\ln S_{t+\Delta t} - \ln S_t \approx \left[ r_t - \tilde{r}_t + \sigma_t \Lambda_t - \frac{1}{2} \sigma_t \sigma_t^\top \right] \Delta t + \sigma_t (W_{t+\Delta t} - W_t).$$

between the short interest rates in each currency, the volatility of the exchange rate, and the market prices of risk. These variables also affect other asset prices; in particular, the term structures of interest rates in both currencies. The price at time  $t$  in domestic currency of a zero coupon bond that pays 1 unit of domestic currency at time  $T$  is

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} 1 \right] = \mathbb{E}_t \left[ e^{-\int_t^T r_u du - \frac{1}{2} \int_t^T \Lambda_u^\top \Lambda_u du - \int_t^T \Lambda_u \cdot dW_u} \right]. \quad (7)$$

Similarly, the price of a zero coupon bond in the foreign currency is

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} \frac{S_T}{S_t} 1 \right] = \mathbb{E}_t \left[ e^{-\int_t^T \tilde{r}_u du - \frac{1}{2} \int_t^T (\Lambda_u^\top - \sigma_u) (\Lambda_u - \sigma_u^\top) du - \int_t^T (\Lambda_u - \sigma_u^\top) \cdot dW_u} \right]. \quad (8)$$

Existing research such as Backus et al. (2001) has struggled to produce an empirical international asset pricing model that captures the term structures of interest rates in both currencies, while simultaneously generating an exchange rate risk premium that is consistent with the forward premium anomaly.

### 3 Model and Valuation

This section presents the specific class of two-currency affine term structure models that I estimate in the paper and discusses the pricing of zero coupon bonds and exchange rate options.

Following the large literature on affine dynamic asset pricing models, I model the minimum variance (nominal) international pricing kernel,  $M$ , as a diffusion process

$$dM_t = -M_t r_t dt - M_t \Lambda_t \cdot dW_t, \quad (9a)$$

where

$$r_t := \rho_0 + \rho_1 \cdot X_t, \quad (9b)$$

$$\Lambda_t := \left( \sqrt{\Delta [\alpha + \beta X_t]} \right)^{-1} [(\mathcal{K}_0^{\mathbb{P}} - \mathcal{K}_0) + (\mathcal{K}_1^{\mathbb{P}} - \mathcal{K}_1) X_t], \quad (9c)$$

and

$$dX_t = [\mathcal{K}_0^{\mathbb{P}} + \mathcal{K}_1^{\mathbb{P}} X_t] dt + \sqrt{\Delta [\alpha + \beta X_t]} dW_t. \quad (9d)$$

I have used the notation  $\Delta [\cdot]$  to denote a square matrix with its vector argument along the diagonal. Dai and Singleton (2000) and Cheridito et al. (2006) provide parameter restrictions so that the process for the risk factors,  $X$ , is admissible and the model parameters can be identified. To extend this model to a two-currency setting, I also model the exchange rate as an affine process

$$dS_t = S_t \left[ r_t - \tilde{r}_t + \underbrace{\Sigma \sqrt{\Delta [\alpha + \beta X_t]} \Lambda_t}_{\sigma_t} \right] dt + S_t \underbrace{\Sigma \sqrt{\Delta [\alpha + \beta X_t]}}_{\sigma_t} dW_t, \quad (9e)$$

where

$$\tilde{r}_t := \tilde{\rho}_0 + \tilde{\rho}_1 \cdot X_t. \quad (9f)$$

This model nests previous two-currency affine term structure models that have been empirically studied in the literature. Previous papers such as Backus et al. (2001) and Dewachter and Maes (2001) model the market prices of risk as

$$\Lambda_t := \sqrt{\Delta [\alpha + \beta X_t]} \Lambda_0. \quad (10)$$

If the market prices of risk are constant multiples,  $\Lambda_0$ , of the volatility of the risk factors, Backus et al. (2001) show that the exchange rate risk premium,  $\sigma_t \Lambda_t$ , cannot have large

variation while the interest rate differential,  $r_t - \tilde{r}_t$ , simultaneously has low variation. As equation (6) illustrates, this condition is required to match the forward premium anomaly documented by Fama (1984). When the market prices of risk are instead modelled more generally according to equation (9c), the exchange rate risk premium is

$$\sigma_t \Lambda_t = \Sigma [(\mathcal{K}_0^{\mathbb{P}} - \mathcal{K}_0) + (\mathcal{K}_1^{\mathbb{P}} - \mathcal{K}_1) X_t] .^5 \quad (11)$$

Brennan and Xia (2005) note that the condition in equation (6) is more easily satisfied in this setting because the risk factors can affect the market prices of risk directly, and not only through factor volatilities.<sup>6</sup> Duffee (2002), Dai and Singleton (2002), and Cheridito et al. (2006) also find that this type of generalization is required to match the empirical properties of risk premia in the term structure of interest rates within a single-currency affine term structure model.

Duffie and Kan (1996) show that in the affine framework characterized by equation (9), domestic zero coupon bond prices are given by

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} 1 \right] = e^{A(T-t) + B(T-t) \cdot X_t} ,$$

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<sup>6</sup>While the theoretical analysis in Brennan and Xia (2005) focuses on two-currency models, their empirical analysis instead uses regression analysis to examine the relationship between exchange rate returns and the pricing kernels they estimate in different currencies using single-currency term structure models. Two-currency (international) pricing models are a significant extension of these single-currency models, as they restrict the joint relationship between the term structure of interest rates in both currencies, and between interest rates and exchange rate returns.

where  $A$  and  $B$  satisfy Riccati ODEs.<sup>7</sup> Similarly, foreign zero coupon bond prices are

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} \frac{S_T}{S_t} 1 \right] = e^{\tilde{A}(T-t) + \tilde{B}(T-t) \cdot X_t},$$

where  $\tilde{A}$  and  $\tilde{B}$  also satisfy Riccati ODEs.<sup>8</sup> Solutions to the Riccati ODEs can be efficiently computed numerically, therefore domestic and foreign zero coupon bond prices can be used to estimate the model.

Exchange rate call option prices (in the domestic currency) are given by

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} (S_T - K)^+ \right] = \mathbb{E}_t \left[ \frac{M_T}{M_t} S_T 1_{\{S_T \geq K\}} \right] - K \mathbb{E}_t \left[ \frac{M_T}{M_t} 1_{\{S_T \geq K\}} \right].$$

In general, this expectation is expensive to compute (for example, using Monte Carlo simulation), which makes it difficult to include option prices in model estimation. I use a cumulant expansion technique to compute option prices quickly and facilitate estimation.<sup>9</sup> The details of this approach are provided in Appendix C.

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<sup>7</sup>The Riccati ODEs (expressed in integral form) for domestic zero coupon bond prices are

$$\begin{aligned} B(\tau) &= \int_0^\tau -\rho_1 + \mathcal{K}_1^\top B(u) + \frac{1}{2} \beta^\top \Delta [B(u)] B(u) du, \\ A(\tau) &= \int_0^\tau -\rho_0 + \mathcal{K}_0^\top B(u) + \frac{1}{2} \alpha^\top \Delta [B(u)] B(u) du. \end{aligned}$$

<sup>8</sup>The Riccati ODEs (expressed in integral form) for foreign zero coupon bond prices are

$$\begin{aligned} \tilde{B}(\tau) &= \int_0^\tau -\tilde{\rho}_1 + [\beta^\top \Delta [\Sigma] + \mathcal{K}_1^\top] \tilde{B}(u) + \frac{1}{2} \beta^\top \Delta [\tilde{B}(u)] \tilde{B}(u) du, \\ \tilde{A}(\tau) &= \int_0^\tau -\tilde{\rho}_0 + [\alpha^\top \Delta [\Sigma] + \mathcal{K}_0^\top] \tilde{B}(u) + \frac{1}{2} \alpha^\top \Delta [\tilde{B}(u)] \tilde{B}(u) du. \end{aligned}$$

<sup>9</sup>This technique was first introduced to option pricing by Jarrow and Rudd (1982) and was applied to swaption pricing by Collin-Dufresne and Goldstein (2002). I use a special case of the approach that was applied in an earlier version of Almeida et al. (2006) for the general class of affine models.

## 4 Data and Estimation Procedure

I estimate a four-factor version of the general model described in equation (9). The complete model specification is provided in Appendix D. This particular empirical specification essentially has two risk factors that follow Feller, or CIR, processes with stochastic volatility and two factors that follow Gaussian processes with constant volatility (conditional on the Feller processes).<sup>10</sup> I allow interest rate volatility, exchange rate volatility, and the market prices of risk to all vary stochastically in the model.

To estimate the model, I use weekly data on U.S., U.K., and Euro zone Libor rates, swap rates, exchange rates, and implied volatilities from at-the-money options on the exchange rate. The data were obtained from Datastream and span the time period from August 15, 2001 to July 6, 2005 (the period for which option implied volatilities are available). I use spot exchange rates and Libor rates in each currency to convert the implied volatilities to option prices using Black's formula. I also convert swap rates to continuously compounded zero coupon rates by bootstrapping the swap zero curve under the assumption that forward rates are constant between observed swap maturities. The spot Dollar/Pound exchange rate over the sample period is plotted in Figure 1 and the 3-month and 5-year U.S. and U.K. zero coupon rates are plotted in Figure 7 in Appendix A. On average over this sample period, 3-month Libor rates in the U.K. were about 2.41% higher than those in the U.S., but the Pound appreciated against the Dollar by 22.25% (an average annual continuously compounded return of 5.16%).

Similarly, 3-month Libor rates in the Euro zone were about 0.78% higher on average than those in the U.S., but the Euro appreciated against the Dollar by 26.78% (an average annual

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<sup>10</sup>In a single-currency setting, the model is similar to the  $A_2(4)$  model described in Dai and Singleton (2000).

continuously compounded return of 6.87%). The spot Dollar/Euro exchange rate over the sample period is plotted in Figure 2 and the 3-month and 5-year U.S. and Euro zero coupon rates are plotted in Figure 8 in Appendix A.

I use quasi-maximum likelihood with the exact mean and variance to estimate the model on two sets of currency pairs (USD/GBP and USD/EUR). For each currency pair, I estimate two versions of the same model. To obtain the first set of parameter estimates I assume that the model correctly prices the 6-month U.S. Libor rate, the 5-year U.S. zero coupon rate, the 6-month Libor rate in either the U.K. or Euro zone (depending on the currency pair being modelled), and the change in the relevant exchange rate. This assumption allows me to invert these quantities for the implied states,  $X_t$ , at each time period.<sup>11</sup> I assume that the interest rates with other maturities for both relevant currencies are priced with error. For the second set of parameter estimates I invert for the implied latent states by assuming that the model correctly prices the 6-month U.S. Libor rate, the 5-year U.S. zero coupon rate, the 6-month Libor rate in either the U.K. or Euro zone, and the 6-month at-the-money USD/GBP exchange rate option. There is no data on the 6-month option on the Euro so instead I assume that the model correctly prices the 3-month option when I estimate it using Euro data. Since either the change in the log of the exchange rate or an exchange rate option price is used to invert for the implied states, one of the risk factors can affect the exchange rate but need not affect interest rates in either currency. Both Brandt and Santa-Clara

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<sup>11</sup>This estimation technique was introduced by Chen and Scott (1993) in the term structure context and is widely used when estimating dynamic asset pricing models with latent states. Zero coupon rates are affine functions of the states and therefore can be easily inverted to obtain the implied states. Changes in the log of the exchange rate are not affine in the states, but can be closely approximated by

$$\begin{aligned} \ln S_{t+\Delta t} - \ln S_t \approx & (\mathcal{K}_0^S - \Sigma \mathcal{K}_0^P) \Delta t + \left\{ \mathcal{K}_1^S \mathcal{K}_1^{P-1} + \left( \Sigma - \mathcal{K}_1^S \mathcal{K}_1^{P-1} \right) \mathcal{K}_1^P \left( e^{\mathcal{K}_1^P \Delta t} - I \right)^{-1} \Delta t \right\} X_{t+\Delta t} \\ & - \left\{ \mathcal{K}_1^S \mathcal{K}_1^{P-1} + \left( \Sigma - \mathcal{K}_1^S \mathcal{K}_1^{P-1} \right) \mathcal{K}_1^P \left[ I + \left( e^{\mathcal{K}_1^P \Delta t} - I \right)^{-1} \right] \Delta t \right\} X_t, \end{aligned}$$

where  $\mathcal{K}_0^S + \mathcal{K}_1^S X_t := r_t - \tilde{r}_t + \sigma_t \Lambda_t - \frac{1}{2} \sigma_t \sigma_t^\top$ .

