

# Valuations and Dynamics of Negotiations

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## Abstract

This paper analyzes three-party negotiations in the presence of externalities, deriving a close form solution for the stationary subgame perfect Nash equilibrium of a standard non-cooperative bargaining model. Players' values are monotonically increasing (or decreasing) in the amount of negative (or positive) externalities that they impose on others. Moreover, players' values are continuous and piecewise linear on the worth of bilateral coalitions, and are inextricably related to their negotiation strategies: the equilibrium value is the Nash bargaining solution when no bilateral coalitions form; the Shapley value when all bilateral coalitions form; or the nucleolus, when either one bilateral coalition among 'natural partners' or two bilateral coalitions including a 'pivotal player' form.

JEL: C71, C72, C78, D62.

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# 1 Introduction

This paper studies multilateral negotiations in the presence of externalities. These problems are important in economics, appearing in such diverse areas as mergers and acquisitions, bankruptcy and international treaty negotiations, as well as the formation of labor unions and coalitional governments. What is the sequencing of negotiations and how do the parties involved form valuations? Our goal in this paper is to analyze a standard strategic model of negotiations and to derive a close form solution answering the question above.

While bilateral negotiations have been extensively studied, much less is known about the more complex problem of negotiations involving three or more parties, where, for example, coalition formation is an important aspect of the negotiations. We restrict our attention here to three-player games and develop an explicit solution for all three-player games with externalities. Having a simple off-the-shelf solution for three-player games is helpful in extending our understanding one step beyond bilateral interactions, and is useful in applications where coalition formation and externalities play an important role.

The bargaining model analyzed herein is a standard non-cooperative model (see literature review below). Our main contribution is to characterize its stationary subgame perfect Nash equilibrium. In our model there are three players who can form bilateral coalitions and the grand coalition. A set of parameters describe the worth of the grand coalition and of all bilateral coalitions, including the amount of externalities they impose on excluded players. The bargaining game evolves over time with players making offers followed by responses every period.

We show that the equilibrium value, referred to as the coalitional bargaining value (CBV), is a continuous and piecewise linear function of the parameters of the game. Specifically, the space of all games is divided into four convex regions (eight including all permutations), and in each region the CBV is a linear function of the parameters of the game. For games without

externalities, in one of the regions the CBV coincides with the Nash bargaining solution, in another region with the Shapley value, and in the other regions with the nucleolus. For games with externalities, a similar treatment applies as long as an adjusted measure for the worth of pairwise coalitions is used to generalize the Shapley value and the nucleolus. This adjustment involves measuring the worth of a pairwise coalition adding the amount of negative externalities (or subtracting the amount of positive externalities) that it creates for the excluded player. Therefore, this paper proposes a way to select a specific cooperative solution concept for all games without externalities, and a generalization to situations with externalities.

The strategies employed by players in each region have an intuitive economic interpretation in terms of credible outside options (see also Sutton (1986)). This interpretation serves to enhance our understanding of the related cooperative solution concepts and shows how valuations are inextricably related to the equilibrium negotiation strategies. First, the CBV is equal to the Nash bargaining solution (equal split of the surplus) and no bilateral coalitions forms, if the (adjusted) worth of all bilateral coalition is less than a third of the grand coalition value. In this region no player is able to demand more than an equal share of surplus because the outside option of forming a bilateral coalition is not credible. Second, the CBV coincides with the (generalized) Shapley value and all bilateral coalitions can form in equilibrium, if the sum of the (adjusted) values of all bilateral coalitions is greater than the grand coalition value. In these games, there is an advantage from being the proposer (first mover advantage) and a disadvantage from being excluded from a bilateral coalition.

Finally, there are two novel cases in which the CBV coincides with the (generalized) nucleolus: games where only the ‘natural coalition’ among two ‘natural partners’ creates significant value, and games where only the two pairwise coalitions including a ‘pivotal player’ create significant value. In the first case, the player excluded from the natural coalition agrees with a payoff lower than an equal split of the surplus, and the natural partners equally split the gains from forming the natural coalition—an outcome that

is driven by the fact that only the natural coalition can credibly form in equilibrium. In the second case, both non-pivotal players agree to form a coalition with the pivotal player, receiving a payoff lower than an equal split of the surplus—an outcome that is driven by the fact that only the pairwise coalitions including the pivotal player can credibly form.

There is now an extensive literature studying non-cooperative coalitional bargaining games—we refer below to the most closely related literature. Earlier papers in the area analyzed the properties of games without externalities: Gul (1989), Chatterjee et al. (1993), Moldovanu and Winter (1995), Okada (1996), Hart and Mas-Colell (1996), Krishna and Serrano (1996), Seidmann and Winter (1998), among others. Later studies considered the extension to coalitional bargaining games with externalities. For example, Jehiel and Moldovanu (1995ab), Bloch (1996), Yi (1997), Ray and Vohra (1999), Montero (1999), and Gomes (2001) showed that a much broader range of applications can be analyzed when allowing for externalities. Moreover, they addressed several general properties (such as efficiency, bargaining delays, existence, stability, uniqueness, and convergence) of the equilibrium for games with an arbitrary number of players.

Similarly to Moldovanu (1990), Serrano (1993), and Cornet (2003), the contribution of this paper to the literature is its in depth analysis of 3-player games. Moldovanu (1990) studies the coalition-proof Nash equilibria of 3-player games without side payments. Among other properties, he shows that if the game is balanced then the equilibrium payoff is in the core. Serrano (1993) studied 3-player bargaining games without externalities in which responders may exit and have endogenous outside options. When the order of proposers corresponds to the power players have in the underlying coalition function, the unique Markov perfect equilibrium outcome of the game is the prenucleolus. Cornet (2003) studied 3-player negotiations with externalities using a model in which bargaining takes place using the demand-making framework originally proposed by Binmore (1985). In this framework players sequentially pose demands and accept or reject standing demands from other player(s)—acceptance leads to the formation of a two or three player

coalition and the game terminates. Cornet (2003) analyzed the (stationary) subgame perfect equilibrium and showed that it is not unique and may be inefficient. His main conclusion is that creation of positive or negative externalities by a coalition is irrelevant to the equilibrium value. The properties of our solution are different because we show that the players' values are increasing (decreasing) on the amount of negative (positive) externality that they can impose on others. These properties as well as the model's predictions for the dynamics of coalition formation have been tested using experimental methods by Croson, et al. (2003), and their results largely support our conclusions.

The remainder of the paper is organized as follows: Section 2 presents the negotiation model, Section 3 characterizes the equilibrium and derives the explicit formula for the CBV, Section 4 studies the relationship of the CBV with cooperative game theory concepts, and Section 5 concludes. The Appendix contains the proofs of all propositions.

## 2 The Bargaining Model

The game has three players  $N = \{1, 2, 3\}$ . Each player owns an indivisible tradeable resource or right, and they can buy or sell resources in exchange of a transfer of utility. Players that acquire resources continue trading, and players that sell their resources leave the game. As a result of trading, different ownership or *coalition structures (c.s.)* may arise, starting from the initial c.s.  $[1|2|3]$  in which all resources are owned by different players: the c.s.  $[ij|k]$ , where one player, either  $i$  or  $j$ , owns both resources  $\{i, j\}$ , or the c.s.  $[N]$  where one player owns all resources. We assume that players owning the same resources play identical strategies, and thus to simplify notation we do not keep track of who owns resources  $C$ , instead referring to this player as *coalition C*.<sup>1</sup>

Players are expected utility maximizers and have a common per period

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<sup>1</sup>Alternatively, we could have interpreted the formation of a coalition as a binding agreement (i.e., not necessarily an ownership agreement), in which the coalition acts as an agent maximizing the aggregate utility of the coalition members.

discount factor equal to  $\delta \in [0, 1)$ . A set of parameters describes the flow of utility generated by the resources for all the possible c.s. Accordingly, if the c.s. is  $[1|2|3]$ , the flow of utility to player  $i$  is  $(1 - \delta) u_i$ ; if the c.s. is  $[ij|k]$ , the flow of utility to coalition  $\{i, j\}$  and player  $k$  are, respectively, equal to  $(1 - \delta) U_{ij}$  and  $(1 - \delta) U_k$ ; and finally if the c.s. is  $[N]$ , the flow of utility to the grand coalition  $N$  is  $(1 - \delta) U$ . Note that the specification above can capture any positive or negative externalities that the coalition  $\{j, k\}$  creates for player  $i$ , whenever  $u_i < U_i$  or  $u_i > U_i$ , respectively. The set of parameters  $u$  is also known as a *partition function form* (see Thrall and Lucas (1963) and Ray and Vohra (1999)). A *characteristic function form* corresponds to a special partition function form where  $u_i = U_i$ , and thus there are no externalities. Without any loss of generality, we consider only 0-normalized partition functions, that is  $u_i = 0$ . Furthermore, all partition functions considered here are such that a three-player agreement is efficient, i.e.,  $U \geq U_i + U_{jk}$  for all distinct  $i, j$ , and  $k$  and  $U \geq 0$ .

We model negotiations as an infinite horizon non-cooperative game with complete information, utilizing the partition function as the basic underlying structure. The negotiation game evolves with players making offers (to acquire the resources of other players) followed by players that have received offers accepting or rejecting it, as in Rubinstein (1982). Specifically, the game is defined recursively by the following extensive form, starting from the c.s.  $[1|2|3]$ : (*Random proposer's choice*) At the beginning of each period one of the players belonging to the current c.s. is randomly chosen to be the proposer. In order to capture the role of the players' opportunity to propose, if the c.s. is  $[1|2|3]$ , player  $i$  is proposer with arbitrary probability  $p_i = p_i[1|2|3]$ , and if the c.s. is  $[ij|k]$ , coalition  $ij$  and player  $k$  are proposers with probabilities  $p_{ij}[ij|k]$  and  $p_k[ij|k]$ . (*Proposal stage*) The proposer then makes one of the following choices: an offer to buy the resources of another player, say  $j$ , at price  $t_j$ ; an offer to buy the resources of both players, say  $j$  and  $k$ , at prices  $t_j$  and  $t_k$ ; or makes no offers (i.e., pass his opportunity to propose). (*Response stage*) The player(s) receiving the offer respond sequentially either accepting or rejecting the offer (the order of response

turns out to be irrelevant). An exchange of ownership then takes place if *all* player(s) receiving the offer accept(s) it. After trading a new c.s. arises (or the c.s. remains the same if any player rejected the offer), with those players selling their resources leaving the game and the proposer remaining in the game. Flow payoffs occur at the end of each period according to the partition function described in the previous paragraph.<sup>2</sup> The game is repeated, after a lapse of one period of time, with a new proposer being randomly chosen as described above, until the game terminates when the c.s.  $[N]$  forms.

