Lifetime Consumption and Investment: Retirement and Constrained Borrowing*

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Extended Abstract

Saving for retirement is a primary end purpose of many parts of the financial sector, including pension plans, life insurance, and indeed much of retail banking and brokerage. This paper and a companion paper develop a group of workhorse models that can be adapted to answer many questions of personal finance. This paper focuses on retirement (modeled as irreversible) and constrained borrowing. This paper has three simple models that can be solved explicitly, at least parametrically up to the determination of a few parameters. All three models have realistic features such as a hazard rate of mortality, preference for working or not, a possible bequest motive, and wage income that can vary stochastically over time. A companion piece has models with time-dependent features, permitting the hazard rate of mortality, average wage level, and preference for working to be exogenous functions of time.

Consumption and portfolio choice can both jump at the endogenous retirement date. Consumption can jump because preferences are different after retirement. This could be due to household production (time to cook instead of buying more expensive prepared food), reduced work-related expenses (for clothes or commuting), or just different preference for consumption when more leisure is available. Portfolio choice jumps at retirement because there is significant hedging of human capital just before retirement. When just below the critical wealth for retirement, the agent knows that increase in wealth implies human capital goes to zero (or a small post-retirement level) but a decrease in wealth implies human capital may be much greater because of a possibly long excursion through the no-retirement region. Hence, human capital has substantial negative dependence on the market return and the portfolio holding in the market is higher just before retirement than just after retirement because of the disappearing hedging demand.

Portfolio choice and consumption depend significantly on whether the agent is free to choose when to retire and on whether it is possible to borrow against labor income. When the agent has free choice of when to retire, the agent chooses to work longer in expensive states in which market returns are low, self-insuring an aggressive investment position. The effectiveness of this strategy is reduced by a nonnegative wealth constraint (which prevents borrowing against future income), since the constraint prevents significant transfer of wealth from labor in expensive market states to cheaper states (since the transfer would require borrowing in the expensive states). The effectiveness of this strategy is also reduced for agents whose working wage is positively related to market returns, since working longer in expensive states will also be at a lower wage.

In general, human capital is a less and less important fraction of total wealth over time, and the fraction of financial wealth in risky assets will vary to counteract the risk exposure in human capital. If the wage is uncorrelated with the market, human capital is underexposed to market risk (and indeed will have negative exposure to the market once we include the impact of varying the duration of employment as the market goes up and down), and the risky asset position will be larger when young than when old. If the wage depends positively enough with the market, then human capital already has more market risk than is optimal and the risky asset position starts negative and increases over time. This second situation contradicts brokers' traditional advice that young investors should be aggressive and older investors should be conservative.

These results only scratch the surface of possible applications of the general class of models. The hope is that this tractable life-cycle model of consumption and investment with realistic mortality and time-dependent preferences will provide a workhorse model for analyzing problems in pensions, life-cycle consumption, and life insurance.

I. Introduction

Retirement is one of the most important economic events in a worker's life. Not surprisingly, retirement is connected to a number of important personal decisions such as consumption and investment and also to policy issues such as those on insurance and pensions, as well as mandatory versus voluntary retirement.[1] In this paper, we extend recent advances in finance to build a tractable optimal consumption and investment model with voluntary or mandatory retirement, and with or without a non-negative wealth constraint (which prevents borrowing against future wages). This paper solves three models for which more or less complete solutions are available, either in the primal or the dual, up to determination of a constant in one case. A companion piece studies explicit dependence of the mortality rate, wage, and preference for working on the stage of life. It is hoped that these models and extensions will be useful for studying policy questions in insurance and retirement.

We consider three models in our analysis. The three models vary in the treatment of retirement and borrowing against future labor income. All three models share a number of common features: a constant hazard rate of mortality, different preferences for consumption before and after retirement, possibly stochastic labor income, bequest, and actuarially fair life insurance. Keeping these features the same makes it easy to perform a parallel comparison of the three models. All three models consider retirement to be irreversible, emphasizing that a worker may be much more valuable to a firm working full-time than when working part-time. This is an extreme alternative to models with a continuously variable labor-leisure choice, as in Liu and Neis [2002]. We have also looked at models with both types of choice, allowing the possibility of a return to part-time work at a lower wage after retirement, but not in this paper.

The first model is a benchmark case with a fixed retirement date, which we interpret as mandatory retirement.[2] Our first model is a close relative of the Merton model with i.i.d. returns and constant relative risk aversion, and it can be solved explicitly.

The second model has voluntary retirement in a model in which the agent is free to borrow against future labor income. The second model is solved explicitly in the dual (as a function of the dual variable which is the marginal utility of wealth in the value function). This is a explicit parametric solution of the original problem which means that we know everything about the solution once we have conducted a one-dimensional numerical search for the value of the dual variable that corresponds with the current wealth level. Before retirement, there is a critical wealth-to-wage ratio at which it is optimal to retire. The expected time to retirement depends a lot on what the wage is today: the agent self-insures against risk in the security market by working more when the security market returns are poor. By working longer in expensive states, the agent generates more income (some of which is transferred to other states) for the average time worked than if the worker worked for the same amount of time in each state.

[1] In the UK, mandatory retirement is still widespread, and this is still an active policy issue (see Meadows [2003]). In the US, the Age Discrimination in Employment Act of 1967 (ADEA) generally prohibits mandatory retirement. One exception is for a qualifying "bona fide executive" or person in a "high policymaking position" who can face mandatory retirement at an age of 65 or above. There used to be an exception for tenured academics who could face mandatory retirement at an age of 70 or above, but that exception expired on January 1, 1994.

[2] A more realistic model of mandatory retirement is given by Panageas and Farhi [2003], who permit retirement at or before mandatory retirement date. Our simpler assumption is better for our benchmarking because we can solve it model exactly and it is easier to compare with the other models.