(2002) and Dewachter and Maes (2001) also estimate models with risk factors that affect the exchange rate but not interest rates, however, they impose that such risk factors are not priced.

For both estimates of the model, I assume that the following quantities are priced with error:

- 1-month and 3-month Libor rates in both relevant currencies (and 6-month Libor rates when I estimate the model with options);
- 2-, 3-, 7-, and 10-year U.S. zero coupon rates;
- 2-, 3-, 5-, 7-, and 10-year zero coupon rates in either the U.K. or the Euro zone.

When I estimate the model with exchange rate options, I assume that the change in the exchange rate and the 1-month (and 3-month for GBP) at-the-money options are also priced with error. I use the following procedure to obtain historical quasi-maximum likelihood estimates:

- Randomly generate 25 sets of feasible starting parameters.
- Starting from the feasible parameters with the highest value of the quasi-likelihood function, I use a gradient search (implemented in Matlab) to obtain a maximum of the quasi-likelihood function.
- Repeat these steps 1000 times to obtain (what I hope is) a global maximum.

## 5 Empirical Results

A priori, one might expect that using options to estimate the model should improve its ability to capture exchange rate volatility. However, recall from equations (7) and (8) that the foreign term structure of interest rates depends on the differences between the market prices of risk and exchange rate volatility,  $\Lambda_t - \sigma_t$ , and the domestic term structure of interest rates depends on the market prices of risk,  $\Lambda_t$ . Therefore, the model's ability to capture the term structure of interest rates in both currencies could decrease when options are included in estimation. The relevant focus is the trade-off between these effects.

Table 1 presents the root mean squared pricing errors (expressed in basis points) for zero coupon Libor and swap rates in the U.S. and U.K. with maturities ranging from one month to ten years. The pricing errors increase when the model is estimated with options, particularly for 1-month and 3-month maturities. At longer U.S. maturities, the pricing errors increase by only a few basis points. The pricing errors for U.K. interest rates increase by only a few basis points across all maturities for the model that is estimated with options.

	Maturity (in Years)							
	1/12	3/12	6/12	2	3	5	7	10
USD w/o Options	10	5	0	13	10	0	6	13
USD w/ Options	27	14	0	18	12	0	7	14
GBP w/o Options	13	7	0	13	12	11	10	11
GBP w/ Options	15	8	0	13	12	13	13	13

**Table 1: Root Mean Squared Errors for U.S. and U.K. Interest Rates**

This table presents the root mean squared errors in basis points for zero coupon Libor and swap rates. The first row provides cross-sectional pricing errors for U.S. Dollar interest rates when the model is estimated without including options. The second row provides cross-sectional pricing errors for U.S. Dollar rates when the model is estimated with exchange rate options. The third row provides cross-sectional pricing errors for U.K. Pound rates when the model is estimated without including options. The fourth row provides cross-sectional pricing errors for U.K. Pound rates when the model is estimated with exchange rate options.

Table 2 presents the root mean squared pricing errors (expressed in basis points) for U.S. and Euro zero coupon Libor and swap rates with maturities ranging from one month to ten years. When options are used to estimate the model, the pricing errors for U.S. interest rates increase for short maturities (1-month and 3-month) and slightly decrease for longer maturities. Pricing errors for interest rates in the Euro zone also increase more for shorter maturities than for longer maturities.

	Maturity (in Years)							
	1/12	3/12	6/12	2	3	5	7	10
USD w/o Options	16	8	0	13	9	0	5	10
USD w/ Options	22	11	0	12	8	0	5	10
Euro w/o Options	7	4	0	10	9	10	11	13
Euro w/ Options	13	7	0	15	15	14	12	11

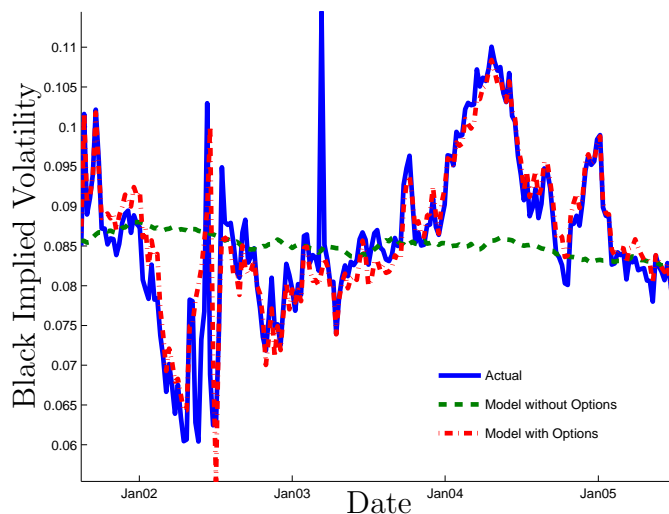
**Table 2: Root Mean Squared Errors for U.S. and Euro Interest Rates**

This table presents the root mean squared errors in basis points for zero coupon Libor and swap rates. The first row provides cross-sectional pricing errors for U.S. Dollar interest rates when the model is estimated without including options. The second row provides cross-sectional pricing errors for U.S. Dollar rates when the model is estimated with exchange rate options. The third row provides cross-sectional pricing errors for Euro interest rates when the model is estimated without including options. The fourth row provides cross-sectional pricing errors for Euro interest rates when the model is estimated with exchange rate options.

Figure 3 plots the option-implied volatility from 3-month at-the-money options on the Dollar/Pound exchange rate when the model is estimated with and without including options.<sup>12</sup> When the model is estimated without including exchange rate options, it matches the average level of the option-implied volatility. However, the option-implied volatility from the model has a correlation of -16.07% with the actual option-implied volatility in the data. When the model is estimated with options it better captures the variation in option-implied volatility and this correlation increases to 89.37%. The mean absolute relative pricing errors for 3-month ATM exchange rate options are 8.92% when options are not included in

<sup>12</sup>The results for 1-month and 6-month implied volatilities are virtually identical and are omitted to conserve space.

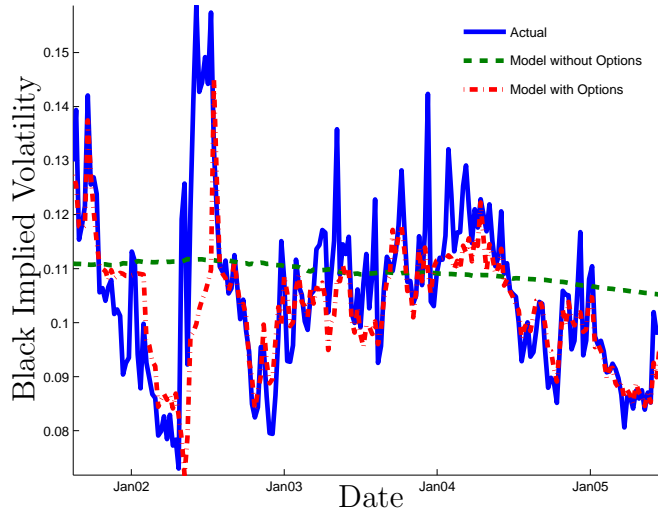
estimation versus 3.29% when options are included in estimation.



**Figure 3: USD/GBP Exchange Rate Option-Implied Volatility**  
This figure plots the implied volatility for a 3-month at-the-money option on the Dollar/Pound exchange rate. The model price of an at-the-money is converted to an option-implied volatility using Black’s formula.

Figure 4 plots the option-implied volatility from 1-month at-the-money options on the Dollar/Euro exchange rate when the model is estimated with and without including options. Again, when the model is estimated without including exchange rate options, it matches the average level of option-implied volatility but it does not capture the variation in the option-implied volatility. The actual option-implied volatility in the data has a correlation of 27.88% with the option-implied volatility from the model that is estimated without including options. This number increases to 67.39% when options are included in estimation. The mean absolute relative pricing errors for 1-month ATM exchange rate options are 13.83% when options are not included in estimation versus 0.12% when options are included in estimation.

Figures 9 and 10 in Appendix A compare the actual (not option-implied) volatility from



**Figure 4: USD/EUR Exchange Rate Option-Implied Volatility**  
 This figure plots the implied volatility for a 1-month at-the-money option on the dollar/Euro exchange rate. The model price of an at-the-money is converted to an option-implied volatility using Black’s formula.

the models with an estimate of volatility I compute using a 6-month rolling window.<sup>13</sup> The story is largely the same: the models that are estimated with options do a better job of capturing the variation in exchange rate volatility. For the Dollar/Pound exchange rate, the mean value of the 6-month rolling window estimate of volatility is 7.68%. The mean model-implied exchange rate volatility when the model is estimated without options is 8.53% and is 8.40% when the model is estimated with options. More importantly, when the model is estimated without options, the correlation of the model-implied exchange rate volatility with the rolling window estimate of exchange rate volatility is -0.21. This correlation improves to

<sup>13</sup>Other papers such as Inci and Lu (2004) compare the realized volatility of the *fitted* exchange rate returns to that of the actual exchange rate returns, but do not examine the volatility from their model of exchange rates. To emphasize this distinction, when the model in this paper is estimated without using options, the fitted values of exchange rate returns exactly match the realized exchange rate returns and therefore any estimate of the volatility of the fitted exchange rate returns exactly matches that of the actual exchange rate returns. However, although the volatility of the fitted exchange rate returns matches the data exactly, the exchange rate volatility in the model *does not* capture the variation in volatility that is observed in the data.

0.32 when the model is estimated with options. The results are qualitatively the same for the Dollar/Euro exchange rate. The mean of the rolling window estimate of volatility for the Dollar/Euro exchange rate is 9.22%. When the model is estimated without options, the mean model-implied exchange rate volatility is 10.93%, and is 10.03% when options are used to estimate the model. When options are not used to estimate the model, the correlation of model-implied exchange rate volatility with the rolling window estimate of exchange rate volatility is 0.04 and this correlation improves to 0.29 when options are used to estimate the model.

To summarize, when exchange rate option prices are used in estimation, the model's fit to the dynamics of both option-implied and actual exchange rate volatility improves significantly. However, there is a slight deterioration in the fit to the term structures of interest rates, particularly at the very short end of the yield curves (less than 3 months).