Our notion of equilibrium is *stationary subgame perfect Nash equilibrium* or *Markov perfect equilibrium (MPE)*. A strategy profile  $\sigma$  is MPE if it is a subgame perfect Nash equilibrium and the strategies are Markovian, i.e., the strategies at each stage of the game do not depend on the history of the game nor on calendar time. Formally, at the proposal stage a Markovian strategy depends only on the current c.s. and who is the proposer; at the response stage a Markovian strategy depends only on the current c.s., the proposer, the offer made by the proposer, and (if a player is the second responding) the response of the first responder.

### 3 The Coalition Bargaining Value

The interesting action happens at the initial stage. After any bilateral agreement, the analysis reduces to a standard two-player bargaining game. It is a well-known result that the random proposer bilateral bargaining game has a unique (stationary) subgame perfect equilibrium (this game is an unessential variation of Rubinstein's (1982) alternating offer bargaining game). In the unique equilibrium, the continuation values of the bilateral game with c.s.  $[ij|k]$  are equal to

$$V_{ij} = U_{ij} + p_{ij}([ij|k]) (U - U_k - U_{ij}) \quad \text{and} \quad V_k = U_k + p_k([ij|k]) (U - U_k - U_{ij}), \quad (1)$$

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<sup>2</sup>For example, the situation in which an offer by player  $i$  is accepted by  $j$  leads to a flow payoff equal to  $(1 - \delta)U_{ij} - t_j$  to player  $i$ ,  $(1 - \delta)U_k$  to player  $k$ , and player  $j$  leaves the game with a final payoff equal to  $t_j$ .

respectively, for coalition  $ij$  and player  $k$ . Note that this is the value at the subgame  $[ij|k]$  before the proposer has been chosen. It is also useful to define the continuation values at the end of subgame  $[ij|k]$ , after offers have been rejected, which are equal to  $V_{ij}^\delta = \delta V_{ij} + (1 - \delta) U_{ij}$  and  $V_k^\delta = \delta V_k + (1 - \delta) U_k$ .

Consider any MPE  $\sigma$  and let  $v_i$  be the equilibrium continuation value of player  $i$  when the c.s. is  $[1|2|3]$ , before the proposer has been chosen. Any player responding to an offer can, by rejecting it, maintain the negotiations on the same state. Therefore, when faced with any offer, responders accept it only if the offer price is above or equal to  $v_i^\delta := \delta v_i$ . On the other hand, proposers choose whom to extend an offer to based on which deal produces the largest gains—always offering the minimum prices that are acceptable. Specifically, say that player  $i$ 's strategy puts probability  $\sigma_i^\delta(S)$  on making an offer to form coalition  $S$  (where  $i \in S$ ), and let the gain associated with the formation of coalition  $S$  be  $e_S^\delta$ , which is equal to  $V_{ij}^\delta - v_i^\delta - v_j^\delta$  if  $S = \{i, j\}$  (symmetric for  $S = \{i, k\}$ ),  $U - v_i^\delta - v_j^\delta - v_k^\delta$  if  $S = N$ , and zero if  $S = \{i\}$  (no coalition forms). Note that this implies that in equilibrium coalition  $S$  forms with probability  $\mu_S^\delta = \sum_{i \in N} p_i \sigma_i^\delta(S)$ . Also note that the value  $v_i$  is endogenously given, and must be equal to  $(1 - \mu_{jk}^\delta) v_i^\delta + \mu_{jk}^\delta V_i^\delta + p_i \max_{i \in S} e_S^\delta$ . This is because, when player  $i$  proposes, his value is  $v_i^\delta + \max_{i \in S \subset N} e_S^\delta$ , and when another player proposes,  $i$ 's value is  $V_i^\delta$ , if  $i$  is excluded from the offer, and  $v_i^\delta$  otherwise.

We obtain the equilibrium for all  $\delta > \bar{\delta}$ , where  $\bar{\delta} < 1$ , by constructing strategies  $\sigma_i^\delta$  satisfying

$$\sigma_i^\delta(S) = 0 \text{ if } e_S^\delta < \max_{i \in S \subset N} e_S^\delta \text{ for all } i, \quad (2)$$

and satisfying

$$v_i^\delta = \delta \left( \mu_{jk}^\delta V_i^\delta + (1 - \mu_{jk}^\delta) v_i^\delta + p_i \max_{i \in S \subset N} e_S^\delta \right) \text{ for all } i. \quad (3)$$

We are particularly interested in the limit equilibrium value when  $\delta$  converges to one (which corresponds to an arbitrarily small interval between

offers). The limit equilibrium value is referred to as the *coalition bargaining value* (CBV) of the game.

**Proposition 1** *Consider any  $\theta$ -normalized three-player game where the grand coalition is efficient (i.e.,  $U \geq 0$  and  $U \geq U_{ij} + U_k$ ) and the opportunities to propose satisfy  $p_i < \frac{1}{2}$ . There exists a MPE  $\sigma^\delta$  for all  $\delta > \bar{\delta}$ , where  $\bar{\delta} < 1$ , such that:*

Part A. *The equilibrium values  $v_i^\delta$  converge to  $v_i$  (the coalition bargaining value) when  $\delta$  converges to one:*

*Case i : If  $V_{12} \leq (p_1 + p_2)U$ ,  $V_{13} \leq (p_1 + p_3)U$ , and  $V_{23} \leq (p_2 + p_3)U$ , then*

$$v_i = p_i U \text{ for all } i \in N,$$

*Case ii : If  $V_{12} \geq (p_1 + p_2)U$ ,  $V_{13} + \frac{p_2}{p_1+p_2}V_{12} \leq U$ , and  $V_{23} + \frac{p_1}{p_1+p_2}V_{12} \leq U$ , then*

$$v_1 = \frac{p_1}{p_1 + p_2}V_{12}, v_2 = \frac{p_2}{p_1 + p_2}V_{12}, \text{ and } v_3 = U - V_{12},$$

*Case iii : If  $V_{12} + \frac{p_3}{p_1+p_3}V_{13} \geq U$ ,  $V_{13} + \frac{p_2}{p_1+p_2}V_{12} \geq U$ , and  $V_{12} + V_{13} + V_{23} \leq 2U$ , then*

$$v_1 = U - V_2 - V_3, v_2 = V_2, \text{ and } v_3 = V_3,$$

*Case iv : If  $V_{12} + V_{13} + V_{23} \geq 2U$ , then*

$$v_i = \frac{1}{3}(U - 2V_{jk} + V_{ij} + V_{ik}) \text{ for all } i \in N.$$

Part B: *The following coalitions form in each of the cases above ( $\mu_S^\delta = \sum_{i \in N} p_i \sigma_i^\delta(S)$  is the probability of coalition  $S$  forming):*

*Case i:  $\mu_N^\delta = 1$  (all  $\mu_{ij}^\delta = 0$ ),*

*Case ii:  $\mu_{12}^\delta + \mu_N^\delta = 1$  ( $\mu_{12}^\delta > 0$  and  $\mu_{13}^\delta = \mu_{23}^\delta = 0$ ),*

*Case iii:  $\mu_{12}^\delta + \mu_{13}^\delta + \mu_N^\delta = 1$  ( $\mu_{12}^\delta, \mu_{13}^\delta > 0$  and  $\mu_{23}^\delta = 0$ ),*

*Case iv:  $\mu_{12}^\delta + \mu_{13}^\delta + \mu_{23}^\delta = 1$  (all  $\mu_{ij}^\delta > 0$  and  $\mu_N^\delta = 0$ ).*

In the proof (see appendix) we consider a partition of the parameter space into regions defined by the inequalities *i*, *ii*, *iii*, and *iv*. In each region, we compute the values  $v_i^\delta$  and the strategies  $\sigma_i^\delta$  satisfying the system of equations and inequalities (2)-(3). We then take the limit of the solution in each region and show that the limit satisfies part A and B of proposition 1 (explicit formulas for  $\mu_S^\delta$ 's are also provided in the appendix).<sup>3</sup>

Note that each case above defines a convex region, and there are a total of eight regions when all permutations are included (three of types *ii* and *iii*). Moreover, all games belong to one of the eight regions and the (interior) of the regions are disjoint.

The sequencing of negotiations is different in each region. Intuitively, region *i* corresponds to the case where no pairwise coalitions create much value, so the threat of forming a pairwise coalition is not credible and would only benefit the excluded player (accordingly the strategies are such that  $\mu_{ij}^\delta = 0$ ). Region *ii* corresponds to the case where  $\{1, 2\}$  is the only bilateral coalition that creates significant value and is the only one that should arise in equilibrium ( $\mu_{13}^\delta = \mu_{23}^\delta = 0$ ). In region *iii* only pairwise coalitions with player 1 create significant value, so they are the only ones that should arise in equilibrium ( $\mu_{23}^\delta = 0$ ). In region *iv*, the sum of values created by pairwise coalitions surpass the grand coalition value, and the solution predicts that all pairwise coalitions arise, but not the grand coalition ( $\mu_N^\delta = 0$ ). Croson, et al. (2003) consider games in each of the four regions. Their experimental results for the transition probabilities are in line with proposition 1, part B.<sup>4</sup>

The close form solution exhibited in proposition 1 allows for the evaluation of *comparative statics* effects associated with changes in the coalitions'

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<sup>3</sup>The equilibrium we obtain in proposition 1 is the unique equilibrium of the game. The uniqueness result is based on Gomes (2001, theorem 6 and corollary 3), who studies local and global uniqueness properties of the equilibria of n-player coalitional bargaining games with externalities.

<sup>4</sup>Croson et al. (2003, table 7) experimental results for the transition probabilities from  $[1|2|3]$  to  $[N]$ ,  $[12|3]$ ,  $[13|2]$ ,  $[23|1]$ , and  $[1|2|3]$  are, respectively: case *i*, 84%, 11%, 4%, 0, and 2%; case *ii*, 60%, 26%, 0, 6%, and 7%; case *iii*, 46%, 20%, 23%, 11%, and 0; case *iv*, 3%, 42%, 40%, 14%, and 0.

worth *and* in the proposers' probabilities. We discuss below how valuations are intrinsically linked to the negotiation strategies, and the intuition for the comparative statics effects.