The third model has voluntary retirement in a model in which the agent cannot borrow against future labor income. This restriction reduces the usefulness of investing in stocks because any significant negative return would wipe out the financial wealth and bring the agent against the borrowing constraint. The borrowing constraint prevents the agent from transferring income across states to the extent that would be optimal and reduces the attractiveness of working longer in expensive states of nature.

The paper contains technical innovations that permit solution up to determination of a few parameters. In particular, we combine the dual approach of He and Pagès (1993) with an analysis of the boundary to obtain a problem we can solve in parametric form even if no known solution exists in the primal problem.

Consumption and portfolio choice can both jump at the endogenous retirement date. Consumption can jump because preferences are different after retirement. This could be due to household production (time to cook instead of buying more expensive prepared food), reduced work-related expenses (for clothes or commuting), or just different preference for consumption when more leisure is available. Portfolio choice jumps at retirement because there is significant hedging of human capital just before retirement. When just below the critical wealth for retirement, the agent knows that increase in wealth implies human capital goes to zero (or a small post-retirement level) but a decrease in wealth implies human capital may be much greater because of a possibly long excursion through the no-retirement region. Hence, human capital has substantial negative dependence on the market return and the portfolio holding in the market is higher just before retirement than just after retirement because of the disappearing hedging demand.

Portfolio choice and consumption depend significantly on whether the agent is free to choose when to retire and on whether it is possible to borrow against labor income. When the agent has free choice of when to retire, the agent chooses to work longer in expensive states in which market returns are low, self-insuring an aggressive investment position. The effectiveness of this strategy is reduced by a nonnegative wealth constraint (which prevents borrowing against future income), since the constraint prevents significant transfer of wealth from labor in expensive market states to cheaper states (since the transfer would require borrowing in the expensive states). The effectiveness of this strategy is also reduced for agents whose working wage is positively related to market returns, since working longer in expensive states will also be at a lower wage.

In general, human capital is a less and less important fraction of total wealth over time, and the fraction of financial wealth in risky assets will vary to counteract the risk exposure in human capital. If the wage is uncorrelated with the market, human capital is underexposed to market risk (and indeed will have negative exposure to the market once we include the impact of varying the duration of employment as the market goes up and down), and the risky asset position will be larger when young than when old. If the wage depends positively enough with the market, then human capital already has more market risk than is optimal and the risky asset position starts negative and increases over time. This second situation contradicts brokers' traditional advice that young investors should be aggressive and older investors should be conservative.

Liu and Neis (2002) consider the optimal consumption and investment problem with endogenous working hours. In contrast to our model, they allow an investor to borrow against future labor income. In addition, they assume that the stock price can never fall below a fixed positive level. Bodie, Merton, and Samuelson (1992) consider the effect of labor choice on optimal investment policy and Basak (1999) develops a continuous-time general equilibrium model to adapt dynamic asset pricing theory to include labor income. Similar to Liu and Neis (2002), both of these papers assume that working hours are infinitely divisible. Sundaresan and Zapatero (1997) examine how pension plans affect the retirement policies with an emphasis on the valuation of pension obligations. They abstract from modelling the disutility of working and the investor's investment opportunities outside the pension.

The rest of the paper is organized as follows. Section II presents the formal choice problems used in most of the paper. Section III presents graphically the solution presented in Section IV as well as a numerical solution of case with locally unspanned labor income described in Section V. Section VI closes the paper. All of the proofs are in the Appendix.

II. Choice Problems

Our general goal is to provide a tractable workhorse model that can be used to analyze various issues related to life cycle consumption and investment, retirement, and insurance. In this paper, we focus on stationary models that can be solved more or less explicitly. A companion piece looks at models with life stages having potentially different hazard rates of mortality, disutilities of working, incidence of sickness, and pure rate of time discount. This section poses the formal decision problems for most of the stationary models considered in this paper.

The choice problems make many of the assumptions that are common in continuous-time financial models, for example the constant riskfree rate and lognormal stock returns. Other assumptions are not standard but seem particularly appropriate for analysis of life-cycle consumption and investment. For example, our model includes mortality and bequest as well as preference for not working. There is labor income with potentially stochastic wage.

All the models in this paper consider pure retirement without flexible hours, return to full-time work in retirement, or part-time work in retirement. There is no reason why these other features cannot be added to the model, but we choose to focus instead on the essential nonconvexity that says half-time work is much less valuable than full-time work in some positions. We do not have anything against more general models, but there is a limit of what can be included in one paper.

In our main analysis we consider the following three cases:

benchmark fixed retirement date and free borrowing against wages (Problem 1 and Theorem 1).

NBC ("No Borrowing Constraint") free choice of retirement date and free borrowing against wages (Problem 2 and Theorem 2).

BC ("Borrowing Constraint") free choice of retirement date but no borrowing against wages (Problem 3 and Theorem 3).

The benchmark case is a close relative of the Merton model with i.i.d. returns and constant relative risk aversion. Moving to the NBC case isolates the impact of making retirement flexible. Subsequently moving to the BC case isolates the impact of the borrowing constraint.

Here are the three choice problems corresponding to the above three cases.