The forward premium anomaly relates the risk premium,  $\sigma_t \Lambda_t$ , in exchange rate returns to the difference in interest rates,  $r_t - \tilde{r}_t$ , between the two currencies. Is the exchange rate risk premium that I estimated also consistent with the forward premium anomaly that is documented in Fama (1984)? Recall from equation (5) in Section 2 that Fama performs the following regression

$$\ln S_{t+\Delta t} - \ln S_t = \alpha + \beta (r_t - \tilde{r}_t) + \varepsilon_t,$$

and finds that  $\hat{\beta} < 0$  for most currency pairs. In the weekly data set that I use in this paper (August 2001 to July 2005),  $\hat{\beta} = -4.50$  for the Dollar/Pound regression. Using the historical quasi-maximum likelihood estimates, I simulate four years of weekly exchange rate and interest rate data 1,000 times. When I simulate from the model that is estimated without options, the mean regression coefficient from the simulations is  $\hat{\beta} = -7.29$  and the corresponding 95% confidence interval is  $[-20.67, 4.53]$ . When I simulate from the model that

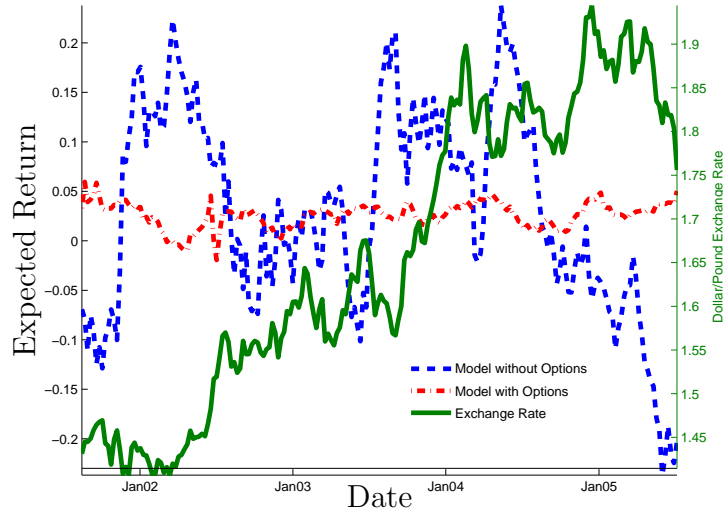
is estimate with options, the mean regression coefficient from the simulations is  $\hat{\beta} = -4.51$  and the corresponding 95% confidence interval is  $[-25.31, 11.82]$ . Therefore both of these models are statistically consistent with the results from Fama's regression for the U.S. and U.K. data.

For the Dollar/Euro regression,  $\hat{\beta} = -10.08$ . When I simulate from the model that is estimated without options, the mean regression coefficient from the simulations is  $\hat{\beta} = 0.80$  and the corresponding 95% confidence interval is  $[-39.58, 39.45]$ . When I simulate from the model that is estimated with options, the mean regression coefficient from the simulations is  $\hat{\beta} = -8.67$  and the corresponding 95% confidence interval is  $[-31.55, 6.72]$ . Again, both of the models are, statistically speaking, consistent with the results from Fama's regression for the U.S. and Euro data. However, the mean regression coefficient is negative only for the model that is estimated with options.

When options are included in estimation, the number of factors in the model remains the same, but the properties of the estimated models change. A natural question is how does the model-implied expected return on the exchange rate change when options are used to estimate the model. To this end, Figures 5 and 6 plot the continuously compounded annual expected return on the Pound and the Euro respectively.<sup>14</sup> On average over the sample period (August 2001 to July 2005), the continuously compounded annual return on the Pound was 5.16%. When the model is estimated without using options, the mean expected rate of return on the Pound is 2.97%. The mean expected rate of return is 2.59% when the model is estimated using options. Although the means are similar, there is a significant difference in the variation of the expected returns. When I estimate the model without using options, the expected return on the Pound varies between a low of -23.67% and a high of 23.78%

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<sup>14</sup>Recall from equation (4) that the continuously compounded instantaneous expected return on a foreign currency is given by  $r_t - \tilde{r}_t + \sigma_t \Lambda_t$ .

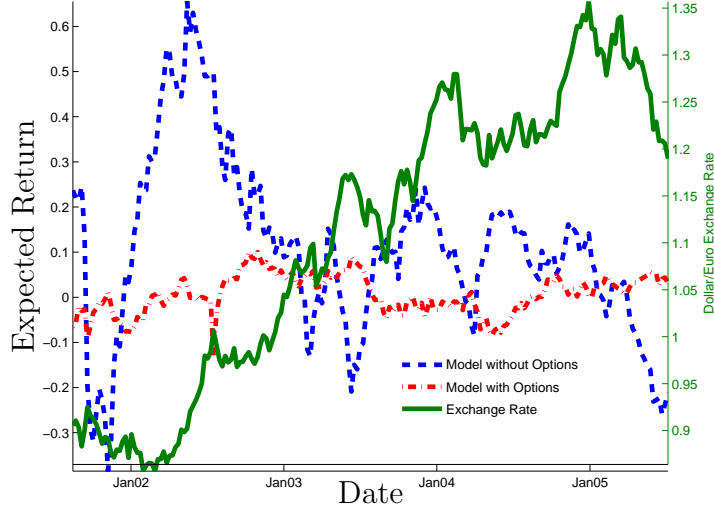


**Figure 5: Expected USD/GBP Exchange Rate Return**

This figure plots the expected exchange rate return,  $r_t - \tilde{r}_t + \sigma_t \Lambda_t$ , when the model is estimated both with and without using options.

(the standard deviation is 10.38% over the sample period). By contrast, when options are used to estimate the model, the expected return on the Pound only varies between a low of -1.94% and a high of 5.95% (the standard deviation over the sample period is 1.28%). Thus, when exchange rate options are included in estimation, the estimated risk premium is much less variable and more economically plausible.

The difference in expected exchange rate returns is even more pronounced when the model is estimated on the Euro. The average continuously compounded annual return on the Euro over the sample period was 6.87%. When the model is estimated without using options, the mean expected return on the Euro is 10.96%, but it varies between a low of -38.49% and a high of 65.52%. By contrast, when options are used to estimate the model, the mean expected return on the Euro is 0.67%. Moreover, the variation in the expected return is again much smaller: the low is -13.00% and the high is 10.02% (the standard deviation over the sample period is 4.29%).



**Figure 6: Expected USD/EUR Exchange Rate Return**

This figure plots the expected exchange rate return,  $r_t - \tilde{r}_t + \sigma_t \Lambda_t$ , when the model is estimated both with and without using options.

To further decompose the properties of the estimated exchange rate risk premiums, I examine how each of the risk factors in the model affects the exchange rate risk premium and interest rate risk premiums in each currency. The instantaneous exchange rate risk premium can be written as

$$\begin{aligned} \sigma_t \Lambda_t &= \Sigma [(\mathcal{K}_0^{\mathbb{P}} - \mathcal{K}_0) + (\mathcal{K}_1^{\mathbb{P}} - \mathcal{K}_1) X_t] , \\ &= \sum_{i=1}^4 \underbrace{\Sigma_i [(\mathcal{K}_0^{\mathbb{P}} - \mathcal{K}_0)_i + (\mathcal{K}_1^{\mathbb{P}} - \mathcal{K}_1)_{\{\text{row } i\}} X_t]}_{\text{dependence on risk factor } i} . \end{aligned}$$

Similarly, the risk premium on a zero coupon bond can be written as

$$\begin{aligned} \sigma_t(T) \Lambda_t &= B(T-t) [(\mathcal{K}_0^{\mathbb{P}} - \mathcal{K}_0) + (\mathcal{K}_1^{\mathbb{P}} - \mathcal{K}_1) X_t] , \\ &= \sum_{i=1}^4 \underbrace{B_i(T-t) [(\mathcal{K}_0^{\mathbb{P}} - \mathcal{K}_0)_i + (\mathcal{K}_1^{\mathbb{P}} - \mathcal{K}_1)_{\{\text{row } i\}} X_t]}_{\text{dependence on risk factor } i} . \end{aligned}$$

Figures 11, 12, 13, and 14 in Appendix A illustrate how each risk factor contributes to the risk premium for swap rates and the exchange rate in each of the four models that I estimate. Figures 15 and 16 in Appendix A plot the sensitivity of interest rates to changes in each of the four factors when the factors are evaluated at their sample means. For each model that I estimate, there is a risk factor that primarily affects the exchange rate risk premium and has little affect on the risk premium for interest rates in either currency. Brandt and Santa-Clara (2002) and Dewachter and Maes (2001) also estimate two-currency term structure models with risk factors that can affect the exchange rate but not interest rates in either currency. However, these papers impose the restriction that the risk factors that do not affect interest rates are not priced and therefore do not affect the risk premium in exchange rate returns. Instead, I find that these risk factors are priced, and exchange rate options are useful for identifying their prices.

Moreover, when exchange rate options are used to estimate the models, the risk premiums attached to the risk factors that primarily affect the exchange rate have much smaller variation. For example, as Figure 11 in Appendix A illustrates, when the model is estimated on Dollar and Pound data without using options, the first risk factor primarily affects the exchange rate risk premium but not interest rates in either currency. The mean expected excess exchange rate return due to this risk factor is 7.19%. However, this expected excess return has an in-sample standard deviation of 10.09% and it varies between a low of -16.77% and a high of 28.01%. Instead, when options are used to estimate the model, the fourth risk factor primarily affects the exchange rate risk premium but not interest rates in either currency. The mean expected excess return on the exchange rate that is due to this risk factor is 10.26%, but the in-sample standard deviation is much smaller at 3.07% (the minimum is 3.89% and the maximum is 18.39%). The story is the same when the model is estimated using U.S. and Euro data. When options are not used to estimate the model, the

second risk factor primarily affects the Dollar/Euro exchange rate, but not interest rates in either the U.S. or the Euro zone. The standard deviation of the expected excess exchange rate return associated with the second risk factor is 19.09%. By contrast, when options are used to estimate the model, the fourth risk factor primarily affects the exchange rate and the standard deviation of the expected excess exchange rate return associated with this risk factor is only 3.15%.

To measure how well the models predict changes in the exchange rate, I compute the following statistic,

$$R^2 = 1 - \frac{\text{mean} [(\text{Actual } \Delta S_t - \text{Model Predicted } \Delta S_t)^2]}{\text{mean} [(\text{Actual } \Delta S_t)^2]}. \quad (12)$$

For the random walk model, the predicted change in the exchange rate,  $\Delta S_t$ , is zero, so this statistic has the convenient feature that it is zero for the random walk model. Any value of  $R^2$  that is above zero outperforms the random walk model, and any value that is below zero underperforms the random walk model.

Table 3 provides the  $R^2$  statistics for each of the models. All of the models outperform the random walk model except the USD/Euro model that is estimated with options. Surprisingly, although the models that are estimated without options have much more variation in expected returns on the Pound and Euro, their  $R^2$  statistics are actually higher than their counterparts that are estimated with options.

Finally, I examine the relationship between macroeconomic fundamentals and the exchange rate risk premiums that I estimate from each of the models. Table 4 in Appendix A provides the regression results when I regress the exchange rate risk premium on the quarterly money supply, current account balance, and expected inflation in the foreign country (either the

	Non-Overlapping			Overlapping		
	1 Mth	2 Mth	3 Mth	1 Mth	2 Mth	3 Mth
USD/GBP w/o Options	0.110	0.283	0.205	0.098	0.190	0.200
USD/GBP w/ Options	0.004	0.016	0.046	-0.002	0.003	0.037
USD/Euro w/o Options	0.176	0.225	-0.010	0.168	0.157	0.014
USD/Euro w/ Options	-0.016	-0.052	-0.151	-0.016	-0.041	-0.107

**Table 3: R<sup>2</sup>'s for In-Sample Predictions of Exchange Rate Changes**

The table presents the R<sup>2</sup> statistic from equation (12) for changes in the exchange rate for each of the models estimated in the paper. The prediction horizon ranges from 1 to 3 months. The R<sup>2</sup> statistic is provided for both non-overlapping and overlapping observations in the sample (July 2001 to August 2005).