In region *i*, the CBV is the split of the surplus according to players' opportunities to propose. The intuition behind this result is that the threat of any pair of players *i* and *j* to form coalition  $\{i, j\}$  is not credible because the most the coalition  $\{i, j\}$  can get by alienating player *k* is  $V_{ij}$ , and  $V_{ij} \leq (p_i + p_j)U$ , which is the amount they can get by conforming to the equilibrium strategies. In other words, the ability of players to demand more than a proportional split of the surplus by threatening to form a pairwise coalition is an outside option that is not credible (see Sutton (1986)). The CBV prediction has the following comparative statics implication in this region: the expected outcome of players should be insensitive to local changes in the coalition's worth and is increasing in the proposer probability, as long as the conditions for belonging to region *i* are maintained.

In region *iv*, the strategy of proposer *i* is to choose a player randomly, say *j*, and offer him the value  $\delta v_j$  to form the pairwise coalition  $\{i, j\}$ . Conditional on player *i* been the proposer and making an offer to player *j*, the value of left out player *k* is equal to  $V_k$ , which is smaller than  $v_k$  because

$$V_k - v_k = \frac{2U - (V_{12} + V_{13} + V_{23})}{3} \leq 0,$$

(obtained after taking into account that  $V_k = U - V_{ij}$ ). Therefore in this region there is an advantage from being the proposer and a disadvantage from being excluded from a pairwise coalition.

The comparative statics with respect to the proposer probability yields a surprising result. Based on the discussion above one would expect that in region *iv* the value of a player would be increasing in his opportunity to propose, but this is not the case (note that the limit equilibrium values are not a function of the proposer probabilities). The explanation comes from the analysis of the equilibrium strategies. In the appendix, we show that coalition *jk* forms in equilibrium with probability  $\mu_{jk} = p_i$  (in the limit when  $\delta \rightarrow 1$ ). Thus increases in player *i* proposer probability  $p_i$  are offset

by players  $j$  and  $k$  who are more likely to form coalition  $jk$  (and conditional on coalition  $jk$  forming, the value of player  $i$  is  $V_i \leq v_i$ ).

It is interesting to note that in region  $iv$ , if there are no externalities, the value of players are exactly equal to Shapley value. Recall that the Shapley value is the concept Shapley (1953) derived from axioms, and, in particular, its value for 3-player games is equal to  $Sh_i = \frac{1}{6}(2U - 2U_{jk} + U_{ij} + U_{ik})$ . Thus the Shapley value arises as the equilibrium in situations where players rush to form any pairwise coalitions and there are significant first mover advantages (see also Gul (1989)).<sup>5</sup>

In the next section, we further discuss the connection with cooperative solution as well as the results in regions  $ii$  and  $iii$ .

## 4 Further Properties of the Solution

A situation of special interest is the one in which all players have an equal opportunity to propose (i.e.,  $p_i([1|2|3]) = \frac{1}{3}$  and  $p_{ij}([ij|k]) = p_k([ij|k]) = \frac{1}{2}$ ). The CBV in this case simplifies to the following expression.

**Proposition 2** *Consider any three-player game where all players have equal probability to propose, and let  $\bar{U}_{ij} = U_{ij} - U_k$ . The CBV is:*

*Case i : If  $\bar{U}_{12} \leq \frac{U}{3}$ ,  $\bar{U}_{13} \leq \frac{U}{3}$ , and  $\bar{U}_{23} \leq \frac{U}{3}$  then*

$$v_i = \frac{U}{3} \quad \text{for all } i;$$

*Case ii : If  $\bar{U}_{12} \geq \frac{U}{3}$ ,  $2\bar{U}_{13} + \bar{U}_{12} \leq U$ , and  $2\bar{U}_{23} + \bar{U}_{12} \leq U$  then*

$$v_1 = v_2 = \frac{1}{4}(U + \bar{U}_{12}), \quad \text{and } v_3 = \frac{1}{2}(U - \bar{U}_{12});$$

*Case iii : If  $\bar{U}_{12} + \bar{U}_{13} + \bar{U}_{23} \leq U$ ,  $2\bar{U}_{13} + \bar{U}_{12} \geq U$ , and  $2\bar{U}_{12} + \bar{U}_{13} \geq U$  then*

$$v_1 = \frac{1}{2}(\bar{U}_{12} + \bar{U}_{13}), \quad v_2 = \frac{1}{2}(U - \bar{U}_{13}), \quad \text{and } v_3 = \frac{1}{2}(U - \bar{U}_{12});$$

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<sup>5</sup>Gul (1989) also obtains the Shapley value as the solution under a similar condition.

Case iv : If  $\bar{U}_{12} + \bar{U}_{13} + \bar{U}_{23} \geq U$  then

$$v_i = \frac{1}{6} (2U - 2\bar{U}_{jk} + \bar{U}_{ij} + \bar{U}_{ik}) \text{ for all } i.$$

Note that the CBV depends only on an adjusted measure  $\bar{U}_{ij}$  of the coalition's worth, where  $\bar{U}_{ij} = U_{ij} - U_k$  is the value that coalition  $\{i, j\}$  creates plus (minus) the amount of negative (positive) externalities that it creates for the excluded player  $k$ . Therefore, players' equilibrium values increase or decrease with the amount of negative or positive externalities they impose on others. Interestingly, any game with externalities has similar value and dynamics compared to a game without externalities (a characteristic function game) once coalitions' values are adjusted,  $\bar{U}_{ij}$ , to take externalities into account.

#### 4.1 Examples

A better understanding of the negotiation strategies can be grasped by analyzing a few specific examples illustrating the cases previously discussed (in all examples we assume that players have equal opportunities to propose).

**Example 1** *Three firms compete in an industry in which there are the following merger gains:  $u_i = U_i = 0$ ,  $U = 1$ ,  $U_{12} = v_H$ ,  $U_{13} = v_{L_1}$ ,  $U_{23} = v_{L_2}$  where  $v_H \in [\frac{1}{3}, 1]$  and  $v_{L_1} \leq v_{L_2} \leq \frac{1-v_H}{2} \leq v_H$ .*

What are the prices at which firms merge? Are there any natural merger partners in this industry? The bargaining value and strategies provide a direct answer to the questions above, as one can easily verify that this game belongs to region *ii* and thus the bargaining value is

$$v_1 = \frac{1 + v_H}{4}, \quad v_2 = \frac{1 + v_H}{4}, \quad \text{and} \quad v_3 = \frac{1 - v_H}{2},$$

where  $v_1 = v_2 \geq v_3$ . In this situation, we only expect to see the bilateral merger between firms 1 and 2. On the contrary, say that firms 1 and 3 merges. Their profitability increases by  $v_{L_1}$ , and there are still gains from further consolidation with firm 2. Firm 2 and conglomerate  $\{1, 3\}$

split the merger gains in a Nash bargaining way, each getting, respectively,  $\frac{1}{2}(1 - v_{L_1})$  and  $\frac{1}{2}(1 + v_{L_1})$ . Note that the value of the conglomerate  $\{1, 3\}$  is  $\frac{1}{2}(1 + v_{L_1}) \leq \frac{1}{4}(3 - v_H) = v_1 + v_3$ . Therefore, one can predict that firms 1 and 3 are not going to merge, and by the same reasoning, one can also rule out a merger between firms 2 and 3.

Consider now a merger between firms 1 and 2. How should the value of the conglomerate  $\{1, 2\}$  be split among firms 1 and 2? Firm 2 has an apparent stronger bargaining position than firm 1 because  $v_{L_1} \leq v_{L_2}$  and thus it seems reasonable that firm 2 should receive a higher share of the value than firm 1. However, this intuitive idea is wrong: Firm 2 does not have any credible outside options other than to merge with firm 1, and thus the Nash bargaining solution is an equal split of the value of the conglomerate  $\{1, 2\}$ .

**Example 2** *Formation of labor unions: In this game  $u_i = U_i = 0$ ,  $U_{12} = v$ ,  $U_{13} = v$ ,  $U_{23} \leq 1 - 2v$  where  $v \in [\frac{1}{3}, 1]$ , and  $U = 1$ . The firm is player 1, and players 2 and 3 are two unions.*

What are the firm's profits and the employee wages? Are the workers better off forming only one union to collectively bargain for wages or bargaining separately as two unions? This game belongs to region *iii*, and thus the bargaining value is equal to

$$v_1 = v, \quad v_2 = \frac{1 - v}{2}, \quad \text{and} \quad v_3 = \frac{1 - v}{2},$$

where  $v_1 \geq v_2 = v_3$ . Note that the unions are not willing to agree to a wage lower than  $\frac{1-v}{2}$ . Otherwise, the union could just wait until the firm signs a contract with the other union and bargain with the firm for a wage equal to half of the extra profits that the union could create, which results in a wage equal to  $\frac{1}{2}(1 - v)$ . Interestingly, the threat of forming only one union to bargain for higher wages is not credible. The larger union can bargain for a total wage package equal to half of the surplus that it creates, which is equal to  $\frac{1}{2}(1 + U_{23}) \leq 1 - v$ .<sup>6</sup>

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<sup>6</sup>Kovenock and Widdows (1989) also analyze negotiations between a firm and two unions. They show that the sequencing of negotiations is related to what contingent contracts are

**Example 3** *One-seller and two-buyer market game:* Consider the negotiation game where  $u_i = U_i = 0$ ,  $U_{12} = v_H = 1$ ,  $U_{13} = v_L$ ,  $U_{23} = 0$ , and  $U = v_H = 1$ , with  $v_L < v_H = 1$ . In this game player 1 is the seller, player 2 is the high valuation buyer, and player 3 is the low valuation buyer.

By proposition 1 we have that the bargaining value is

$$v_1 = \frac{v_H}{2} + \frac{v_L}{6}, \quad v_2 = \frac{v_H - v_L}{2} + \frac{v_L}{6} = \frac{1}{2}v_H - \frac{1}{3}v_L, \quad \text{and} \quad v_3 = \frac{v_L}{6},$$

because  $\bar{U}_{12} + \bar{U}_{13} + \bar{U}_{23} = v_H + v_L = 1 + v_L > 1$ .<sup>7</sup> Note that if the valuation of the buyers are the same  $v_H = v_L = 1$  then the only point in the core of the market game is  $(1, 0, 0)$ , where the seller extracts all the surplus from the two buyers. In this case the CBV, which is equal to  $(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$ , does not belong to the core.