Problem 1 (benchmark) *Given initial wealth W^0, initial income from working y^0, and time-to-retirement T with associated retirement indicator $R_t = \iota(T \leq t)$ for some fixed time-to-retirement T,* *choose adapted nonnegative consumption $\{c_t\}$, adapted portfolio $\{\theta_t\}$, and adapted nonnegative bequest $\{B_t\}$, to maximize expected utility of lifetime consumption and bequest*

$$E\left[\int_{t=0}^{\infty} e^{-(\rho+\delta)t}\left((1-R_t)\frac{c_t^{1-\gamma}}{1-\gamma}+R_t\frac{(Kc_t)^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_t)^{1-\gamma}}{1-\gamma}\right)dt\right] \tag{1}$$

subject to the budget constraint

$$\begin{aligned} W_t &= W^0+\int_{s=0}^{t}(rW_s ds+\theta_s^{\top}((\mu-r\mathbf{1})ds+\sigma dZ_s)+\delta(W_s-B_s)ds-c_s ds \\ &\quad +(1-R_s)y_s ds), \end{aligned} \tag{2}$$

the labor income process

$$y_t = y^0\exp\left((\mu_y-\frac{\sigma_y^{\top}\sigma_y}{2})t+\sigma_y^{\top}Z_t\right), \tag{3}$$

and no-borrowing-without-repayment

$$W_t \geq -g(t)y_t, \tag{4}$$

where

$$g(t)=\begin{cases}\left(\frac{1-e^{-\beta_2(T-t)}}{\beta_2}\right)^{+} & \beta_2\neq 0\\ (T-t)^{+} & \beta_2=0,\end{cases} \tag{5}$$

$$\beta_2 = r+\delta-\mu_y+\sigma_y^{\top}\kappa \tag{6}$$

is the effective discount rate for labor income, and

$$\kappa=\sigma^{-1}(\mu-r\mathbf{1})$$

is the price of risk.

Problem 2 **(NBC)** *Given initial wealth W^0, initial income from working y^0, and initial retirement status R^0, choose adapted nonnegative consumption $\{c_t\}$, adapted portfolio $\{\theta_t\}$, adapted nonnegative bequest $\{B_t\}$, and nondecreasing adapted retirement indicator[3] $\{R_t\}$, to maximize expected utility of lifetime consumption and bequest (1) subject to the budget constraint (2), labor income before retirement (3), and no-borrowing-without-repayment*

$$W_t \geq -(1-R_t)\frac{y_t}{\beta_2}, \tag{7}$$

where β_2 is assumed to be positive.

Problem 3 **(BC)** *The same as Problem 2, except that the no-borrowing-without-repayment constraint is replaced by the stronger non-negative wealth constraint*

$$W_t \geq 0. \tag{8}$$

[3]By "indicator," we mean a right-continuous process taking values 0 and 1.

The uncertainty in the model comes from two sources: the standard Wiener process Z_t and the Poisson arrival of mortality at a fixed hazard rate δ. These are drawn independently. The Wiener process Z_t has dimensionality equal to the number of linearly independent risky returns, and maps into security returns through the constant mean vector μ and the constant nonsingular standard deviation matrix σ. For most of the paper, we will assume that local changes in the labor income y are spanned by local returns on the assets, but we will generalize this result (requiring a numerical solution) in Section V and provide some plots in Section III. The common objective function (1) for the problems has already integrated out the impact of mortality risk: utility is discounted at the rate $\rho+\delta$ where ρ is the pure rate of time discount and δ is the hazard rate of mortality. Some early work on utility theory suggested that the pure rate of time discount is positive only because of the effect of mortality; if this is true then we could take ρ to be 0. Still, the problem would not be the same as the traditional problem due to the presence of bequest and insurance. Insurance is assumed to be fairly priced at the rate δ per unit of coverage, both long and short. When $W-B>0$, this is term life coverage purchased for a premium of $\delta(W-B)$ per unit time. If $W-B<0$, then this is a short position in term life coverage, which is like a term version of a life annuity since it trades wealth in the event of death for more consumption when living.

The utility function (1) is a standard time-separable von Neumann-Morgenstern utility function with mortality and bequest. The utility function features the same constant level $\gamma>0$ of risk aversion for consumption before retirement, after retirement, and for bequests. Felicity of consumption or bequest is discounted using a pure rate of time discount ρ plus the mortality rate δ. The indicator function R_t is 1 after retirement and 0 before retirement. The constant $K>1$ indicates how much more consumption before retirement must be increased to compensate for having to work. Preference for not working could be due to a disutility of work, or it could be due indirectly to household production and cost savings. For example, when working there may not be enough time to shop for bargains or prepare meals or take a cruise. The constant $k>0$ measures the intensity of preference for leaving a large bequest, the limit $k^{1-\gamma}\rightarrow 0$ implements no preference for bequest.

The terms in the integrand of the wealth equation (2) are mostly familiar. The first term says that if all wealth is invested in the riskfree asset, the rate of return is r. For the dollar investment θ_t in the risky asset, there is risk exposure $\theta_t\sigma dZ$ and the mean return $\theta_t\mu$ is substituted for the corresponding riskfree return $r\theta_t\mathbf{1}$ (where $\mathbf{1}$ is a vector of 1's with dimension equal to the number of risky assets). The term $\delta(B_t-W_t)dt$ is the insurance premium we have already discussed, $c_t dt$ is payment for consumption, and $(1-R_t)y_t dt$ is labor income. The factor $(1-R_t)$ multiplying wage income says income disappears after retirement.

In general, it is a subtle question what kind of constraint to add in an infinite-horizon portfolio problem to rule out borrowing without repayment and doubling strategies. Fortunately, there is a simple and reasonable constraint that suffices in our problem. The no-borrowing-without-repayment constraints (4) or (7) specify that indebtedness ($=\max(-W_t,0)$) can never be larger than earnings potential. The two constraints differ because the earnings potential is different when there is a fixed retirement date than when retirement is a choice and it is possible to work until death. In Problem 3, the no-borrowing-without-repayment constraint is replaced by the stronger no-borrowing constraint (8).

To review the differences in the problems, moving from the benchmark Problem 1 to the NBC Problem 2, the fixed retirement date T ($R_t=\iota(t\geq T)$) is replaced by free choice of when to retire (R_t a choice variable), along with an inessential change in the calculation of the maximum value of future labor income ((4) to (7)). Moving from the NBC Problem 2 to the BC Problem 3 replaces the no-borrowing-without-repayment constraint (7) with a no-borrowing constraint (8).