U.K. or the Euro zone) and the U.S.<sup>15</sup> These results, using quarterly data, are consistent with a long-standing puzzle in international economics that fundamental macroeconomic variables do not help predict future changes in exchange rates.<sup>16</sup> Table 5 in Appendix A provides the regression results when I regress the estimated exchange rate risk premium from each of the models on monthly expected inflation in the foreign country (either the U.K. or the Euro zone) and the U.S. (data on the other macroeconomic variables is only available quarterly). Using this monthly data, the regression coefficients on expected inflation in the U.K. and the U.S. are statistically significant at the 1% confidence level for the model that is estimated with options. For all of the other estimated models, none of the regression coefficients are statistically significant at conventional confidence levels.

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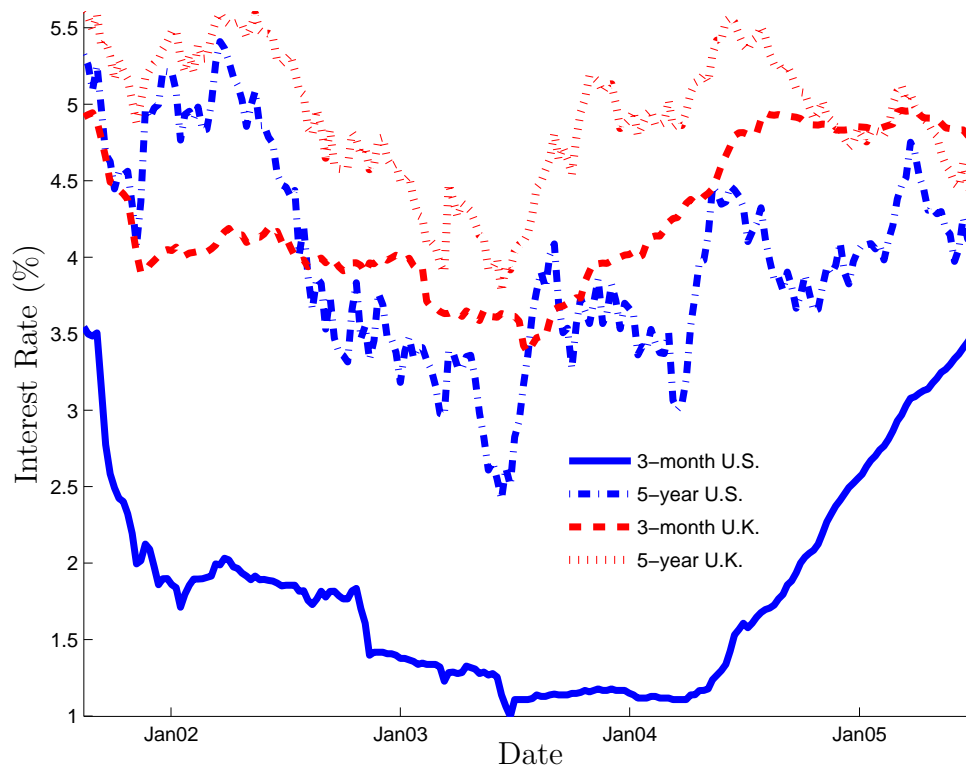
<sup>15</sup>One should exercise caution when interpreting the regression results in Table 4 since they rely on only 16 quarterly observations

<sup>16</sup>See Meese and Rogoff (1983) and Engel and West (2005).

## 6 Conclusion

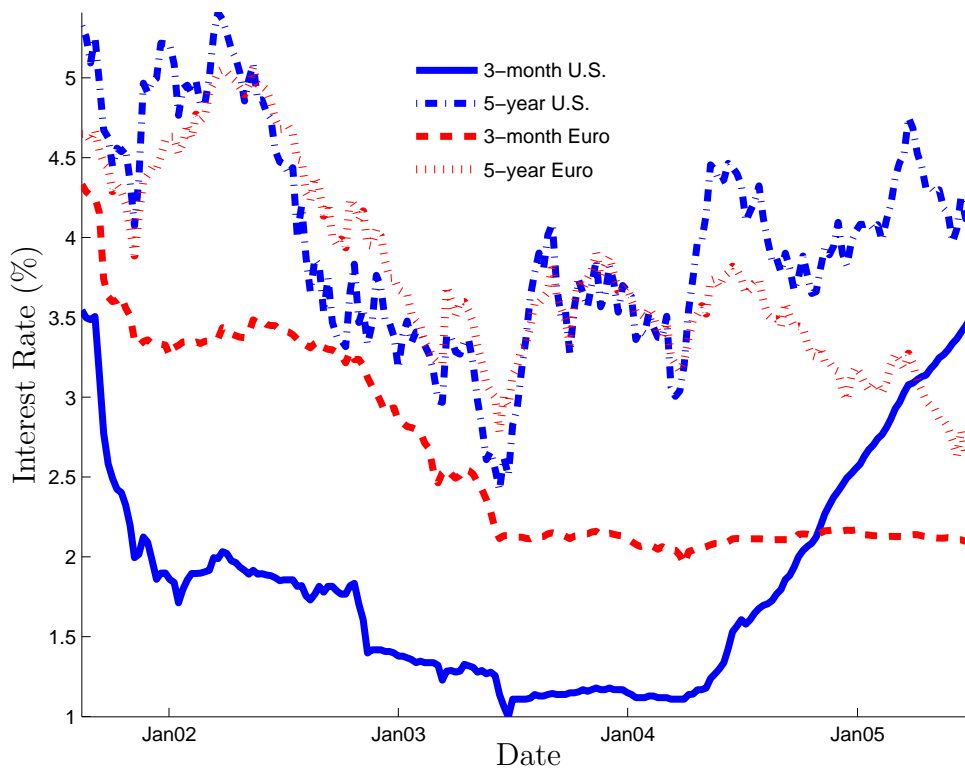
This paper provides an arbitrage-free empirical model that generalizes previous models that have been studied in the literature. I estimate the model for two sets of currency pairs: the U.S. Dollar and British Pound, and the U.S. Dollar and Euro. I show that exchange rate option prices provide valuable information about exchange rate volatility that is much harder to identify in a two-currency term structure model using only time-series data on foreign exchange rates and interest rates in each currency. Moreover, option prices also help to identify the price of risk factors that affect exchange rates but not interest rates in either currency. When I include options, the estimated exchange rate risk premium is much less variable and more economically plausible than when options are not included in estimation. Using data simulated from the model at the parameter estimates, I show that the model is consistent with the forward premium anomaly and it replicates the empirical findings in Fama (1984).

## A Tables and Figures



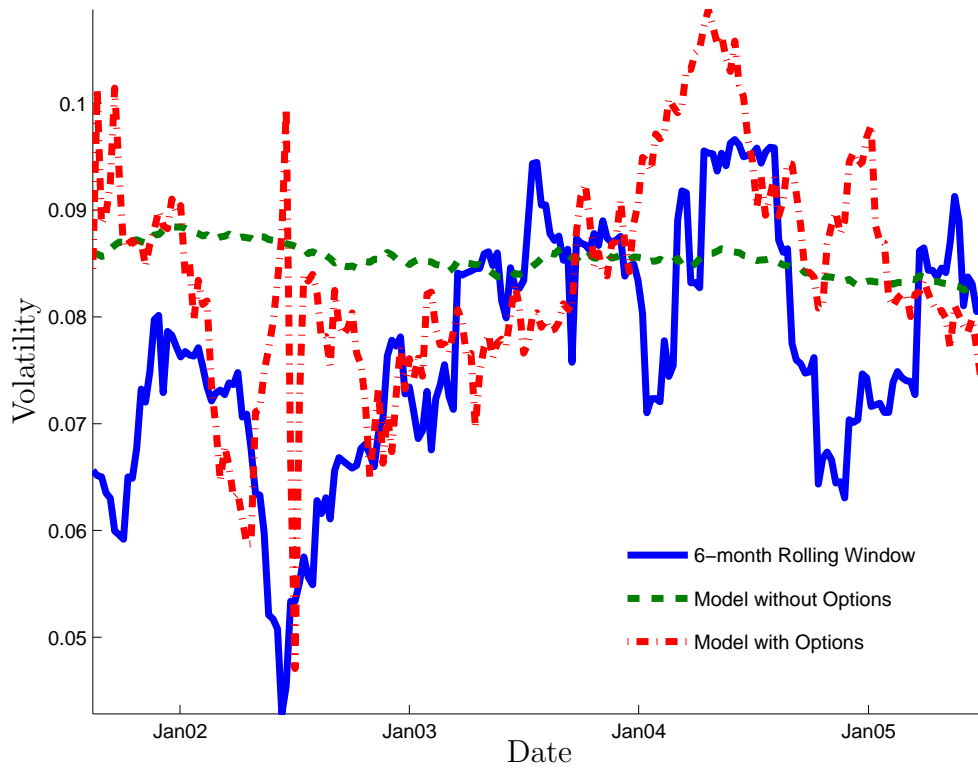
**Figure 7: U.S. and U.K. Interest Rates**

This figure plots the 3-month and 5-year U.S. and U.K. zero coupon swap rates for the sample period from August 15, 2001 to July 6, 2005. The swap and Libor data are from Datastream. The swap zero coupon curve was bootstrapped under the assumption that forward rates are constant between observed swap maturities.



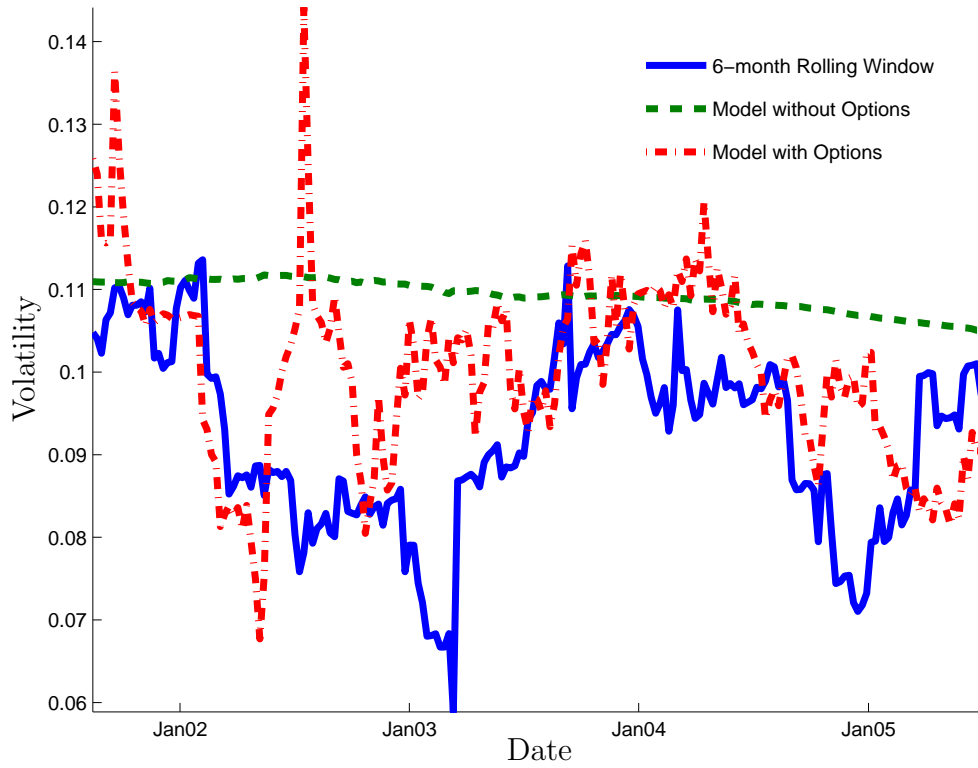
**Figure 8: U.S. and Euro Interest Rates**

This figure plots the 3-month and 5-year U.S. and Euro zero coupon swap rates for the sample period from August 15, 2001 to July 6, 2005. The swap and Libor data are from Datastream. The swap zero coupon curve was bootstrapped under the assumption that forward rates are constant between observed swap maturities.