Are the CBV predictions reasonable? Shouldn't we expect competition between the two buyers to drive the good's price to 1, as the core predicts? The main reason why the seller can't extract the entire surplus from the buyers is that both buyers have the option of forming a cartel to bid for the good and then buy it at a very low price (0.5), rather than initiating a bidding war. The seller knows about that all too well, and, rather than auctioning the good, the seller prefers to negotiate an intermediate price (between 0.5 and 1) with one buyer, leaving the second buyer with nothing. Because all agreements are binding after a deal is sealed (i.e., either a buyers' cartel is formed or the good is sold) there is no way for the excluded player to undo the deal by enticing one of the players with a slightly better offer.

## 4.2 Cooperative Solutions and the CBV

We now discuss the CBV's relationship with cooperative solution concepts. The CBV is closely related to classic cooperative game theory solutions for

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available to unions.

<sup>7</sup>This solution generalizes the solution of the one-seller two-buyer market game in Osborne and Rubinstein (1990) when players are allowed to use contracts and resell the resource.

characteristic function games (i.e., partition function games without externalities, where  $\overline{U}_{ij} = U_{ij}$ ).

Our next result shows that the CBV in regions *i*, *ii*, and *iii* coincides with the nucleolus. We recall that the nucleolus is the concept introduced by Schmeidler (1969), who proved that the nucleolus always exists and is a unique point belonging to the core of the game, whenever the core is non-empty. Kohlberg (1971) then showed that the nucleolus is a piecewise linear function of the characteristic function of the game, and Brune (1983) computed the nucleolus with its regions of linearity for three-person games (see appendix). We have seen in section 3 that the CBV coincides with the Shapley value in region *iv*.<sup>8</sup>

**Proposition 3** *The CBV of any 0-normalized superadditive characteristic function game is the nucleolus, if  $U_{12} + U_{13} + U_{23} \leq U$ , or the Shapley value, if  $U_{12} + U_{13} + U_{23} \geq U$ .*

Note first that belonging to regions *i*, *ii*, or *iii* is indeed equivalent to the constraint  $U_{12} + U_{13} + U_{23} \leq U$ . Comparing the formula for the nucleolus derived by Brune (1983) with the formula for the CBV yields the above result (see appendix). While the nucleolus is a concept that is mathematically very attractive and simple, economists have had difficulties in developing a motivation for it. The strategies employed by players in region *ii* and *iii*, where the nucleolus arise,<sup>9</sup> have an intuitive economic interpretation in terms of credible outside options which we now discuss.

For games that satisfy the conditions of case *ii*, there exists a pair of players  $\{i, j\}$  (*natural partners*) that are willing to form a pairwise coalition. According to proposition 2, the outcome of negotiations when *i* and *j* are natural partners is  $v_k = V_k$  and  $v_i = v_j = \frac{V_{ij}}{2}$  whenever case *ii* holds, which

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<sup>8</sup>It is also worth pointing out the relationship between the CBV and the core. It is straightforward that the core of a three-player superadditive characteristic function game is non-empty if and only if  $U_{12} + U_{13} + U_{23} \leq 2U$ . Therefore, we conclude that the Shapley value is the CBV of all games with an empty core (because whenever the core is empty the game belongs to region *iv*).

<sup>9</sup>See section 3 for a discussion of the equilibrium in region *i*.

one can easily see is equivalent to  $V_k \leq \frac{V}{3}$ ,  $V_{ik} \leq v_i + v_k$ , and  $V_{jk} \leq v_j + v_k$  (these inequalities can be verified by substituting expression (1) for  $V_i$  and  $V_{jk}$ ). Here is the intuition for the result. Note first that the proposed solution is consistent with coalition  $\{i, j\}$  being the only pairwise coalition forming; consider the alternative and suppose that the coalition  $\{i, k\}$  forms; then the payoff for the coalition is  $V_{ik}$  and the payoff of the player left out is  $V_j$ . But since  $V_{ik} \leq v_i + v_k$ , then the coalition  $\{i, k\}$  is worse off (with respect to the proposed equilibrium). The payoffs of the players  $i$  and  $j$ ,  $v_i = v_j = \frac{V_{jk}}{2}$ , are also consistent with the fact that only the pairwise coalition  $\{i, j\}$  may form: players  $i$  and  $j$  bargain over  $V_{ij}$  using as disagreement points their zero status quo values.

For games that satisfy the conditions of case *iii*, a *pivotal* player is included in all pairwise coalitions that are proposed, and the pairwise coalition between the non-pivotal players is never proposed. According to proposition 2, the outcome of negotiations when player  $i$  is pivotal is  $v_i = V - V_j - V_k$ ,  $v_j = V_j$ , and  $v_k = V_k$  whenever case *iii* holds, which one can easily see is equivalent to  $V_{jk} \leq V_j + V_k$ ,  $V_j \leq \frac{V_{ij}}{2}$ , and  $V_k \leq \frac{V_{ik}}{2}$ . The intuition for this result is that players  $j$  and  $k$  cannot demand a higher payoff than  $V_j$  and  $V_k$  from player  $i$  by threatening to form the coalition  $\{j, k\}$ , since they would be worse off pursuing this strategy ( $V_{jk} \leq V_j + V_k$ ). Also, note that players  $j$  and  $k$  are not willing to accept any offer lower than  $V_j$  and  $V_k$  because they can guarantee this amount by credibly holding out. This is so because if  $j$  holds out then  $i$  would successfully bargain with  $k$  to form a coalition;  $k$ 's gains are  $\frac{V_{ik}}{2} \geq V_k$ , and thus  $k$  does not want to hold out when  $j$  holds out.

## 5 Conclusion

This paper explicitly derives the solution of a standard model for three-player negotiations with externalities. The analysis shows how the players' values are inextricably related to the equilibrium sequencing of negotiations. A simple way to deal with externalities is developed: add to the worth of

a bilateral coalition the amount of negative externalities (or subtract the amount of positive externalities) that it creates for the excluded player. Players' equilibrium values are monotonically increasing or decreasing in the amount of negative or positive externalities they impose on others.

We show that the equilibrium value can be any of the following: the Nash bargaining solution, in the case where the value of all (adjusted) pairwise coalitions are less than a third of the grand coalition value; the Shapley value, in the case where the sum of the (adjusted) values created by all pairwise coalitions is greater than the grand coalition value; or the nucleolus, in the case where only the 'natural coalition' among two 'natural partners' creates significant value, and in the case where only the two pairwise coalitions including a 'pivotal player' create significant value.

We believe that the solution is economically intuitive and is an interesting candidate to be an off-the-shelf solution for applications, filling a gap in the literature. A natural (difficult) next step for future research is to derive close form solutions for games with an arbitrary number of players and externalities. Moreover, it would be interesting to establish links between the solution and existing or novel cooperative solution concepts (see for example Maskin (2003)). The results in this paper suggest that any plausible solution concept that applies to all games is likely to be a piecewise linear function, and it would be important to explicitly indicate the negotiation strategies associated with each region of linearity.

## Appendix

PROOF OF PROPOSITION 1: First note that cases  $i$ - $iv$  form a partition of the parameter space: the union of cases  $i$ ,  $ii$ , and  $iii$  is the half-space  $V_{12}+V_{13}+V_{23} \leq 2U$  (and so it follows that  $i$ ,  $ii$ , and  $iii$  is a partition of the half-space) and case  $iv$  corresponds to the complementary half-space  $V_{12} + V_{13} + V_{23} \geq 2U$ .

Denote limits when  $\delta$  converges to one without the superscript  $\delta$ :  $v_i = \lim_{\delta \rightarrow 1} v_i^\delta$ ,  $V_{ij} = \lim_{\delta \rightarrow 1} V_{ij}^\delta$ , and  $e'_S = \lim_{\delta \rightarrow 1} \frac{de_S^\delta}{d\delta}$ , etc. We have that  $\frac{dV_{ij}^\delta}{d\delta} + \frac{dV_k^\delta}{d\delta} = (U - W_{ij})$  where  $W_{ij} = U_{ij} + U_k$  and  $V_k^\delta + V_{ij}^\delta = \delta U + (1 - \delta) W_{ij}$ .

We consider below a partition of the parameter space into open regions defined by strict inequalities  $i$ ,  $ii.a$ - $b$ ,  $iii.a$ - $d$  (subdivisions of cases  $ii$  and  $iii$  in which the strategies are slightly different but the limit solutions turn out to be the same), and  $iv$ . Consider the analysis of each of the regions separately in the remainder of the proof, and assume momentarily that  $U > 0$  and  $U - U_{ij} + U_k > 0$  (strict superadditivity).

$$\begin{aligned} V_{12} &< (p_1 + p_2)U \\ V_{13} &< (p_1 + p_3)U \\ V_{23} &< (p_2 + p_3)U \end{aligned} \tag{i}$$

Whenever (i) holds let  $\sigma_i^\delta(N) = 1$  and  $v_i^\delta = \delta p_i U$  (so  $\mu_N = 1$ ). Eqs. (3) hold because  $v_i^\delta = \delta p_i e_N^\delta + \delta v_i^\delta$ ,  $e_N^\delta = U - \sum_{i \in N} v_i^\delta$  (note that  $v_i = p_i U$ ). Ineqs. (2) hold because  $e_N^\delta \geq e_S^\delta$  for all  $S \subset N$ : the excesses are equal to  $e_N^\delta = (1 - \delta)U$ ,  $e_{ij}^\delta = V_{ij}^\delta - \delta(p_i + p_j)U$ ,  $e_i^\delta = 0$  and the inequalities hold because  $V_{ij} < (p_i + p_j)U$  for all  $i, j \in N$ .