III. Graphical Solution

Before proceeding to the more general model with stochastic wages, we explore graphically the solution to the simple model. We present many of the results normalized by total wealth, equal to financial wealth plus human capital, where human capital is the market value of future labor income in the optimal solution. The formulas for human capital are given at the end of Section IV., specialized to the simple case in which the wage is constant. While the market's valuation of the individual's human capital may be different from the individual's own valuation (due to the borrowing constraint), this is still a useful normalization for interpreting the results.

Recall that he results consider three cases:

benchmark fixed retirement date and free borrowing against wages (Problem 1 and Theorem 1)

NBC ("No Borrowing Constraint") free choice of retirement date and free borrowing against wages (Problem 2 and Theorem 2)

BC ("Borrowing Constraint") free choice of retirement date but no borrowing against wages (Problem 3 and Theorem 3)

The benchmark case is a close relative of the Merton model with i.i.d. returns and constant relative risk aversion. Moving to the NBC case isolates the impact of making retirement flexible. Subsequently moving to the BC case isolates the impact of the borrowing constraint.

Figure 1 shows the optimal stock position in the three cases, per unit of total wealth, as a function of financial wealth. The horizontal line shows the optimal portfolio choice for the benchmark case. In this case, it is as if all future wage income is capitalized and then there is fixed-proportions investment as in the Merton model. The time to retirement in the plot is 20 years, but the portfolio proportion would be the same constant whatever the time to retirement (and even after retirement). Moving to the NBC case, permitting flexibility in the retirement date permits a larger equity position because working longer can insure against variation in the stock market. The agent works longer in expensive states (when the market is down) and the wealth from these states is transferred to other states by taking a significant position in equities. Adding the borrowing constraint in the BC case, transfer of wealth across states is restricted and indeed there is not much point of taking a significant position in equities when financial wealth is low, since that would just imply bumping into the borrowing constraint much of the time.

Figure 2 shows the consumption rate, normalized by total wealth, as a function of financial wealth. An interesting point not visible in the picture is that consumption jumps at retirement, in a direction that depends on whether risk aversion is larger or smaller than 1 (log utility). In the benchmark case, the consumption rate does not depend on financial wealth, but it does depend on time to maturity (not shown). Moving to the NBC case, adding flexible retirement leads to a higher consumption rate when wealth is higher (due to risk aversion greater than 1 in the example), since high wealth implies retirement is expected soon and demand for consumption is less after retirement. Moving to the BC case, consumption is significantly less at low wealth levels, which represents precautionary savings against market declines.

The critical wealth level at which the agent chooses to retire as a function of relative risk aversion is plotted in Figure 3. There is no curve for the benchmark case, since in that case retirement is at a fixed date, not at a freely chosen wealth boundary. The critical wealth is lower when there is a borrowing constraint than when there is none: while retirement is equally desirable in both cases, continuing to work is less desirable when borrowing is limited. The critical wealth level decreases in risk aversion because it is valuable to earn more money to take advantage of market returns when risk aversion is small.

The value of human capital can vary inversely with wealth as a form of insurance. Figure 4 shows the dependence of human capital on financial wealth. This is a constant over wealth in the benchmark case, but would vary from zero to the maximum on the NBC curve as maturity increases. In the BC and NBC cases, we see the insurance effect of how the agent hedges financial risk by working longer (and thereby increasing the value of human capital) when financial wealth is low.

Having flexible retirement and borrowing (the NBC case) is the least constrained of the three cases. It is interesting to measure the loss in value in the other two cases. Figure 5 gives the value loss from losing retirement flexibility, as a function of the fixed time to retirement. The value loss is measured as a certainty-equivalent fraction of total wealth corresponding to moving from the NBC case to the benchmark case. When wealth is high (=20), retiring soon is optimal and the loss is least when forced retirement comes soon. As wealth decreases, it becomes optimal to work longer and longer and the minimum loss is at larger and larger fixed times to retirement. Figure 6 shows the value of being able to borrow measured in certainty-equivalent units as a function of total wealth. The value is small when wealth is large and large when the borrowing constraint is nearly binding. The value of being able to borrow is greatest when the mean return on the stock is high, since being able to borrow makes it possible to make full use of equity.

Stock brokers have traditionally advised customers that young people should take on more risk than older people who are close to retirement. Our analysis can be used to generalize and confirm analysis of Jagannathan and Kocherlakota [1996] that calls into question the traditional rule. We have already seen in Figure 1 that in both the NBC case and BC case, the risky asset holding increases as a function of wealth, and higher wealth corresponds to being nearer to retirement. Arguably this result is not a fair criticism of the traditional advice because of the normalization by total wealth. Figure 8 shows the portfolio choice normalized by financial wealth (both curves NBC), which may be closer to the intent of the traditional advice. When wages are less risky (top curve), the proportion of financial wealth put in stock does indeed fall as financial wealth increases. However, drawing on results from Section IV. below, this result can be reversed if wages are stochastic and move in the same direction as stocks, as illustrated by the bottom curve.

Part of the problem with the traditional advice can be illustrated by the benchmark case in which risk exposure is chosen to be a fixed proportion of total wealth. When young, most of the agent's total wealth is in human capital, and if the wage is stochastic this may already represent too much exposure to risk, implying a desire for to take a short position in stocks to hedge the excess risk. When retirement is discretionary, the stock choice is less for the direct reason we have been discussing that wages are riskier, and also because it is less attractive to work more in expensive states (since those are also states of low wage when wages move in the same direction as stocks). Together with result that a borrowing constraint reduces stock demand at low levels even more, these results question the usefulness of the traditional advice, at least in the absence of a lot of qualifications of when the advice should be applied.