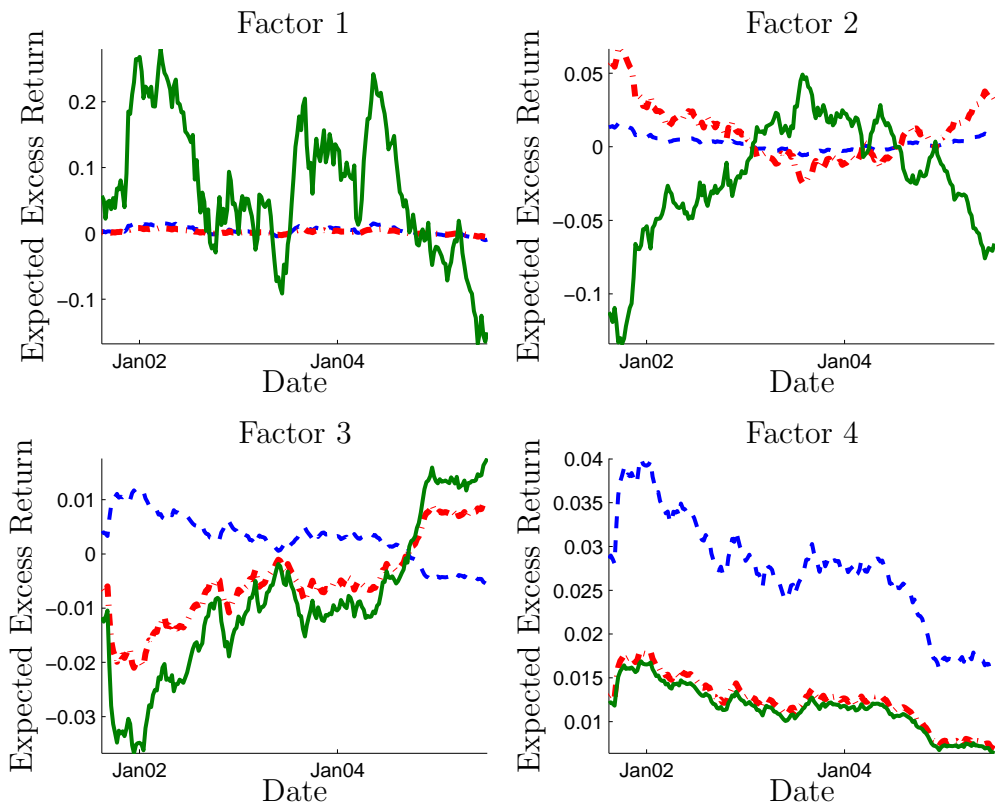
**Figure 9: USD/GBP Exchange Rate Volatility**

This figure plots the 6-month rolling window estimate of volatility from the Dollar/Pound exchange rate data, and the volatility given by the model when it is estimated both with and without using options.

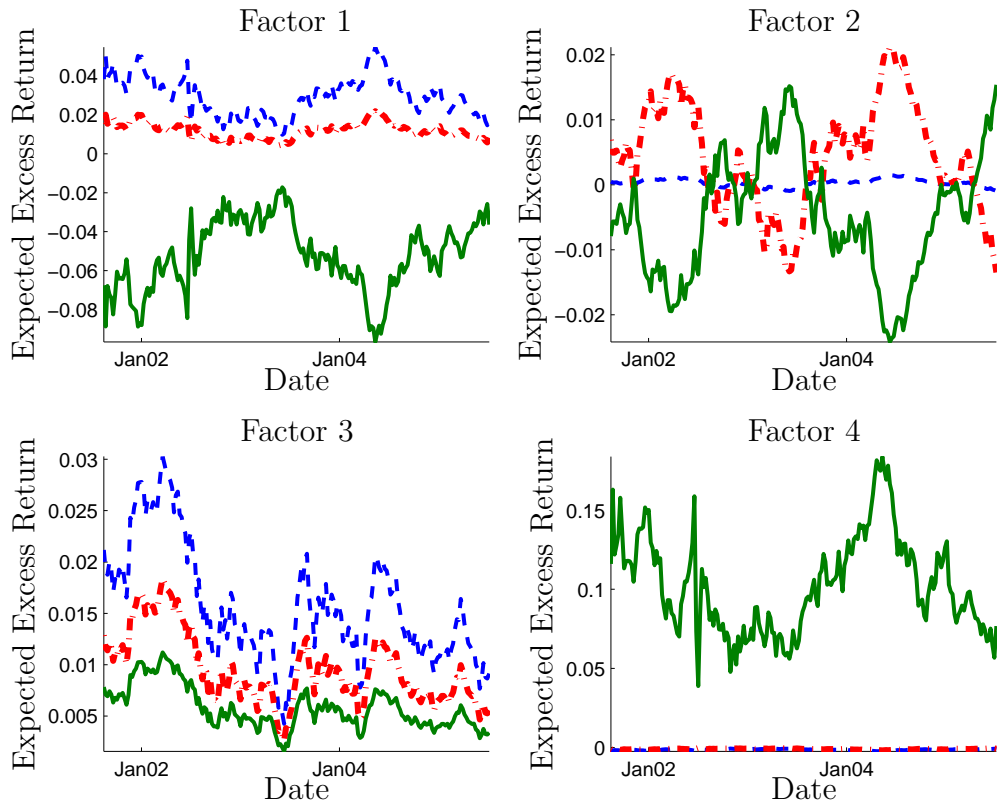


**Figure 10: USD/EUR Exchange Rate Volatility**

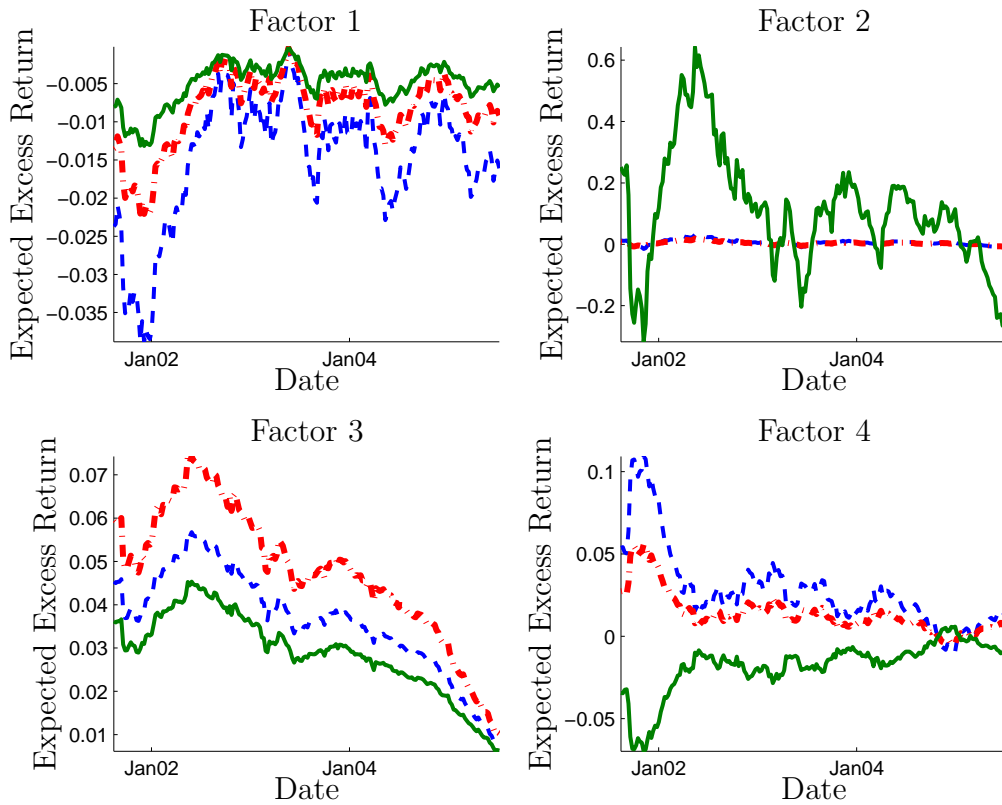
This figure plots the 6-month rolling window estimate of volatility from the dollar/Euro exchange rate data, and the volatility given by the model when it is estimated both with and without using options.



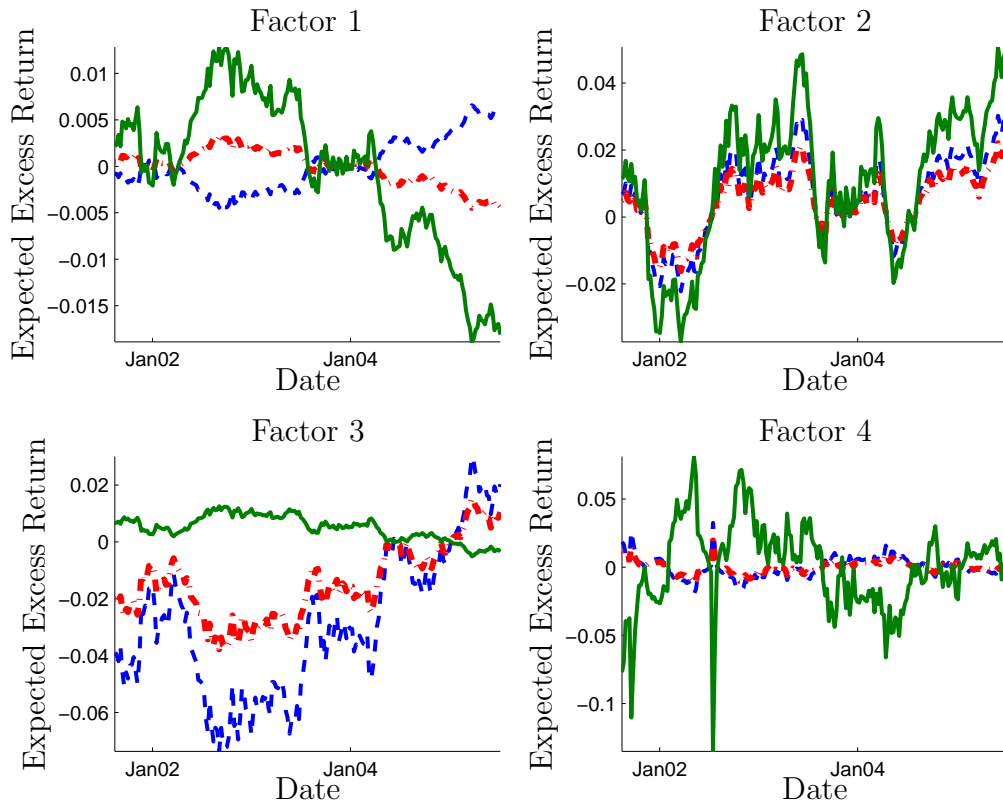
**Figure 11: U.S. and U.K. Risk Premiums Estimated w/o Options**  
 This figure plots the risk premium, or expected excess return, due to each latent factor when I estimate the model without using options. The solid (green) line is the risk premium for the USD/GBP exchange rate. The dashed (blue) line is the risk premium for the 5-year USD zero coupon bond. The dotted (red) line is the risk premium for the 5-year GBP zero coupon bond.