Consider separately subregions  $ii.a$  and  $ii.b$  of region  $ii$ .

$$\begin{aligned} V_{ij} &> (p_i + p_j)U \\ (p_i + p_j)V_{ik} + p_j V_{ij} &< (p_i + p_j)U \\ (p_i + p_j)V_{jk} + p_i V_{ij} &< (p_i + p_j)U \\ V_{ij} - (p_i + p_j)^2(U - W_{ij}) &> (p_i + p_j)U \end{aligned} \tag{ii.a}$$

(There are a total of three symmetric cases): Whenever  $(ii.a)$  holds let  $\sigma_i^\delta(\{i, j\}) = 1$ ,  $\sigma_j^\delta(\{i, j\}) = 1$  and  $\sigma_k^\delta(N) = 1$  (so  $\mu_N = p_k$  and  $\mu_{ij} = p_i + p_j$ ), and  $v^\delta$  be the solution of the system of linear eqs. (3):

$$\begin{aligned} v_i^\delta &= \delta p_i e_{ij}^\delta + \delta v_i^\delta \\ v_j^\delta &= \delta p_j e_{ij}^\delta + \delta v_j^\delta \\ v_k^\delta &= \delta p_k e_N^\delta + \delta((p_i + p_j)V_k^\delta + p_k v_k^\delta) \\ e_N^\delta &= U - \sum_{l \in N} v_l^\delta \\ e_{ij}^\delta &= V_{ij}^\delta - v_i^\delta - v_j^\delta \end{aligned}$$

Eqs. (3) have only one solution for all  $\delta$  (that can be obtained applying, for example, Cramer's rule) and this solution converges to  $v_i = \frac{p_i V_{ij}}{p_i + p_j}$  and  $v_j = \frac{p_j V_{ij}}{p_i + p_j}$ ,  $v_k = V_k = U - V_{ij}$ . Ineqs. (2) holds if there exists  $\bar{\delta} < 1$  such that for all  $\delta \in [\bar{\delta}, 1)$ ,  $e_N^\delta \geq \max\{e_{ik}^\delta, e_{jk}^\delta, 0\}$  and  $e_{ij}^\delta \geq \max\{e_N^\delta, e_{ik}^\delta, e_{jk}^\delta, 0\}$ : both excesses  $e_N^\delta$  and  $e_{ij}^\delta$  converge to zero when  $\delta$  converges to one. To show that  $e_{ij}^\delta \geq e_N^\delta$ , for all  $\delta \in [\bar{\delta}, 1)$ , when both excesses converge to the same value, we prove that  $e'_{ij} < e'_N$ . Applying L'Hôpital's rule to the expression  $v_i^\delta = \frac{e_{ij}^\delta}{p_i(1-\delta)}$ , derived from the first eq. in the system, yields  $e'_{ij} = -\frac{v_i}{p_i}$ , a result that will be used repeatedly in the remainder of the proof. From the solution to the system of equations,  $e_N^\delta = (1-\delta)(\delta(p_i+p_j)(U-W_{ij})+U)$ , and thus  $\frac{de_N^\delta}{d\delta} = -((p_i+p_j)(U-W_{ij})+U)$ . The inequality above holds because it is equivalent to  $\frac{V_{ij}}{p_i+p_j} > (p_i+p_j)(U-W_{ij})+U$ , (last inequality in *ii.a*). The condition  $e_N^\delta \geq e_{ik}^\delta$  also holds as long as  $\lim_{\delta \rightarrow 1} e_N^\delta = 0 > \lim_{\delta \rightarrow 1} e_{ik}^\delta$ , which corresponds to  $V_{ik} - \frac{p_i V_{ij}}{p_i+p_j} - V_k < 0 \Leftrightarrow (p_i+p_j)V_{ik} + p_j V_{ij} < (p_i+p_j)U$ . By symmetry,  $e_N^\delta \geq e_{jk}^\delta$  holds whenever,  $(p_i+p_j)V_{jk} + p_i V_{ij} < (p_i+p_j)U$ .

$$\begin{aligned}
V_{ij} &> (p_i+p_j)U \\
(p_i+p_j)V_{ik} + p_j V_{ij} &< (p_i+p_j)U \\
(p_i+p_j)V_{jk} + p_i V_{ij} &< (p_i+p_j)U \\
V_{ij} - (p_i+p_j)^2(U-W_{ij}) &< (p_i+p_j)U
\end{aligned} \tag{ii.b}$$

(There are a total of three symmetric cases) *ii.b* strategy: players  $i$  and  $j$  randomize over the choices  $\{i, j\}$  and  $N$ , and player  $k$  chooses  $N$  (let the associated transition probability be  $\mu^\delta$ ). The transition probability is such that  $\mu_{ij}^\delta \geq 0$ ,  $\mu_N^\delta \geq 0$ , and  $\mu_N^\delta + \mu_{ij}^\delta = 1$ , and, in addition,  $\mu_N^\delta \in [p_k, 1]$ , because player  $k$ 's only choice is  $N$ , and  $k$  proposes with probability  $p_k$ . Moreover, given any transition probability  $\mu^\delta$  satisfying the conditions above, we can always find a strategy profile  $\sigma^\delta$  with associated transition probability  $\mu^\delta$ . Let  $\mu^\delta$  and  $v^\delta$  be a solution of the (non-linear) system of eqs. (3):

$$\begin{aligned}
v_i^\delta &= \delta p_i e^\delta + \delta v_i^\delta \\
v_j^\delta &= \delta p_j e^\delta + \delta v_j^\delta \\
v_k^\delta &= \delta p_k e^\delta + \delta (\mu_{ij}^\delta V_k^\delta + \mu_N^\delta v_k^\delta) \\
e^\delta &= U - \sum_{l \in N} v_l^\delta \\
e^\delta &= V_{ij}^\delta - v_i^\delta - v_j^\delta \\
1 &= \mu_N^\delta + \mu_{ij}^\delta
\end{aligned}$$

The system of equations excluding the third and last eqs. and the variables  $\mu_{ij}^\delta$  and  $\mu_N^\delta$ , is linear and can be solved using Cramer's rule. The solution converges to  $v_i = \frac{p_i V_{ij}}{p_i+p_j}$ , and  $v_k = V_k = U - V_{ij}$ . After the expressions for  $v_k^\delta$  and  $e^\delta$  have

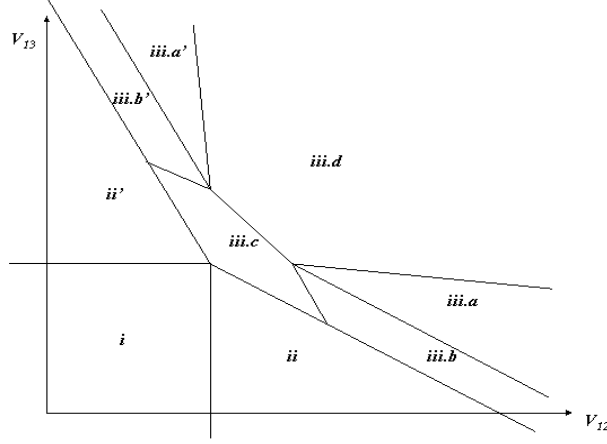


Figure 1: Case *iii* subcases.

been obtained we can solve for  $\mu_{ij}^\delta$  and  $\mu_N^\delta$  considering the third and last eqs. (The solution also converges to  $\mu_N = \frac{(p_i+p_j)(2U-W_{ij})-V_{ij}}{(p_i+p_j)(U-W_{ij})}$  and  $\mu_{ij} = \frac{V_{ij}-(p_i+p_j)U}{(p_i+p_j)(U-W_{ij})}$ ). The restriction  $\mu_N > p_k = (1-p_i-p_j)$  on the transition probability corresponds to  $V_{ij} < (p_i+p_j)U + (p_i+p_j)^2(U-W_{ij})$ , and the restriction  $\mu_{ij} > 0$ , corresponds to,  $V_{ij} > (p_i+p_j)U$ , and both holds. Moreover, ineqs. (2) hold: (observe that  $\lim_{\delta \rightarrow 1} e^\delta = 0$  and  $e^\delta \geq 0$  for  $\delta < 1$ )  $e^\delta \geq e_{ik}^\delta$  holds, for  $\delta$  close enough to one, because  $\lim_{\delta \rightarrow 1} e^\delta = 0 > \lim_{\delta \rightarrow 1} e_{ik}^\delta = V_{ik} - \frac{p_i V_{ij}}{p_i+p_j} - V_k < 0 \Leftrightarrow (p_i+p_j)V_{ik} + p_j V_{ij} < (p_i+p_j)U$ . Symmetrically,  $e^\delta \geq e_{jk}^\delta$  also holds, because  $\lim_{\delta \rightarrow 1} e_{jk}^\delta = V_{jk} - \frac{p_j V_{ij}}{p_i+p_j} - V_k < 0 \Leftrightarrow (p_i+p_j)V_{jk} + p_i V_{ij} < (p_i+p_j)U$ .

Consider the decomposition of case *iii* into four subcases *iii.a-iii.d*. Figure 1 illustrates each of the subcases (projected in the  $V_{12}$ - $V_{13}$  space). Note that, as figure 3 illustrates, all four subcases have a common intersection point.

$$\begin{aligned}
 (p_2 - p_3) V_{12} + (1 - 2p_3) V_{13} &< (1 - 2p_3) U + p_1 p_3 (-U + (1 - p_3) (W_{12} - W_{13})) \\
 (p_1 + p_2) V_{13} + p_2 V_{12} &> (p_1 + p_2) U + p_1 p_3 (U - W_{13}) & \text{(iii.a)} \\
 V_{12} + V_{13} + V_{23} &< 2U
 \end{aligned}$$

(there are a total of six symmetric cases corresponding to all permutations of the players). *iii.a* strategy: players 1 and 2 choose  $\{1, 2\}$  and player 3 chooses  $\{1, 3\}$ ,

and  $v^\delta$ 's are the (unique) solution of the system of linear eqs. (3)

$$\begin{aligned}
v_1^\delta &= \delta p_1 e_{12}^\delta + \delta v_1^\delta \\
v_2^\delta &= \delta p_2 e_{12}^\delta + \delta ((p_1 + p_2)v_2^\delta + p_3 V_2^\delta) \\
v_3^\delta &= \delta p_3 e_{13}^\delta + \delta ((p_1 + p_2)V_3^\delta + p_3 v_3^\delta) \\
e_{12}^\delta &= V_{12}^\delta - v_1^\delta - v_2^\delta \\
e_{13}^\delta &= V_{13}^\delta - v_1^\delta - v_3^\delta
\end{aligned}$$