Table 1 contains a number of solutions illustrating the sensitivity of the equilibrium to various

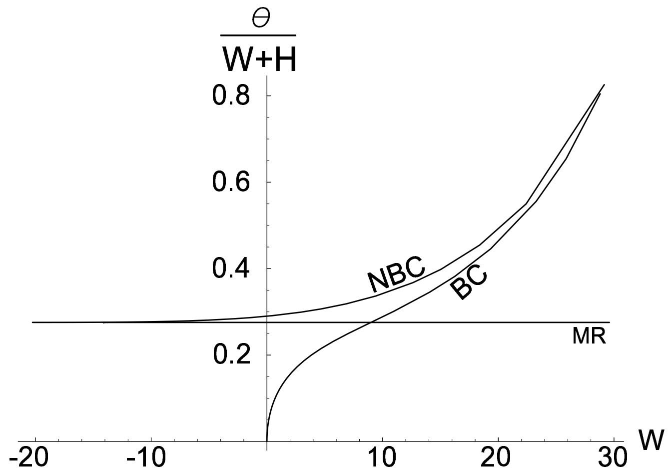

Figure 1: Equity as a fraction of total wealth (= financial wealth plus human capital), as a function of the financial wealth W **for parameters:** $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $y_0 = 1$**. The horizontal line is for the benchmark case with a fixed retirement date 20 years from now and free borrowing against future wages. The NBC ("No Borrowing Constraint") case adds free choice of retirement date but still allows free borrowing. The BC ("Borrowing Constraint") case adds a nonnegative wealth constraint that restricts borrowing against future wages.**

parameters. In particular, it shows the significant drop of the consumption and the stock investment at retirement date. In addition, as the mortality rate decreases, the investor requires a higher critical wealth-to-income ratio to retire and consumes less to save for after-retirement.

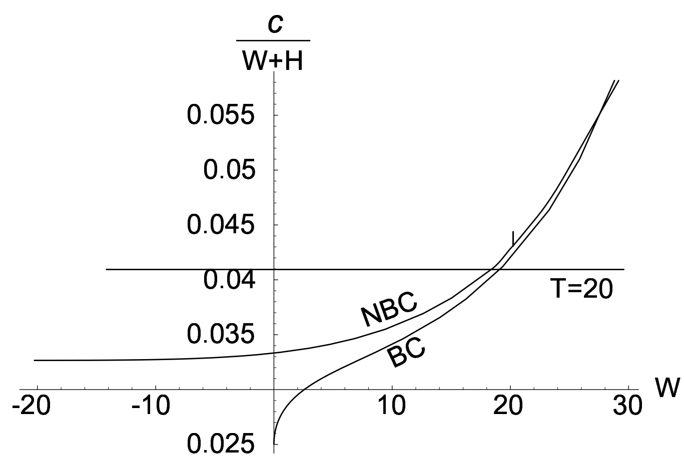

Figure 2: Consumption rate as a fraction of total wealth, as a function of financial wealth for parameters $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, and $y_0 = 1$.

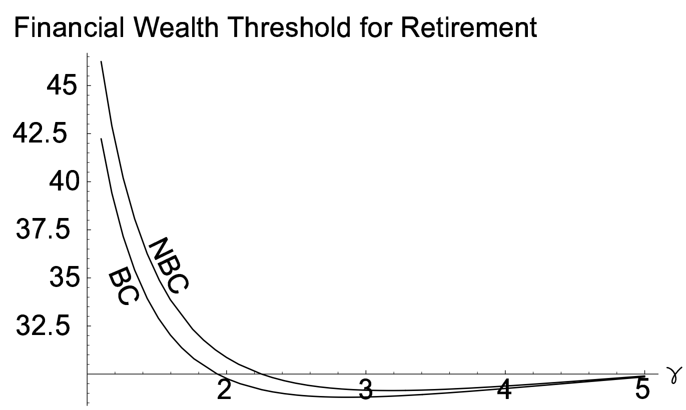

Figure 3: Financial wealth threshold for retirement as a function of relative risk aversion γ for the NBC and BC cases with parameters $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, and $y_0 = 1$.

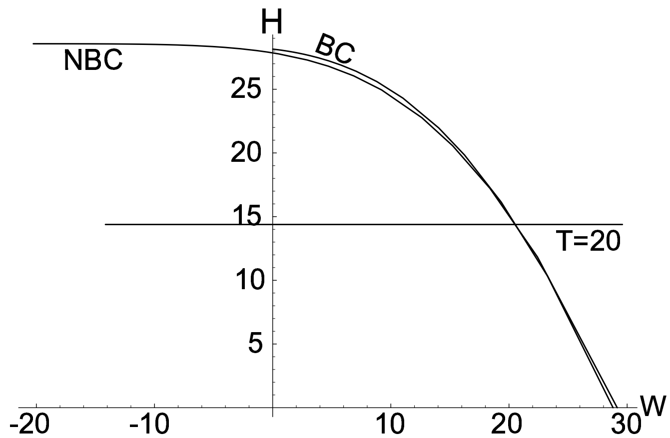

Figure 4: Human capital given financial wealth W **for parameters** $\mu = 0.05$**,** $\sigma = 0.22$**,** $r = 0.01$**,** $\delta = 0.025$**,** $\rho = 0.01$**,** $\gamma = 3$**,** $K = 3$**,** $k = 0.05$**, and** $y_0 = 1$**.**