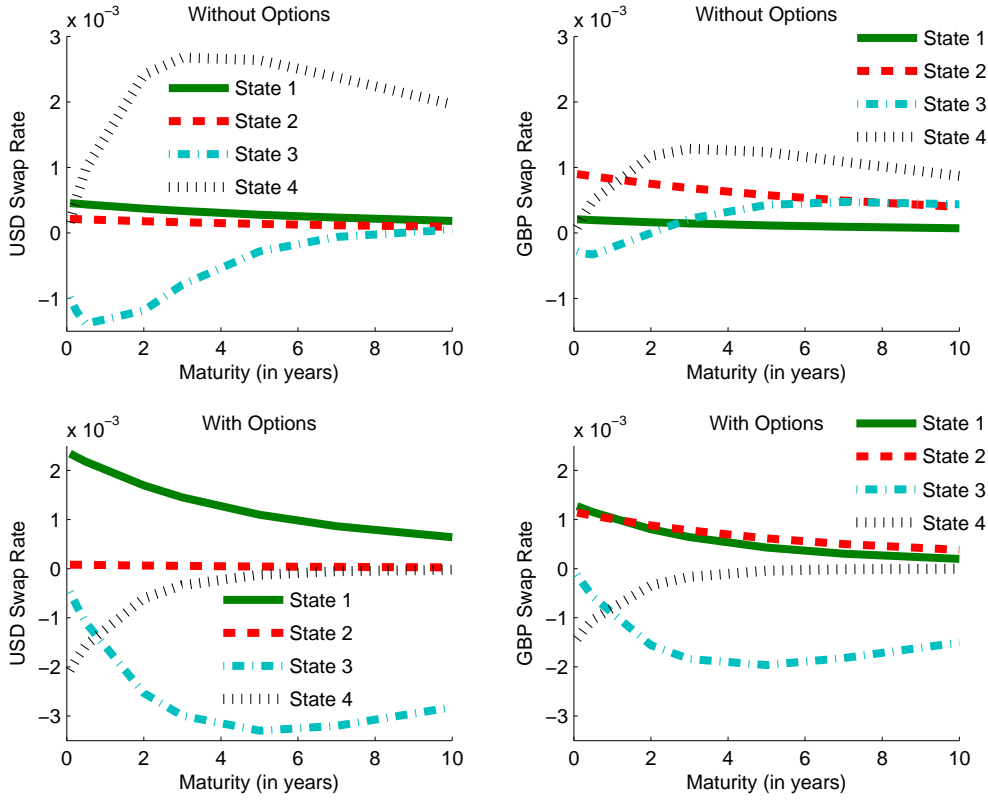
**Figure 12: U.S. and U.K. Risk Premiums Estimated w/ Options**  
 This figure plots the risk premium, or expected excess return, due to each latent factor when I estimate the model using options. The solid (green) line is the risk premium for the USD/GBP exchange rate. The dashed (blue) line is the risk premium for the 5-year USD zero coupon bond. The dotted (red) line is the risk premium for the 5-year GBP zero coupon bond.



**Figure 13: U.S. and Euro Risk Premiums Estimated w/o Options**  
 This figure plots the risk premium, or expected excess return, due to each latent factor when I estimate the model without using options. The solid (green) line is the risk premium for the USD/EUR exchange rate. The dashed (blue) line is the risk premium for the 5-year USD zero coupon bond. The dotted (red) line is the risk premium for the 5-year EUR zero coupon bond.



**Figure 14: U.S. and Euro Risk Premiums Estimated w/ Options**  
 This figure plots the risk premium, or expected excess return, due to each latent factor when I estimate the model using options. The solid (green) line is the risk premium for the USD/EUR exchange rate. The dashed (blue) line is the risk premium for the 5-year USD zero coupon bond. The dotted (red) line is the risk premium for the 5-year EUR zero coupon bond.

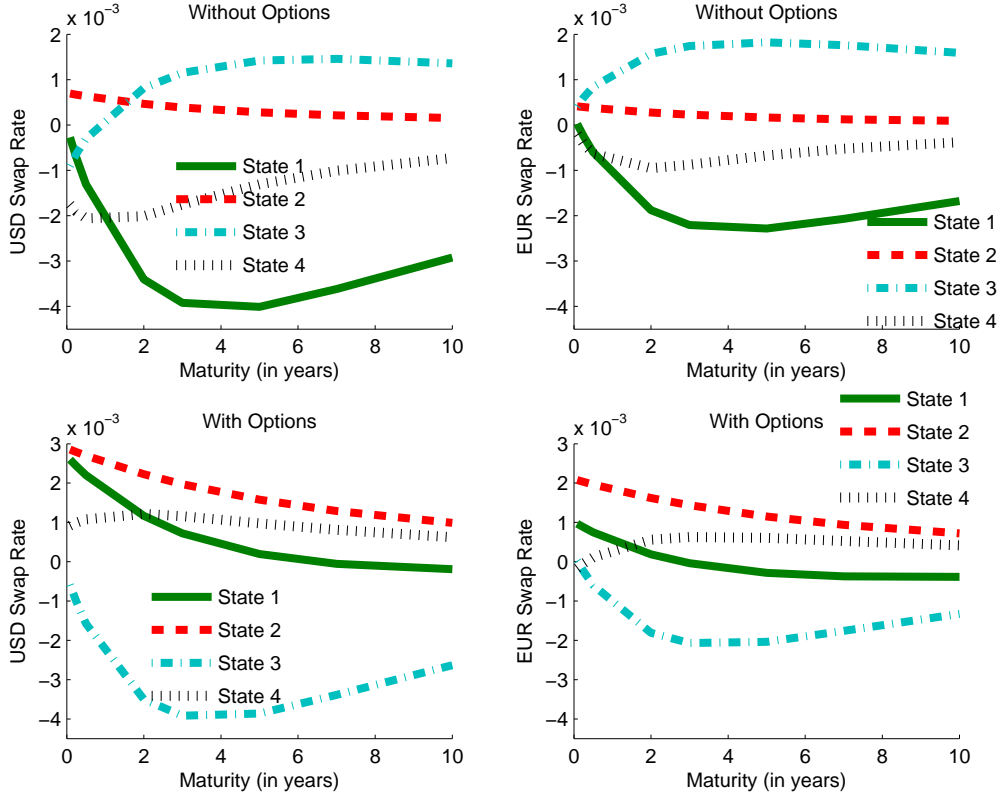


**Figure 15: Sensitivity of USD and GBP Swap Rates**

This figure plots the unexpected change in the zero coupon swap rate over one month when each of the four risk factors in the model changes by one standard deviation. Formally, the figure plots

$$-\frac{B(\tau)}{\tau} \cdot e^{\kappa_1^p \Delta t} \sqrt{\Delta [\alpha + \beta X]} (W_{t+\Delta t} - W_t) \approx Y_{t+\Delta t}^\tau - \mathbb{E}_t [Y_{t+\Delta t}^\tau] ,$$

evaluated at  $[W_{t+\Delta t} - W_t]_i = \sqrt{\Delta t}$  for each factor  $i = 1, 2, 3, 4$  (where  $Y_{t+\Delta t}^\tau$  is the  $\tau$ -maturity zero coupon swap rate and  $X$  is the sample mean of  $X_t$ ). The top left plot shows changes in the USD swap rates for different maturities when I estimate the model without including exchange rate options. The top right plot shows changes in the GBP swap rates for different maturities when I estimate the model without including exchange rate options. The bottom left plot shows changes in the USD swap rates for different maturities when I estimate the model with exchange rate options. The bottom right plot shows changes in the GBP swap rates for different maturities when I estimate the model without including exchange rate options.



**Figure 16: Sensitivity of USD and Euro Swap Rates**

The figure plots the unexpected change in the zero coupon swap rate over one month when each of the four risk factors in the model changes by one standard deviation. Formally, the figure plots

$$-\frac{B(\tau)}{\tau} \cdot e^{\kappa_1^p \Delta t} \sqrt{\Delta [\alpha + \beta X]} (W_{t+\Delta t} - W_t) \approx Y_{t+\Delta t}^\tau - \mathbb{E}_t [Y_{t+\Delta t}^\tau],$$

evaluated at  $[W_{t+\Delta t} - W_t]_i = \sqrt{\Delta t}$  for each factor  $i = 1, 2, 3, 4$  (where  $Y_{t+\Delta t}^\tau$  is the  $\tau$ -maturity zero coupon swap rate and  $X$  is the sample mean of  $X_t$ ). The top left plot shows changes in the USD swap rates for different maturities when I estimate the model without including exchange rate options. The top right plot shows changes in the Euro swap rates for different maturities when I estimate the model without including exchange rate options. The bottom left plot shows changes in the USD swap rates for different maturities when I estimate the model with exchange rate options. The bottom right plot shows changes in the Euro swap rates for different maturities when I estimate the model without including exchange rate options.

	Money Supply		Current Accnt Balance		Expected Inflation	
	Frgn	U.S.	Frgn	U.S.	Frgn	U.S.
USD/GBP w/o Options	-0.27 8.8%	0.25 11.2%	-0.01 60.0%	0.01 52.5%	-0.02 32.8%	-0.03 70.9%
USD/GBP w/ Options	-0.01 34.1%	0.03 11.4%	0.01 17.3%	0.02 83.6%	-0.00 45.6%	0.01 18.8%
USD/EUR w/o Options	-0.51 11.4%	-0.01 50.6%	0.06 16.4%	-0.20 12.9%	-0.30 4.7%	0.15 12.6%
USD/EUR w/ Options	0.08 79.4%	-0.12 91.1%	-0.01 73.8%	-0.04 14.5%	0.01 59.9%	-0.04 87.2%

**Table 4: FX Risk Premium and Macroeconomic Fundamentals**

This table provides the OLS regression coefficients when the exchange rate risk premium from each of the models is regressed on the money supply, current account balance, and expected inflation in the foreign country (either the U.K. or the Euro zone) and the U.S. The data is quarterly from August 2001 to May 2005. The regressors are standardized to have mean 0 and standard deviation 1. One-sided  $p$ -values are provided below the regression coefficients.

	Expected Inflation	
	Foreign	U.S.
USD/GBP w/o Options	-0.01 23.2%	-0.03 96.8%
USD/GBP w/ Options	-0.01 0.6%	0.01 1.0%
USD/EUR w/o Options	0.07 99.5%	-0.11 100.0%
USD/EUR w/ Options	-0.01 20.1%	0.00 45.3%

**Table 5: FX Risk Premium and Expected Inflation**

This table provides the OLS regression coefficients when the exchange rate risk premium from each of the models is regressed on the expected inflation in the foreign country (either the U.K. or the Euro zone) and the U.S. The data is monthly from August 2001 to June 2005. The regressors are standardized to have mean 0 and standard deviation 1. One-sided  $p$ -values are provided below the regression coefficients.

## B Pricing Kernels in the Foreign Currency

This paper characterizes international asset pricing models by their joint specification for the pricing kernel,  $M$ , (denominated in the domestic currency) and the exchange rate,  $S$ . Previous research has instead characterized international asset pricing models by specifying the joint dynamics of the domestic and foreign pricing kernels  $M$  and  $\tilde{M}$ . This section shows that these two approaches are equivalent.