The limit solution is  $v_1 = U - V_2 - V_3$ , and  $v_j = V_j$  for  $j = 2, 3$ . Ineqs. (2) are  $e_{12}^\delta \geq e_{13}^\delta$ ,  $e_{13}^\delta \geq e_N^\delta$ , and  $e_{13}^\delta \geq e_{23}^\delta$  for all  $\delta \in [\bar{\delta}, 1)$ : all the excesses  $e_{12}^\delta$ ,  $e_{13}^\delta$ , and  $e_N^\delta$  converge to zero as  $\delta$  converges to one. So we analyze the derivatives of the excesses evaluated at  $\delta = 1$  (see also case *ii.a*) which are equal to  $e'_{12} = -(p_1)^{-1}(V_{12} + V_{13} - U)$ ,  $e'_{13} = -(p_1 p_3)^{-1}((p_1 + p_2)(V_{13} - U) + p_2 V_{12}) + (-U + (1 - p_3)(W_{12} - W_{13}))$ ,  $e'_N = p_3 W_{13} + (1 - p_3)W_{12} - 2U$ . The ineq.  $e_{12}^\delta \geq e_{13}^\delta$  holds for  $\delta \in [\bar{\delta}, 1)$ , because  $p_1 + p_2 + p_3 = 1$  and first inequality in *iii.a*, imply  $e'_{13} - e'_{12} = -(p_1 p_3)^{-1} \begin{pmatrix} (p_2 - p_3)V_{12} + (1 - 2p_3)(V_{13} - U) \\ -p_1 p_3(-U + (1 - p_3)(W_{12} - W_{13})) \end{pmatrix} > 0$ , and the inequality  $e^\delta(13) \geq e_N^\delta$  holds for  $\delta \in [\bar{\delta}, 1)$ , because the inequality in *iii.a* imply  $e'_N - e'_{13} = (p_1 p_3)^{-1}((p_1 + p_2)(V_{13} - U) + p_2 V_{12} - p_1 p_3(U - W_{13})) > 0$ , and finally  $e_{13}^\delta \geq e_{23}^\delta$  holds because  $\lim_{\delta \rightarrow 1} e_{13}^\delta = 0 > \lim_{\delta \rightarrow 1} e_{23}^\delta$ :  $\lim_{\delta \rightarrow 1} e_{23}^\delta = V_{23} - v_2 - v_3 = V_{23} - (U - V_{13}) - (U - V_{12}) = V_{12} + V_{13} + V_{23} - 2U < 0$

$$\begin{aligned}
(p_1 + p_2)V_{13} + p_2 V_{12} &> (p_1 + p_2)U \\
(p_1 + p_2)V_{13} + p_2 V_{12} &< (p_1 + p_2)U + p_1 p_3(U - W_{13}) & \text{(iii.b)} \\
(p_1 + p_3)V_{12} + p_3 V_{13} &> (p_1 + p_3)U + p_1(p_1 + p_2)(U - W_{12}) \\
V_{12} + V_{13} + V_{23} &< 2U
\end{aligned}$$

(there are six symmetric cases corresponding to all permutations of the players). *iii.b* strategy: players 1 and 2 choose  $\{1, 2\}$ , and player 3 randomizes over the choice of  $\{1, 3\}$  or  $N$ . The transition prob. must satisfy  $\mu_N^\delta + \mu_{13}^\delta = p_3$ , because player 3 is the only player choosing  $\{1, 3\}$  and  $N$  (and 3 is proposer with prob.  $p_3$ ), and  $\mu_N^\delta \geq 0$  and  $\mu_{13}^\delta \geq 0$  (reciprocally, given any  $\mu^\delta$  satisfying the restrictions above, a strategy profile with transition prob. equal to  $\mu^\delta$  can be constructed). Let  $\mu^\delta$  and  $v^\delta$ 's be a solution of the (non-linear) system of eqs. (3),

$$\begin{aligned}
v_1^\delta &= \delta p_1 e_1^\delta + \delta v_1^\delta \\
v_2^\delta &= \delta p_2 e_2^\delta + \delta ((1 - \mu_{13}^\delta)v_2^\delta + \mu_{13}^\delta V_2^\delta) \\
v_3^\delta &= \delta p_3 e_2^\delta + \delta (p_3 v_3^\delta + (p_1 + p_2)V_3^\delta) \\
e_1^\delta &= V_{12}^\delta - v_1^\delta - v_2^\delta \\
e_2^\delta &= V_{13}^\delta - v_1^\delta - v_3^\delta \\
e_2^\delta &= U - v_1^\delta - v_2^\delta - v_3^\delta
\end{aligned}$$

The system of equations excluding the second and last eqs. and the variables  $\mu_{13}^\delta$  and  $\mu_N^\delta$ , is a linear system of equations that can be solved using Cramer's rule. The solutions converges to  $v_1 = U - V_2 - V_3$  and  $v_j = V_j$  for  $j = 2, 3$ . After the expressions for  $v_2^\delta$  and  $e_2^\delta$  have been obtained we can solve for  $\mu_{13}^\delta$  and  $\mu_N^\delta$  considering the third and last eqs. The solution converges to  $\mu_{13} = \frac{(p_1+p_2)V_{13}+p_2V_{12}-(p_1+p_2)U}{p_1(U-W_{13})}$ , and the restrictions that  $\mu_{13} \geq 0$  and  $\mu_N \geq 0$  corresponds to inequalities one and two in iii.b. Ineqs. (2)  $e_1^\delta \geq e_2^\delta$  and  $e_2^\delta \geq e_{23}^\delta$ , for all  $\delta \in [\bar{\delta}, 1)$ : Observe that  $\lim_{\delta \rightarrow 1} e_1^\delta = \lim_{\delta \rightarrow 1} e_2^\delta = 0$ , and so we analyze the derivatives of  $e_1^\delta$  and  $e_2^\delta$  at  $\delta = 1$ . Differentiating with respect to  $\delta$  the expressions for  $e_1^\delta$  and  $e_2^\delta$  obtained from the solutions of the system of equations above yield  $e_1' = -p_1^{-1}v_1 = -p_1^{-1}(V_{12} + V_{13} - U)$  and  $e_2' = -p_1^{-1}((p_1 + p_2)V_{13} + p_2V_{12} - p_2U + p_1(p_1 + p_2)(U - W_{12}))$ . The ineq.  $e_2' - e_1' = p_1^{-1}((p_1 + p_3)(V_{12} - U) + p_3V_{13} - p_1(p_1 + p_2)(U - W_{12})) > 0$  holds (because  $p_1 + p_2 + p_3 = 1$  and the third inequality in iii.b), which implies that  $e_1^\delta \geq e_2^\delta$  for all  $\delta \in [\bar{\delta}, 1)$  for some  $\bar{\delta} < 1$  close enough to one. As we have already argued (see iii.a),  $e_2^\delta \geq e_{23}^\delta$  holds whenever  $V_{12} + V_{13} + V_{23} < 2U$ .

$$\begin{aligned}
(p_1 + p_3)V_{12} + p_3V_{13} &> (p_1 + p_3)U \\
(p_1 + p_2)V_{13} + p_2V_{12} &> (p_1 + p_2)U \\
\frac{(p_1 + p_3)V_{12} + p_3V_{13} - (p_1 + p_3)U}{p_1(U - W_{12})} + \frac{(p_1 + p_2)V_{13} + p_2V_{12} - (p_1 + p_2)U}{p_1(U - W_{13})} &< 1 \\
(p_1 + p_3)V_{12} + p_3V_{13} &< (p_1 + p_3)U + p_1(p_1 + p_2)(U - W_{12}) & \text{(iii.c)} \\
(p_1 + p_2)V_{13} + p_2V_{12} &< (p_1 + p_2)U + p_1(p_1 + p_3)(U - W_{13}) \\
V_{12} + V_{13} + V_{23} &< 2U
\end{aligned}$$

(there are total of three such symmetric cases): all players randomize over the choices of  $\{1, 2\}$ ,  $\{1, 3\}$  and  $N$ . The transition probabilities  $\mu_{12}^\delta$  and  $\mu_{13}^\delta$  (and  $\mu_N^\delta = 1 - \mu_{13}^\delta - \mu_{12}^\delta$ ) must satisfy  $\mu_{13}^\delta \geq 0$ ,  $\mu_{12}^\delta \geq 0$  and  $\mu_N^\delta = 1 - \mu_{13}^\delta - \mu_{12}^\delta \geq 0$ , and in addition, because players 1 and 2 are the only players that can choose  $\{1, 2\}$ , the weight assigned to  $\mu_{12}$  must satisfy,  $\mu_{12}^\delta \leq p_1 + p_2$ . Also, the same considerations applies to  $\{1, 3\}$ , and thus  $\mu_{13}^\delta \leq p_1 + p_3$ . Let  $\mu^\delta$  and  $v^\delta$ 's be a solution of the non-linear system of eqs. (3):

$$\begin{aligned}
v_1^\delta &= \delta p_1 e^\delta + \delta v_1^\delta \\
v_2^\delta &= \delta p_2 e^\delta + \delta ((1 - \mu_{13}^\delta) v_2^\delta + \mu_{13}^\delta V_2^\delta) \\
v_3^\delta &= \delta p_3 e^\delta + \delta ((1 - \mu_{12}^\delta) v_3^\delta + \mu_{12}^\delta V_3^\delta) \\
e^\delta &= V_{12}^\delta - v_1^\delta - v_2^\delta \\
e^\delta &= V_{13}^\delta - v_1^\delta - v_3^\delta \\
e^\delta &= U - v_1^\delta - v_2^\delta - v_3^\delta \\
\mu_N^\delta &= 1 - \mu_{13}^\delta - \mu_{12}^\delta
\end{aligned}$$

The system of four equations (eqs. 1, 4, 5, and 6 above) and four variables ( $v_1^\delta$ ,  $v_2^\delta$ ,  $v_3^\delta$ , and  $e^\delta$ ) is a linear system that has a unique solution that converges to

$v_1 = U - V_2 - V_3$ ,  $v_j = V_j$  for  $j = 2, 3$ . The solution for  $\mu_{12}^\delta$  and  $\mu_{13}^\delta$  (and  $\mu_N^\delta = 1 - \mu_{13}^\delta - \mu_{12}^\delta$ ) can be directly obtained from eqs. 2 and 3 above and converge to,  $\mu_{12} = \frac{(p_1+p_3)V_{12}+p_3V_{13}-(p_1+p_3)U}{p_1(U-W_{12})}$  and  $\mu_{13} = \frac{(p_1+p_2)V_{13}+p_2V_{12}-(p_1+p_2)U}{p_1(U-W_{13})}$ . Note that the conditions  $\mu_{13} > 0$ ,  $\mu_{12} > 0$  and  $1 - \mu_{13} - \mu_{12} > 0$  correspond to the first three inequalities in *iii.c*. Moreover the restrictions  $\mu_{12} < p_1 + p_2$  and  $\mu_{13} < p_1 + p_3$ , correspond respectively to the fourth and fifth inequalities in *iii.c*. The last inequality guarantees that  $e_2^\delta \geq e_{23}^\delta$ .