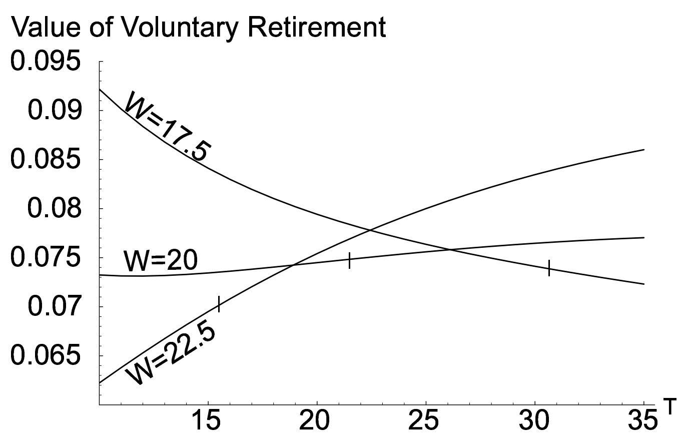

Figure 5: Value of voluntary retirement as a fraction of the total wealth, as a function of retirement horizon T **for parameters** $\mu = 0.05$**,** $\sigma = 0.22$**,** $r = 0.01$**,** $\delta = 0.025$**,** $\rho = 0.01$**,** $\gamma = 3$**,** $K = 3$**,** $k = 0.05$**, and** $y_0 = 1$**.**

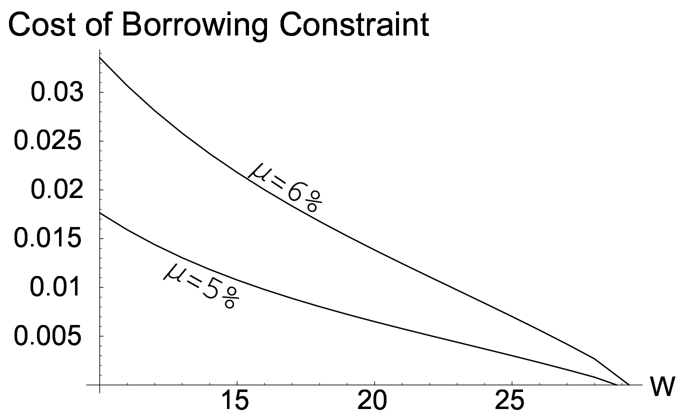

Figure 6: Value of borrowing as a fraction of the total wealth, as a function of financial wealth W **for parameters** $\sigma = 0.22$**,** $r = 0.01$**,** $\delta = 0.025$**,** $\rho = 0.01$**,** $\gamma = 3$**,** $K = 3$**,** $k = 0.05$**, and** $y_0 = 1$**.**

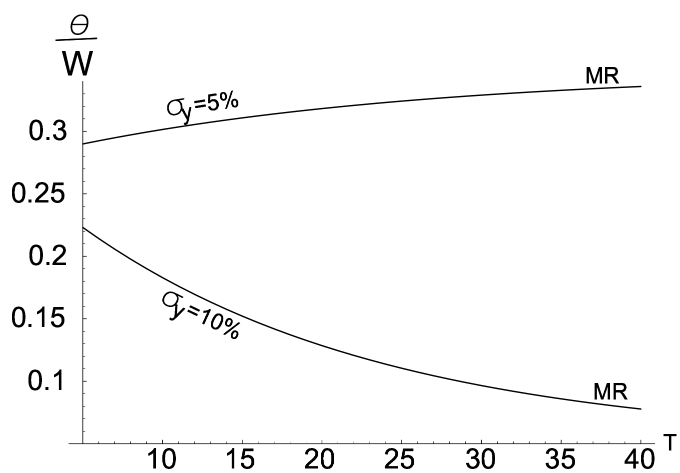

Figure 7: Impact of risky human capital on equity holdings for parameters $\mu = 0.05$**,** $\sigma = 0.22$**,** $\mu_y = 0$**,** $r = 0.01$**,** $\delta = 0.025$**,** $\rho = 0.01$**,** $\gamma = 3$**,** $K = 3$**,** $k = 0.05$**, and** $y_0 = 1$**.**

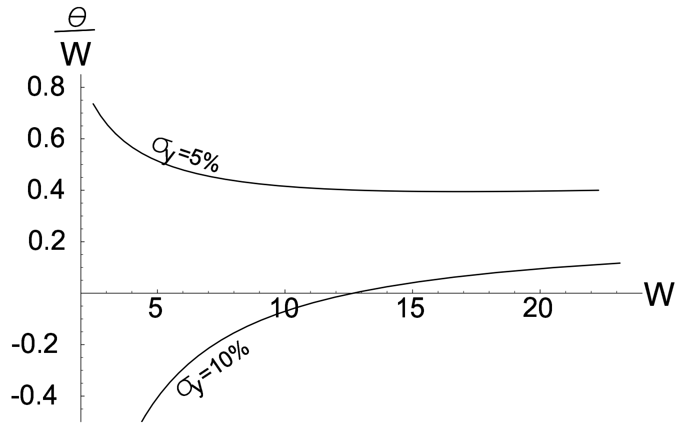

Figure 8: Impact of risky human capital on equity holdings for parameters $\mu = 0.05$**,** $\sigma = 0.22$**,** $\mu_y = 0$**,** $r = 0.01$**,** $\delta = 0.025$**,** $\rho = 0.01$**,** $\gamma = 3$**,** $K = 3$**,** $k = 0.05$**,** $y_0 = 1$**.**

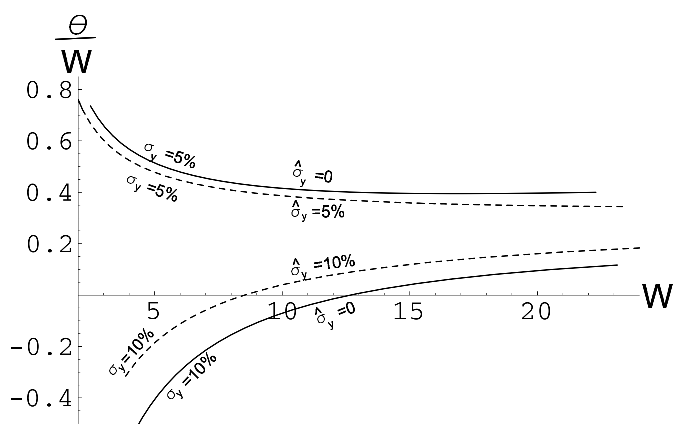

Figure 9: Impact of risky human capital on equity holdings for parameters $\mu = 0.05$**,** $\sigma = 0.22$**,** $\mu_y = 0$**,** $r = 0.01$**,** $\delta = 0.025$**,** $\rho = 0.01$**,** $\gamma = 3$**,** $K = 3$**,** $k = 0.05$**,** $y_0 = 1$**.**