**Lemma 1.** *Let  $S$  be the exchange rate expressed in units of domestic currency per unit of foreign currency. Let  $M$  be the minimum variance nominal pricing kernel (denominated in the domestic currency) such that the price in domestic currency of any payoff  $P_T$  in domestic currency is*

$$P_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} P_T \right],$$

*and the price of any payoff  $\tilde{P}_T$  in foreign currency exchanged to domestic currency is*

$$S_t \tilde{P}_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} \left( S_T \tilde{P}_T \right) \right].$$

*Then*

$$\frac{\tilde{M}_T}{\tilde{M}_t} := \frac{M_T}{M_t} \frac{S_T}{S_t}, \quad (13)$$

*is the minimum variance nominal pricing kernel (denominated in the foreign currency) such that*

$$\tilde{P}_t = \mathbb{E}_t \left[ \frac{\tilde{M}_T}{\tilde{M}_t} \tilde{P}_T \right], \quad \text{and} \quad \frac{1}{S_t} P_t = \mathbb{E}_t \left[ \frac{\tilde{M}_T}{\tilde{M}_t} \left( \frac{1}{S_T} P_T \right) \right]. \quad (14)$$

It is easy to verify that the pricing kernel in equation (13) satisfies the price relationships in equation(14). Let  $\tilde{M}^*$  be the minimum variance such pricing kernel. Then  $\tilde{M}_t = \tilde{M}_t^* \xi_t$  where  $\xi_t$  is a martingale that is independent of  $\tilde{M}_t^*$  and all payoffs  $\tilde{P}_T$  and  $P_T/S_T$ . If  $\xi_t$  is

not constant, then  $M_t^* = M_t/\xi_t$  has lower variance than  $M_t$ , but

$$\mathbb{E}_t \left[ \frac{M_T^*}{M_t^*} P_T \right] = S_t \mathbb{E}_t \left[ \frac{\tilde{M}_T^*}{\tilde{M}_t^*} \frac{1}{S_T} P_T \right] = P_t,$$

and

$$\mathbb{E}_t \left[ \frac{M_T^*}{M_t^*} S_T \tilde{P}_T \right] = S_t \mathbb{E}_t \left[ \frac{\tilde{M}_T^*}{\tilde{M}_t^*} \tilde{P}_T \right] = S_t \tilde{P}_t.$$

Therefore  $M^*$  is a valid nominal domestic pricing kernel with a lower variance than  $M$ , which contradicts the assumption that  $M$  is the minimum variance nominal domestic pricing kernel. Thus,  $\xi_t$  is constant and  $\tilde{M}$  is in fact the minimum variance nominal pricing kernel denominated in the foreign currency.

Lemma 1 implies that any two of  $S$ ,  $M$ , and  $\tilde{M}$  completely determines the third. In particular, it cannot be the case that

$$S_t = S_0 \frac{\tilde{M}_t/\tilde{M}_0}{M_t/M_0} \frac{\xi_t}{\xi_0}.$$

where  $\xi_t$  is independent of both  $M_t$  and  $\tilde{M}_t$ .<sup>17</sup>

It is also important to note that the result in Lemma 1 only needs to hold for the minimum

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<sup>17</sup>To see this more clearly, suppose that

$$S_t = S_0 \frac{\tilde{M}_t/\tilde{M}_0}{M_t/M_0} \frac{\xi_t}{\xi_0}.$$

where  $\xi_t$  is independent of both  $M_t$  and  $\tilde{M}_t$ . Then

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} \right] = \mathbb{E}_t \left[ \frac{\tilde{M}_T}{\tilde{M}_t} \frac{S_t}{S_T} \right] = \mathbb{E}_t \left[ \frac{M_T}{M_t} \frac{\xi_t}{\xi_T} \right] = \mathbb{E}_t \left[ \frac{M_T}{M_t} \right] \mathbb{E}_t \left[ \frac{\xi_t}{\xi_T} \right] \Rightarrow \mathbb{E}_t \left[ \frac{\xi_t}{\xi_T} \right] = 1,$$

and

$$\mathbb{E}_t \left[ \frac{\tilde{M}_T}{\tilde{M}_t} \right] = \mathbb{E}_t \left[ \frac{M_T}{M_t} \frac{S_T}{S_t} \right] = \mathbb{E}_t \left[ \frac{\tilde{M}_T}{\tilde{M}_t} \frac{\xi_T}{\xi_t} \right] = \mathbb{E}_t \left[ \frac{\tilde{M}_T}{\tilde{M}_t} \right] \mathbb{E}_t \left[ \frac{\xi_T}{\xi_t} \right] \Rightarrow \mathbb{E}_t \left[ \frac{\xi_T}{\xi_t} \right] = 1.$$

variance pricing kernels that price assets in both currencies. In particular, Lemma 1 does not relate the the minimum variance pricing kernels that price assets in only one currency. To illustrate this distinction, suppose that the price in domestic currency of a domestic asset follows a process of the form

$$dP_t = P_t \mu dt + P_t \sigma dW_{1t},$$

and the price in foreign currency of a foreign asset follows a similar process

$$d\tilde{P}_t = \tilde{P}_t \tilde{\mu} dt + P_t \tilde{\sigma} dW_{2t},$$

where  $W_1$  and  $W_2$  are independent Brownian motions. Then the domestic minimum variance pricing kernel that prices only domestic assets can be of the form

$$dM_t^d = -M_t^d r dt - M_t^d \Lambda_1 dW_{1t},$$

and the foreign minimum variance pricing kernel that prices only foreign assets can be of the form

$$d\tilde{M}_t^f = -\tilde{M}_t^f \tilde{r} dt - \tilde{M}_t^f \tilde{\Lambda}_2 dW_{2t}.$$

If  $M^d$  and  $\tilde{M}^f$  also price assets in both currencies, then Lemma 1 implies that the dynamics

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However, if  $\xi_T/\xi_t$  is not constant, then by Jensen's inequality,

$$1 = \mathbb{E}_t \left[ \frac{\xi_t}{\xi_T} \right] > 1/\mathbb{E}_t \left[ \frac{\xi_T}{\xi_t} \right] = 1,$$

which is a contradiction.

of the exchange rate are

$$dS_t = S_t [r - \tilde{r} + \Lambda_1^2] + S_t \Lambda_1 dW_{1t} - S_t \tilde{\Lambda}_2 dW_{2t}. \quad (15)$$

However, although the price in domestic currency of the domestic asset does not depend on  $W_2$ , the domestic market price of this risk need not be zero.<sup>18</sup> That is, the domestic pricing kernel that prices both domestic and foreign assets can be of the form

$$dM_t = -M_t r dt - M_t \Lambda_1 dW_{1t} - M_t \Lambda_2 dW_{2t}. \quad (16a)$$

Similarly, the foreign pricing kernel that prices both domestic and foreign assets can be of the form

$$d\tilde{M}_t = -\tilde{M}_t \tilde{r} dt - \tilde{M}_t \tilde{\Lambda}_1 dW_{1t} - \tilde{M}_t \tilde{\Lambda}_2 dW_{2t}.$$

In this more general case, Lemma 1 implies that the dynamics of the exchange rate are

$$\begin{aligned} dS_t = & S_t \left[ r - \tilde{r} + \left( \Lambda_1 - \tilde{\Lambda}_1 \right) \Lambda_1 + \left( \Lambda_2 - \tilde{\Lambda}_2 \right) \Lambda_2 \right] dt \\ & + S_t \left( \Lambda_1 - \tilde{\Lambda}_1 \right) dW_{1t} + S_t \left( \Lambda_2 - \tilde{\Lambda}_2 \right) dW_{2t}. \end{aligned} \quad (17)$$

The pricing kernels  $M$  and  $\tilde{M}$  have the same implications as  $M^d$  and  $\tilde{M}^f$  for prices in domestic currency of the domestic asset and prices in foreign currency of the foreign asset. However, the implied dynamics of the exchange rate in equations (17) and (15) can be different. Brennan and Xia (2005) study the empirical relationship between the exchange rate and single-currency pricing kernels but, as this example illustrates, single-currency pricing

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<sup>18</sup>Intuitively,  $\hat{M}$  is the projection of  $M$  onto the space of payoffs of domestic securities. Since  $W_2$  is orthogonal to the space of payoffs on domestic securities,  $\Lambda_2$  is not present in the the minimum variance pricing kernel  $\hat{M}$  that prices all domestic payoffs. This does not mean that  $\Lambda_2 = 0$ , it simply means that we cannot empirically identify  $\Lambda_2$  by only using asset prices that do not depend on  $W_2$ .

kernels can differ from their counterparts that price assets in two currencies.

## C Exchange Rate Option Pricing

This section describes the cumulant expansion technique used in this paper to efficiently compute exchange rate option prices and facilitate estimation. This technique was first introduced to option pricing by Jarrow and Rudd (1982) and was applied to swaption pricing by Collin-Dufresne and Goldstein (2002). The development in this section is a special case of the results developed in an earlier version of Almeida et al. (2006) for the general class of affine models, and is included here for completeness.

Recall from Section 3 that exchange rate option prices (in domestic currency) are given by

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} (S_T - K)^+ \right] = \mathbb{E}_t \left[ \frac{M_T}{M_t} S_T 1_{\{S_T \geq K\}} \right] - K \mathbb{E}_t \left[ \frac{M_T}{M_t} 1_{\{S_T \geq K\}} \right].$$

By the Lévy inversion formula,

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} S_T^b 1_{\{S_T \geq K\}} \right] = \frac{1}{2} \mathbb{E}_t \left[ \frac{M_T}{M_t} S_T^b \right] - \frac{1}{\pi} \int_0^\infty \frac{1}{v} \text{Im} \left\{ K^{iv} \mathbb{E}_t \left[ \frac{M_T}{M_t} S_T^{b-iv} \right] \right\} dv,$$

where

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} S_T^\delta \right] = e^{A^S(\delta, T-t) + B^S(\delta, T-t) \cdot X_t} S_t^\delta,$$

and  $A^S$  and  $B^S$  satisfy the Riccati ODEs (expressed in integral form)

$$\begin{aligned}
B^S(\delta, \tau) &= \int_0^\tau \left\{ \begin{aligned} &\delta [\rho_1 - \tilde{\rho}_1 - \frac{1}{2} (1 - \delta) \beta^\top \Delta [\Sigma] \Sigma^\top] - \rho_1 \\ &+ [\delta \beta^\top \Delta [\Sigma] + \mathcal{K}_1^\top] B^S(\delta, u) \\ &+ \frac{1}{2} \beta^\top \Delta [B^S(\delta, u)] B^S(\delta, u) \end{aligned} \right\} du, \\
A^S(\delta, \tau) &= \int_0^\tau \left\{ \begin{aligned} &\delta [\rho_0 - \tilde{\rho}_0 - \frac{1}{2} (1 - \delta) \alpha^\top \Delta [\Sigma] \Sigma^\top] - \rho_0 \\ &+ [\delta \alpha^\top \Delta [\Sigma] + \mathcal{K}_0^\top] B^S(\delta, u) \\ &+ \frac{1}{2} \alpha^\top \Delta [B^S(\delta, u)] B^S(\delta, u) \end{aligned} \right\} du.
\end{aligned}$$

If the model parameters are restricted so that the solutions  $A^S$  and  $B^S$  to the Riccati ODEs are known in closed form, then currency option valuation only requires numerical evaluation of a 1-dimensional integral. However, in the most flexible models, the Riccati ODEs must be solved numerically and thus valuing currency options using the Lévy inversion formula can be computationally expensive.