$$\begin{aligned} (p_2 - p_3) V_{12} + (1 - 2p_3) V_{13} &> (1 - 2p_3) U + p_3 p_1 (-U + (1 - p_3) (W_{12} - W_{13})) \\ (p_3 - p_2) V_{13} + (1 - 2p_2) V_{12} &> (1 - 2p_2) U + p_2 p_1 (-U + (1 - p_2) (W_{13} - W_{12})) \\ \frac{(p_1 + p_2) V_{13} + p_2 V_{12} - (p_1 + p_2) U}{p_1 (U - W_{13})} + \frac{(p_1 + p_3) V_{12} + p_3 V_{13} - (p_1 + p_3) U}{p_1 (U - W_{12})} &> 1 \\ V_{12} + V_{13} + V_{23} &< 2U \end{aligned} \quad (\text{iii.d})$$

(there are total of three such symmetric cases): player 1 randomizes over the choices  $\{1, 2\}$  and  $\{1, 3\}$ , and players 2 and 3 choose  $\{1, 2\}$  and  $\{1, 3\}$ , respectively. The transition probabilities associated with the strategy profile are such that  $\mu_{1j}^\delta \geq p_j$  for  $j = 2, 3$  because player  $j$ 's only choice is coalition  $\{1, j\}$  and player  $j$  proposes with probability  $p_j$ . Consider the non-linear system of eqs. (3):

$$\begin{aligned} v_1^\delta &= \delta p_1 e^\delta + \delta v_1^\delta \\ v_2^\delta &= \delta p_2 e^\delta + \delta (\mu^\delta v_2^\delta + (1 - \mu^\delta) V_2^\delta) \\ v_3^\delta &= \delta p_3 e^\delta + \delta (\mu^\delta V_3^\delta + (1 - \mu^\delta) v_3^\delta) \\ e^\delta &= V_{12}^\delta - v_1^\delta - v_2^\delta \\ e^\delta &= V_{13}^\delta - v_1^\delta - v_3^\delta \end{aligned}$$

(where  $\mu^\delta = \mu_{12}^\delta$  and  $\mu_{13}^\delta = 1 - \mu^\delta$ ). We first prove that there exist a solution of the system with  $\mu^\delta \in (p_2, 1 - p_3)$  for all  $\delta \in [\bar{\delta}, 1)$ , for some  $\bar{\delta} < 1$ . The first step is to solve the system of (linear) eqs. composed of eqs. 1, 2, 4, and 5 for the variables  $v_1^\delta$ ,  $v_2^\delta$ ,  $v_3^\delta$ , and  $e^\delta$  as a function of  $\mu^\delta$ . Now, replacing the expressions for  $v_3^\delta$  and  $e^\delta$  into eq. 3, yields a quadratic equation in  $\mu$ ,  $q^\delta(\mu) = 0$ . The quadratic expression  $q^\delta(\mu)$  evaluated at  $\delta = 1$  yields,  $q(\mu) = p_1 (W_{12} - W_{13}) \mu^2 + (V_{13} + V_{12} - U - p_1 (U + W_{12} - W_{13})) \mu - ((p_1 + p_3) (V_{12} - U) + p_3 V_{13})$ . Developing the expressions for  $q(p_2)$  and  $q(1 - p_3)$  yields  $q(p_2) = -(p_3 - p_2) V_{13} - (1 - 2p_2) (V_{12} - U) + p_2 p_1 (-U + (1 - p_2) (W_{13} - W_{12})) < 0$ , and  $q(1 - p_3) = (p_2 - p_3) V_{12} + (1 - 2p_3) (V_{13} - U) - p_3 p_1 (-U + (1 - p_3) (W_{12} - W_{13})) > 0$ . Therefore, by the continuity of  $q(\mu)$  with respect to  $\mu$ , the quadratic equation  $q(\mu) = 0$  has one solution in the interval  $(p_2, 1 - p_3)$ . Moreover, by continuity with respect to  $\delta$ , there exists  $\bar{\delta} < 1$ , such that all equations  $q^\delta(\mu) = 0$  (for all  $\delta \in [\bar{\delta}, 1)$ ) also have one solution  $\mu^\delta \in (p_2, 1 - p_3)$ . The solutions  $v^\delta$  and  $e^\delta$  obtained are such that  $\lim_{\delta \rightarrow 1} e^\delta = 0$  (this can be obtained directly from the first equation), and  $v_j = \lim_{\delta \rightarrow 1} v_j^\delta = V_j$  for  $j = 2, 3$ , and  $v_1 = \lim_{\delta \rightarrow 1} v_1^\delta = U - V_2 - V_3$ , (the second equation, in the limit, is  $v_2 = \mu_{12} v_2 + \mu_{13} V_2$ , which combined with  $\mu_{12} + \mu_{13} = 1$  and the third equation yields the expressions above for the limit equilibrium payoffs).

Ineqs. (2):  $e^\delta \geq e_N^\delta$ , for all  $\delta \in [\bar{\delta}, 1)$ , where  $e_N^\delta = U - \sum_{i=1}^3 v_i^\delta$ . Because  $\lim_{\delta \rightarrow 1} e^\delta = \lim_{\delta \rightarrow 1} e_N^\delta = 0$ , it is sufficient to show that  $e' < e_N'$ . Adding up the first three eqs. yields  $\sum_{i=1}^3 v_i^\delta = \delta \mu_{12}^\delta (V_{12}^\delta + V_3^\delta) + \delta \mu_{13}^\delta (V_{13}^\delta + V_2^\delta)$ , and the derivative with respect to  $\delta$  is  $\sum_{i=1}^3 v_i' = 2U - \mu_{12} W_{12} - \mu_{13} W_{13}$ , which implies that  $g \triangleq e_N' - e' = \frac{v_1}{p_1} - 2U + \mu_{12} W_{12} + \mu_{13} W_{13}$ . Differentiating the second eq. in the system with respect to  $\delta$  yields (at  $\delta = 1$ ),  $-p_2 \frac{v_1}{p_1} = \mu_{13} (v_2' - V_2')$  –  $V_2$ . But note that  $\frac{dv_2^\delta}{d\delta} = \sum_{i=1}^3 \frac{dv_i^\delta}{d\delta} - \left( \frac{dv_1^\delta}{d\delta} + \frac{dv_3^\delta}{d\delta} \right)$  and from the fifth equation in the system,  $v_2' - V_2' = -(g + (U - W_{13}))$ . Therefore,  $\mu_{13} = \frac{\frac{p_2 v_1}{p_1} - V_2}{g + (U - W_{13})} = \frac{(p_1 + p_2) V_{13} + p_2 V_{12} - (p_1 + p_2) U}{p_1 g + p_1 (U - W_{13})}$  and a symmetric equation holds for  $\mu_{12}$ . But  $g$  is the solution of  $\frac{(p_1 + p_2) V_{13} + p_2 V_{12} - (p_1 + p_2) U}{g + p_1 (U - W_{13})} + \frac{(p_1 + p_3) V_{12} + p_3 V_{13} - (p_1 + p_3) U}{g + p_1 (U - W_{12})} = 1$ , because  $\mu_{13} + \mu_{12} = 1$ , thus  $g > 0$  (because of the third inequality in *iii.d*) which shows  $e^\delta \geq e_N^\delta$  and concludes this case.

$$V_{12} + V_{13} + V_{23} > 2U \quad (\text{iv})$$

Strategy: all players randomize over pairwise coalitions  $\{i, j\}$  (including the proposers). The associated transition probabilities satisfy  $\mu_{12}^\delta + \mu_{13}^\delta + \mu_{23}^\delta = 1$ , and moreover, the transition probabilities must satisfy the inequalities  $\mu_{12}^\delta + \mu_{13}^\delta \geq p_1$  (because 1's only choices are  $\{1, 2\}$  and  $\{1, 3\}$ , and 1 is proposer with probability  $p_1$ ) and, similarly,  $\mu_{12}^\delta + \mu_{23}^\delta \geq p_2$  and  $\mu_{13}^\delta + \mu_{23}^\delta \geq p_3$ . Note also that given any transition probability  $\mu$  satisfying the conditions above, we can always find a strategy profile  $\sigma$  with associated transition probability  $\mu^\delta$ . Let  $\mu^\delta$  and  $v^\delta$  be a solution of the system of non-linear eqs. (3):

$$\begin{aligned} v_1^\delta &= \delta p_1 e^\delta + \delta ((\mu_{12}^\delta + \mu_{13}^\delta) v_1^\delta + \mu_{23}^\delta V_1^\delta) \\ v_2^\delta &= \delta p_2 e^\delta + \delta ((\mu_{12}^\delta + \mu_{23}^\delta) v_2^\delta + \mu_{13}^\delta V_2^\delta) \\ v_3^\delta &= \delta p_3 e^\delta + \delta ((\mu_{13}^\delta + \mu_{23}^\delta) v_3^\delta + \mu_{12}^\delta V_3^\delta) \\ e^\delta &= V_{12}^\delta - v_1^\delta - v_2^\delta \\ e^\delta &= V_{13}^\delta - v_1^\delta - v_3^\delta \\ e^\delta &= V_{23}^\delta - v_2^\delta - v_3^\delta \\ 1 &= \mu_{12}^\delta + \mu_{13}^\delta + \mu_{23}^\delta \end{aligned}$$

Substituting  $v_i = \frac{1}{3}(U + V_{ij} + V_{ik} - 2V_{jk})$ ,  $\mu_{ij} = p_k$ , and  $e = \frac{1}{3}(V_{12} + V_{13} + V_{23} - 2U)$  into eqs. show that it is a solution for  $\delta = 1$ . By the implicit function theorem (IFT) a solution of the system for all  $\delta \in [\bar{\delta}, 1)$ , for some  $\bar{\delta} < 1$ , is also guaranteed because the Jacobian evaluated at the solution point and  $\delta = 1$  (where the Jacobian is the natural one associated the system of the equations) is a non-singular matrix. Thus the problem of finding solutions for  $\delta$  in a neighborhood of  $\delta = 1$  satisfies all conditions of the IFT.<sup>10</sup> Moreover, for  $\bar{\delta}$  close enough to one,

<sup>10</sup>Note that the Jacobian associated with all other cases considered before, evaluated at  $\delta = 1$ , are singular, and thus we cannot apply the IFT to the previous cases.

the solution also satisfies the inequalities such as  $\mu_{12}^\delta + \mu_{13}^\delta > p_1$  (the inequality  $\mu_{12} + \mu_{13} = p_3 + p_2 > p_1$  is strict because  $p_1 < \frac{1}{2}$  and  $p_1 + p_2 + p_3 = 1$ ) and  $e^\delta > 0$  (because  $e^1 = \frac{1}{3}(V_{12} + V_{13} + V_{23} - 2U) > 0$  is strict). Ineq. (2) is  $e^\delta \geq e_N^\delta$ . Since  $\lim_{\delta \rightarrow 1} e_N^\delta = 0$  and  $\lim_{\delta \rightarrow 1} e^\delta > 0$  we can guarantee that there exists  $\bar{\delta} < 1$  such that  $e^\delta \geq e_N^\delta$  for all  $\delta \in [\bar{\delta}, 1)$ .