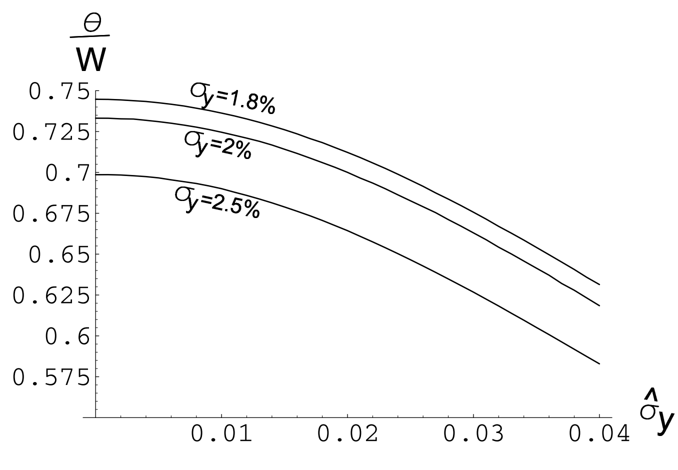

Figure 10: The effect of idiosyncratic risk on on equity holdings for parameters $\mu =$ 0.05, $\sigma = 0.22$, $\mu_y = 0$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $y_0 = 1$.

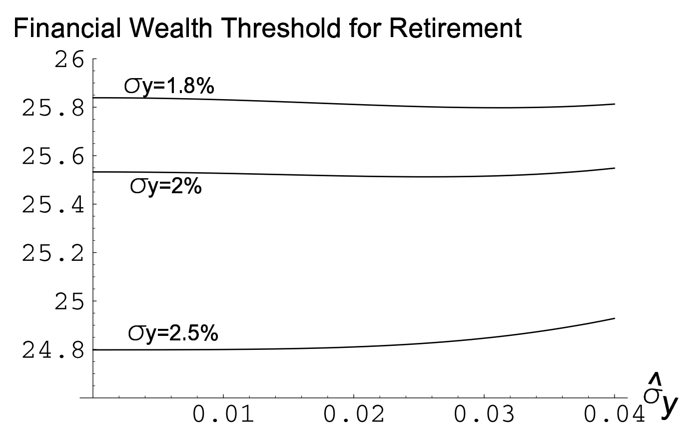

Figure 11: The effect of idiosyncratic risk on retirement boundary for parameters $\mu = 0.05$, $\sigma = 0.22$, $\mu_y = 0$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $y_0 = 1$.

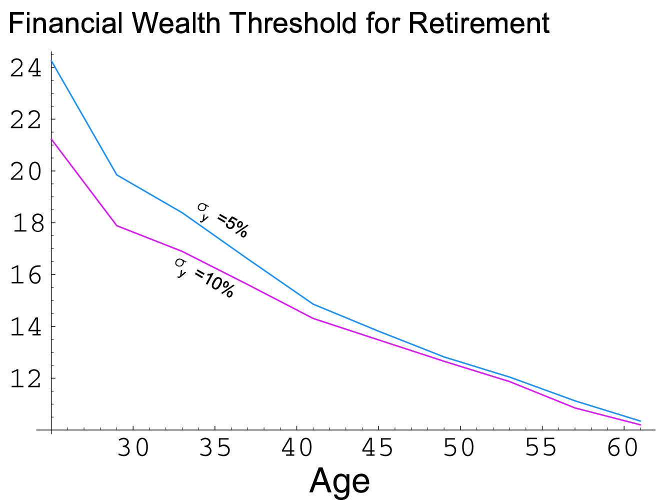

Figure 12: The wealth threshold against life stages.

Parameter	$\bar{W}$	$\frac{c(\tau)}{W}$	$\frac{c(\tau^+)}{W}$	$\frac{\theta(\tau)}{W}$	$\frac{\theta(\tau^+)}{W}$	$\bar{W}_{NBC}$	$\frac{\theta_{NBC}(\tau^+_{NBC})}{W_{NBC}}$
Base Case	28.81	0.058	0.028	0.8	0.28	29.16	0.83
$\gamma = 3.5$	28.98	0.057	0.026	0.69	0.24	29.19	0.71
$\gamma = 3.5$	28.91	0.058	0.03	0.95	0.33	29.52	0.98
$\mu = 0.04$	28.03	0.056	0.027	0.72	0.21	28.14	0.73
$\mu = 0.06$	29.26	0.061	0.029	0.89	0.34	30.06	0.92
$\sigma = 0.15$	29.38	0.065	0.031	1.40	0.59	30.76	1.48
$\sigma = 0.30$	27.97	0.060	0.03	0.53	0.15	28.07	0.53
$\delta = 0.02$	28.81	0.058	0.028	0.80	0.28	29.16	0.83
$\delta = 0.03$	20.96	0.075	0.036	0.81	0.28	21.01	0.82
$\rho = 0.008$	29.08	0.057	0.027	0.78	0.28	29.39	0.80
$\rho = 0.012$	28.54	0.059	0.028	0.83	0.28	28.94	0.85
$K = 2.5$	35.93	0.053	0.029	0.71	0.28	36.22	0.72
$K = 3.5$	24.45	0.063	0.027	0.89	0.28	24.85	0.92
$k = 0.025$	33.59	0.050	0.024	0.74	0.28	33.91	0.75
$k = 0.075$	26.87	0.062	0.030	0.84	0.28	17.86	0.76
$\mu_y = 0.01$	34.82	0.058	0.028	1.10	0.28	38.02	1.23
$\mu_y = -0.01$	25.56	0.058	0.028	0.58	0.28	25.57	0.58

Table 1: Comparative Statics

Base case parameters: $\mu = 0.05$, $\sigma = 0.22$, $r = 0.01$, $\delta = 0.025$, $\rho = 0.01$, $\gamma = 3$, $K = 3$, $k = 0.05$, $\mu_y = 0$, $\sigma_y = 0$, and $y_0 = 1$.