Instead, this paper uses a more computationally efficient cumulant expansion technique to compute currency option prices. The cumulant expansion requires that we compute the Taylor series expansion of

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} S_T^{b-iv} \right] = e^{A^S(b-iv, T-t) + B^S(b-iv, T-t) \cdot X_t} S_t^{b-iv},$$

about  $v = 0$ . Define the cumulants  $c_m$  by

$$c_m := \underbrace{\frac{\partial^m A^S(b-iv, T-t)}{\partial (iv)^m} \Big|_{v=0}}_{i^{3m} \partial_v^m A^S(b, T-t)} + \underbrace{\frac{\partial^m B^S(b-iv, T-t)}{\partial (iv)^m} \Big|_{v=0}}_{i^{3m} \partial_v^m B^S(b, T-t)} \cdot X_t,$$

so that

$$\ln \mathbb{E}_t \left[ \frac{M_T}{M_t} S_T^{b-iv} \right] = A^S(b, T-t) + B^S(b, T-t) \cdot X_t + (b-iv) \ln S_t + \sum_{m=1}^{\infty} \frac{(iv)^m}{m!} c_m.$$

This cumulant expansion technique is especially well-suited to an affine framework because the cumulants are also affine in the state vector  $X_t$  with coefficients that again satisfy Riccati ODEs,

$$\partial_v^0 B^S(b, \tau) := B^S(b, \tau),$$

$$\partial_v^0 A^S(b, \tau) := A^S(b, \tau),$$

$$\partial_v^1 B^S(b, \tau) = \int_0^\tau \left\{ \begin{array}{l} -i [\rho_1 - \tilde{\rho}_1 + (b - \frac{1}{2}) \beta^\top \Delta [\Sigma] \Sigma^\top] \\ -i \beta^\top \Delta [\Sigma] B^S(b, u) + [b \beta^\top \Delta [\Sigma] + \mathcal{K}_1^\top] \partial_v^1 B^S(b, u) \\ + \beta^\top \Delta [B^S(b, u)] \partial_v^1 B^S(b, u) \end{array} \right\} du,$$

$$\partial_v^1 A^S(b, \tau) = \int_0^\tau \left\{ \begin{array}{l} -i [\rho_0 - \tilde{\rho}_0 + (b - \frac{1}{2}) \alpha^\top \Delta [\Sigma] \Sigma^\top] \\ -i \alpha^\top \Delta [\Sigma] B^S(b, u) + [b \alpha^\top \Delta [\Sigma] + \mathcal{K}_0^\top] \partial_v^1 B^S(b, u) \\ + \alpha^\top \Delta [B^S(b, u)] \partial_v^1 B^S(b, u) \end{array} \right\} du,$$

$$\partial_v^2 B^S(b, \tau) = \int_0^\tau \left\{ \begin{array}{l} -i 2 \beta^\top \Delta [\Sigma] \partial_v^1 B^S(b, u) \\ + [b \beta^\top \Delta [\Sigma] + \mathcal{K}_1^\top] \partial_v^2 B^S(b, u) - \beta^\top \Delta [\Sigma] \Sigma^\top \\ + \beta^\top \sum_{k=0}^2 \varphi_k^2 \Delta [\partial_v^{2-k} B^S(b, u)] \partial_v^k B^S(b, u) \end{array} \right\} du,$$

$$\partial_v^2 A^S(b, \tau) = \int_0^\tau \left\{ \begin{array}{l} -i 2 \alpha^\top \Delta [\Sigma] \partial_v^1 B^S(b, u) \\ + [b \alpha^\top \Delta [\Sigma] + \mathcal{K}_0^\top] \partial_v^2 B^S(b, u) - \alpha^\top \Delta [\Sigma] \Sigma^\top \\ + \alpha^\top \sum_{k=0}^2 \varphi_k^2 \Delta [\partial_v^{2-k} B^S(b, u)] \partial_v^k B^S(b, u) \end{array} \right\} du,$$

and for  $m > 2$ ,

$$\begin{aligned} \partial_v^m B^S(b, \tau) &= \int_0^\tau \left\{ \begin{aligned} &-im \beta^\top \Delta [\Sigma] \partial_v^{m-1} B^S(b, u) \\ &+ [b \beta^\top \Delta [\Sigma] + \mathcal{K}_1^\top] \partial_v^m B^S(b, u) \\ &+ \beta^\top \sum_{k=0}^m \varphi_k^m \Delta [\partial_v^{m-k} B^S(b, u)] \partial_v^k B^S(b, u) \end{aligned} \right\} du, \\ \partial_v^m A^S(b, \tau) &= \int_0^\tau \left\{ \begin{aligned} &-im \alpha^\top \Delta [\Sigma] \partial_v^{m-1} B^S(b, u) \\ &+ [b \alpha^\top \Delta [\Sigma] + \mathcal{K}_0^\top] \partial_v^m B^S(b, u) \\ &+ \alpha^\top \sum_{k=0}^m \varphi_k^m \Delta [\partial_v^{m-k} B^S(b, u)] \partial_v^k B^S(b, u) \end{aligned} \right\} du, \end{aligned}$$

where  $\varphi_k^m = \binom{m}{k}$  if  $m \neq 2k$  and  $\varphi_k^m = \frac{1}{2} \binom{m}{k}$  if  $m = 2k$ .

Once we have computed the cumulants, we can use the accurate approximation

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} (S_T - K)^+ \right] \approx \sum_{m=0}^M [\chi_{-1}^m \Phi_{-1}(-\ln K - c_1) + \chi_0^m \Phi_0(-\ln K - c_1)],$$

where

$$\begin{aligned} \Phi_{-1}(y) &= \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{y^2}{2c_2}}, \\ \Phi_0(y) &= \int_{-\infty}^y \Phi_{-1}(z) dz, \end{aligned}$$

and the coefficients  $\chi_{-1}^m$  and  $\chi_0^m$  are related to the cumulants as described below.  $\Phi_{-1}$  and  $\Phi_0$  are just the density and cumulative distribution of the Normal distribution. There exist accurate approximations to the cumulative Normal density, therefore computation of currency prices using a cumulant expansion does not require any numerical integration (aside from solving Riccati ODEs).

I now turn to determining the coefficients  $\chi_{-1}^m$  and  $\chi_0^m$ . Define  $a_m$  to be the coefficients in

a Taylor series expansion of

$$\underbrace{e^{A^S(b-iv, T-t)+B^S(b-iv, T-t) \cdot X_t} S_t^{b-iv}}_{\mathbb{E}_t \left[ \frac{M_T}{M_t} S_T^{b-iv} \right]} e^{-[c_1(iv) + \frac{1}{2}c_2(iv)^2]}$$

about  $v = 0$ , so that

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} S_T^{b-iv} \right] = e^{c_1(iv) - \frac{1}{2}c_2 v^2} \sum_{m=0}^{\infty} a_m v^m.$$

Then

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} \mathbb{E}_t \left[ \frac{M_T}{M_t} S_T^b \right] dv &= \sum_{m=0}^{\infty} a_m \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(z-c_1)v - \frac{1}{2}c_2 v^2} v^m dv \\ &= \sum_{m=0}^{\infty} a_m \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial^m e^{uv - \frac{1}{2}c_2 v^2}}{\partial u^m} \Big|_{u=-i(z-c_1)} dv \\ &= \sum_{m=0}^{\infty} \frac{\partial^m}{\partial u^m} \left\{ a_m \frac{1}{\sqrt{2\pi c_2}} e^{\frac{u^2}{2c_2}} \right\} \Big|_{u=-i(z-c_1)} \\ &\approx \sum_{m=0}^M \frac{\partial^m}{\partial u^m} \left\{ a_m \frac{1}{\sqrt{2\pi c_2}} e^{\frac{u^2}{2c_2}} \right\} \Big|_{u=-i(z-c_1)} \\ &=: \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(z-c_1)^2}{2c_2}} \sum_{m=0}^M \lambda_m (z-c_1)^m, \end{aligned}$$

where the last line defines the coefficients  $\lambda_m$ .

Then by the inverse Fourier transform,

$$\begin{aligned} \mathbb{E}_t \left[ \frac{M_T}{M_t} (S_T - K)^+ \right] &= \int_{-\infty}^{-\ln K} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} \underbrace{\mathbb{E}_t \left[ \frac{M_T}{M_t} S_T^{b-iv} \right]}_{\hat{G}_t(v; b)} dv dz \\ &\approx \sum_{m=0}^M \lambda_m \underbrace{\int_{-\infty}^{-\ln K} \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(z-c_1)^2}{2c_2}} (z-c_1)^m dz}_{\Phi_m(-\ln K - c_1)}, \end{aligned}$$

$\Phi_m(y)$  can be expressed in terms of  $\Phi_{-1}(y)$  and  $\Phi_0(y)$  via the recursive relationship

$$\begin{aligned}\Phi_{-1}(y) &= \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{y^2}{2c_2}}, \\ \Phi_0(y) &= \int_{-\infty}^y \Phi_{-1}(z) dz, \\ \Phi_m(y) &= -c_2 \int_{-\infty}^y z^{m-1} d\Phi_{-1}(z) \\ &= -c_2 [y^{m-1}\Phi_{-1}(y) - (m-1)\Phi_{m-2}(y)].\end{aligned}$$

Therefore,

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} (S_T - K)^+ \right] \approx \sum_{m=0}^M [\chi_{-1}^m \Phi_{-1}(-\ln K - c_1) + \chi_0^m \Phi_0(-\ln K - c_1)],$$

as desired.

Finally,  $M$  must be chosen to balance accuracy and computational speed. I follow Collin-Dufresne and Goldstein (2002) and choose  $M = 7$ .

## D Detailed Model Specification

I estimate a four-factor version of the general model described in equation (9), with

$$\begin{aligned}\mathcal{K}_0^{\mathbb{P}} &:= \begin{bmatrix} 0 & 0 & \mathcal{K}_{03}^{\mathbb{P}} & \mathcal{K}_{04}^{\mathbb{P}} \end{bmatrix}^{\top}, \\ \mathcal{K}_0 &:= \begin{bmatrix} \mathcal{K}_{01} & \mathcal{K}_{02} & \mathcal{K}_{03} & \mathcal{K}_{04} \end{bmatrix}^{\top},\end{aligned}$$

$$\mathcal{K}_1^{\mathbb{P}} := \begin{bmatrix} \mathcal{K}_{111}^{\mathbb{P}} & 0 & \mathcal{K}_{113}^{\mathbb{P}} & \mathcal{K}_{114}^{\mathbb{P}} \\ \mathcal{K}_{121}^{\mathbb{P}} & \mathcal{K}_{122}^{\mathbb{P}} & \mathcal{K}_{123}^{\mathbb{P}} & \mathcal{K}_{124}^{\mathbb{P}} \\ 0 & 0 & \mathcal{K}_{133}^{\mathbb{P}} & 0 \\ 0 & 0 & \mathcal{K}_{143}^{\mathbb{P}} & \mathcal{K}_{144}^{\mathbb{P}} \end{bmatrix},$$

$$\mathcal{K}_1 := \begin{bmatrix} \mathcal{K}_{111} & 0 & \mathcal{K}_{113} & \mathcal{K}_{114} \\ \mathcal{K}_{121} & \mathcal{K}_{122} & \mathcal{K}_{123} & \mathcal{K}_{124} \\ 0 & 0 & \mathcal{K}_{133} & 0 \\ 0 & 0 & \mathcal{K}_{143} & \mathcal{K}_{144} \end{bmatrix},$$

$$\alpha + \beta X_t := \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \beta_{13} & \beta_{14} \\ 0 & 0 & \beta_{23} & \beta_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}_t,$$

$$\Sigma := \begin{bmatrix} \Sigma_1 & \Sigma_2 & \Sigma_3 & \Sigma_4 \end{bmatrix},$$

$$\rho_1 := \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \end{bmatrix}^{\top}, \text{ and}$$

$$\tilde{\rho}_1 := \begin{bmatrix} \tilde{\rho}_{11} & \tilde{\rho}_{12} & \tilde{\rho}_{13} & \tilde{\rho}_{14} \end{bmatrix}^{\top}.$$

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