Consider now any game in the frontier of any of the regions we considered above (i.e., assume that some of the strict inequalities are binding). Note that such game can be approximated by a sequence interior games. Because the results holds for all games in the interior, and the MPE correspondence is an upper hemi-continuous correspondence of the parameters of the game, it implies that the results also hold for all games in the frontier. Q.E.D.

**PROOF OF PROPOSITION 3:** Let  $v$  and  $\eta$  denote the CBV and the nucleolus, respectively. According to Brune (1983), the nucleolus of a three-person 0-normalized superadditive game satisfying  $U_{12} \geq U_{13} \geq U_{23}$ :

If  $U_{12} \leq \frac{U}{3}$  then  $\eta = (\frac{U}{3}, \frac{U}{3}, \frac{U}{3})$ ,

If  $U_{12} \geq \frac{U}{3}$  and  $U_{12} + 2U_{13} \leq U$  then  $\eta = (\frac{U+U_{12}}{4}, \frac{U+U_{12}}{4}, \frac{U-U_{12}}{2})$ ,

If  $U_{12} + 2U_{23} \leq U$  and  $U_{12} + 2U_{13} \geq U$  then  $\eta = (\frac{U_{12}+U_{13}}{2}, \frac{U-U_{13}}{2}, \frac{U-U_{12}}{2})$ ,

If  $2(U_{13}+U_{23})-U_{12} \geq U$  then  $\eta = (\frac{U+U_{12}+U_{13}-2U_{23}}{3}, \frac{U+U_{12}+U_{23}-2U_{13}}{3}, \frac{U+U_{13}+U_{23}-2U_{12}}{3})$ ,

If  $U_{12}+2U_{23} \geq U$ ,  $2(U_{13}+U_{23})-U_{12} \leq U$  then  $\eta = (\frac{U+2U_{13}+U_{12}-2U_{23}}{4}, \frac{U+2U_{23}+U_{12}-2U_{13}}{4}, \frac{U-U_{12}}{2})$ .

We have argued before that the union of cases *i*, *ii*, and *iii* is equal to the half-space  $V_{12} + V_{13} + V_{23} \leq 2U$ , which, after using expression (1), is equal to  $U_{12} + U_{13} + U_{23} \leq U$ .

First note that the CBV and the nucleolus coincide for games in regions *i-iii*: if case *i* holds then obviously  $v = \eta$ ; if case *ii* holds, which is equivalent to  $U_{12} \geq \frac{U}{3}$ ,  $2U_{13} + U_{12} \leq U$ , and  $2U_{23} + U_{12} \leq U$ , then  $v = (\frac{U+U_{12}}{4}, \frac{U+U_{12}}{4}, \frac{U-U_{12}}{2})$ . Note that the inequalities imply that  $U_{12} \geq U_{13}$  and  $U_{12} \geq U_{23}$ . Now if  $U_{13} \geq U_{23}$  then  $v = \eta$ , and similarly, by the symmetry in the nucleolus formula, if  $U_{23} \geq U_{13}$  then also  $v = \eta$ ; if case *iii* holds, which corresponds to (a)  $2U_{13} + U_{12} \geq U$ , (b)  $U_{12} + U_{13} + U_{23} \leq U$ , and  $U_{12} + U_{13} \geq U$  then  $v = (\frac{U_{12}+U_{13}}{2}, \frac{U-U_{13}}{2}, \frac{U-U_{12}}{2})$ . Note that the inequalities imply that  $U_{12} \geq U_{13}$  and  $U_{12} \geq U_{23}$ . Now suppose that  $U_{13} \geq U_{23}$ . Combining inequalities *a* and *b* above (more precisely consider (a)+2(b)  $\geq 0$ ) yields  $U_{12} + 2U_{23} \leq U$ , which implies that  $v = \eta$ . A similar argument applies to the symmetric case where  $U_{23} \geq U_{13}$ . Thus the CBV coincides with the nucleolus whenever  $U_{12} + U_{13} + U_{23} \leq U$ .

Now, if condition  $U_{12} + U_{13} + U_{23} \geq U$  holds, which is equivalent to  $V_{12} + V_{13} + V_{23} \geq 2U$ , then the CBV is equal to  $v = \frac{1}{6}(2(U - U_{jk}) + U_{ij} + U_{ik})$ , which is equal to the Shapley value. Q.E.D.

## References

- [1] BINMORE, K. (1985), "Bargaining and Coalitions," in Roth, A. (ed.) *Game-theoretic models of bargaining*, pp. 269-304. Cambridge University Press.
- [2] BLOCH, F. (1996), "Sequential Formation of Coalitions in Games with Externalities and Fixed Payoff Division," *Games and Economic Behavior*, 14, 90-123.
- [3] BRUNE, S. (1983), "On the Regions of Linearity for the Nucleolus and their Computation," *International Journal of Game Theory*, 12, 47-80.
- [4] CHATTERJEE, K., DUTTA, B., RAY, D., AND SENGUPTA, K. (1993), "A Noncooperative Theory of Coalitional Bargaining," *Review of Economic Studies*, 60, 463-477.
- [5] CORNET, M. (2003), "Partition Function Bargaining with Public Demands," *Economic Theory*, 22, 845-862.
- [6] CROSON, R., GOMES, A., MCGINN, K., AND NOTH, M. (2003), "Mergers and Acquisitions: An Experimental Analysis of Synergies, Externalities and Dynamics," Harvard Business School working paper, No. 02-101.
- [7] GOMES, A. (2001), "Multilateral Negotiations and Formation of Coalitions," Rodney White Center for Financial Research, WP #12-01.
- [8] GUL, F. (1989), "Bargaining Foundations of Shapley Value," *Econometrica*, 57, 81-95.
- [9] HART, S., AND MAS-COLELL, A. (1996), "Bargaining and Value," *Econometrica*, 64, 357-380.
- [10] JEHIEL, P., AND MOLDOVANU, B. (1995a), "Cyclical Delay in Bargaining with Externalities," *Review of Economic Studies*, 62, 619-637.
- [11] JEHIEL, P., AND MOLDOVANU, B. (1995b), "Negative Externalities May Cause Delay in Negotiation", *Econometrica* 63(6), 1321-1337.
- [12] KRISHNA, V., AND SERRANO, R. (1996), "Multilateral Bargaining," *Review of Economic Studies*, 63, 61-80.
- [13] KOHLBERG, E. (1971), "On the Nucleolus of a Characteristic Function Game," *SIAM Journal of Applied Mathematics*, 20, 62-66.

- [14] KOVENOCK, D., AND WIDDOWS, K. (1989), “The Sequencing of Union Contract Negotiations,” *Managerial and Decision Economics*, 10-4, 283-289.
- [15] LUCAS, W., AND THRALL, R. (1963), “ $n$ -Person games in Partition Function Form,” *Naval Research Logistics Quarterly*, 10, 281-298.
- [16] MASKIN, E. (2003), “Bargaining, Coalitions, and Externalities,” mimeo, Institute for Advanced Studies.
- [17] MOLDOVANU, B. (1992), “Coalition-Proof Nash Equilibria and the Core in Three-Player Games,” *Games and Economic Behavior*, 4, 565-581.
- [18] MOLDOVANU, B., AND WINTER, E. (1995), “Order Independent Equilibria,” *Games and Economic Behavior* 9, 1995, 21-35.
- [19] MONTERO, M. (1999), “Coalition Formation in Games with Externalities,” *Center Discussion Paper*, 99121.
- [20] OKADA, A. (1996), “A Noncooperative Coalitional Bargaining Game with Random Proposers,” *Games and Economic Behavior*, 16, 97-108.
- [21] OSBORNE, M., AND RUBINSTEIN, A. (1990), *Bargaining and Markets*. Academic Press.
- [22] RAY, D., AND VOHRA, R. (1999), “A Theory of Endogenous Coalition Structures,” *Games and Economic Behavior*, 26, 268-336.
- [23] RUBINSTEIN, A. (1982), “Perfect Equilibrium in a Bargaining Model,” *Econometrica*, 50, 97-109.
- [24] SEIDMANN, D., AND WINTER, E. (1998), “A Theory of Gradual Coalition Formation,” *Review of Economic Studies*, 65, 793-815.
- [25] SELTEN, R. (1981), “A Non-Cooperative Model of Characteristic Function Bargaining,” in *Essays in Game Theory and Mathematical Economics*, ed. by V. Bohn and Nachtkamp.
- [26] SERRANO, R. (1993), “Non-Cooperative Implementation of the Nucleolus: The 3-Player Case,” *International Journal of Game Theory*, 22, 345-357.

- [27] SHAPLEY, L. (1953), "A Value for n-Person Games," in *Contributions to the Theory of Games II*, ed. by H.W. Kuhn and A.W. Tucker. Princeton University Press, 307-317.
- [28] SUTTON, J. (1986), "Non-Cooperative Bargaining Theory: An Introduction," *Review of Economic Studies*, 53, 709-724.
- [29] SCHMEIDLER, D. (1969), "The Nucleolus of a Characteristic Function game," *SIAM Journal of Applied Mathematics*, 17, 1163-1170.
- [30] YI, S-S. (1997), "Stable Coalition Structures with Externalities," *Games and Economic Behavior*, 20, 201-237.