IV. The Analytical Solution

In this section we provide the analytical solutions to the investor's problem.

Let

$$\nu = \frac{\gamma}{\rho + \delta - (1-\gamma)(r + \delta + \frac{\kappa^\top \kappa}{2\gamma})}. \tag{9}$$

Theorem 1 (benchmark) *Suppose* $\nu > 0$. *Then in the solution to the investor's Problem 1, the optimal wealth process is*

$$W_t^* = f(t) y_t x_t^{-1/\gamma} - g(t) y_t, \tag{10}$$

the optimal consumption policy is

$$c_t^* = K^{-bR_t} f(t)^{-1} (W_t^* + g(t) y_t), \tag{11}$$

the optimal trading strategy is

$$\theta_t^* = \frac{\sigma^{-2}(\mu - r)}{\gamma}(W_t^* + g(t) y_t) - \sigma^{-1} \sigma_y g(t) y_t, \tag{12}$$

and the optimal bequest policy is

$$B_t^* = k^{-b} f(t)^{-1} (W_t^* + g(t) y_t), \tag{13}$$

where

$$b = 1 - 1/\gamma, \tag{14}$$

$$f(t) = (\hat{\eta} - \eta) \exp(-(1 - R_t) \frac{1 + \delta k^{-b}}{\eta}(T - t)) + \eta,$$

$$\eta = (1 + \delta k^{-b}) \nu, \tag{15}$$

$$\hat{\eta} = (K^{-b} + \delta k^{-b}) \nu. \tag{16}$$

$$x_t = \left(\frac{W^0 + g(0) y^0}{y^0 f(0)} \right)^{-\gamma} e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)t + \sigma_x^\top Z_t}, \tag{17}$$

$$\mu_x = -(r - \rho) - \frac{1}{2}\gamma(1-\gamma)\sigma_y^\top \sigma_y + \gamma \mu_y - \gamma \sigma_y^\top \kappa, \tag{18}$$

and

$$\sigma_x = \gamma \sigma_y - \kappa. \tag{19}$$

In addition, the value function is

$$v(W, y, t) = f(t)^\gamma \frac{(W + g(t) y)^{1-\gamma}}{1-\gamma}.$$

Unlike Problem 1, we cannot provide explicit solutions to Problems 2 and 3 in terms of the primal variables. However, we provide explicit solutions (up to at most one constant) in terms of marginal utility in Theorems 2 and 3. Let

$$\beta_3 = \rho + \delta + \frac{1}{2}\gamma(1-\gamma)\sigma_y^\top \sigma_y - (1-\gamma)\mu_y. \tag{20}$$

Theorem 2 (NBC) *Suppose* $\nu > 0$, $\beta_2 > 0$, *and* $\beta_3 > 0$. *The solution to the investor's Problem 2 can be written in terms of the dual variable* x_t *(a normalized marginal utility of consumption). Specifically, let the dual variable be defined by*

$$x_t = x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)t + \sigma_x^\top Z_t}, \tag{21}$$

where x_0 *solves*

$$-y^0 \varphi_x(x_0, R^0) = W^0,$$

where

$$\varphi(x,R) = \begin{cases} -\hat{\eta}\frac{x^b}{b} & \text{if } R = 1 \text{ or } x \le \underline{x} \\ A_+ x^{\alpha_-} - \eta\frac{x^b}{b} + \frac{1}{\beta_2}x & \text{otherwise,} \end{cases} \tag{22}$$

where

$$A_+ = \frac{1}{\gamma(b-\alpha_-)\beta_2}\underline{x}^{1-\alpha_-},$$

the optimal retirement boundary is

$$\underline{x} = \left(\frac{(\eta-\hat{\eta})(b-\alpha_-)\beta_2}{b(1-\alpha_-)}\right)^\gamma,$$

where

$$\alpha_- = \frac{\beta_2 - \beta_3 + \frac{1}{2}\beta_1 - \sqrt{(\beta_2 - \beta_3 + \frac{1}{2}\beta_1)^2 + 2\beta_3\beta_1}}{\beta_1}, \tag{23}$$

and

$$\beta_1 = \kappa^\top \kappa + \gamma^2 \sigma_y^\top \sigma_y - 2\gamma\sigma_y^\top \kappa, \tag{24}$$

Then the optimal consumption policy is

$$c_t^* = K^{-bR_t^*} y_t x_t^{-1/\gamma},$$

the optimal trading strategy is

$$\theta_t^* = y_t[\sigma^{-2}(\mu - r)x_t\varphi_{xx}(x_t, R_t^*) - \sigma^{-1}\sigma_y(\gamma x_t \varphi_{xx}(x_t, R_t^*) + \varphi_x(x_t, R_t^*))],$$

the optimal bequest policy is

$$B_t^* = k^{-b} y_t x_t^{-1/\gamma},$$

the optimal retirement policy is

$$R_t^* = \iota\{t \geq \tau^*\},$$

the corresponding retirement wealth threshold is

$$\overline{W}_t = -y_t \varphi_x(\underline{x}, 0),$$

and the optimal wealth is

$$W_t^* = -y_t \varphi_x(x_t, R_t^*),$$

where

$$\tau^* = (1 - R^0) \inf\{t \geq 0 : x_t \leq \underline{x}\}.$$

In addition, the value function is

$$v(W, y, R) = y^{1-\gamma} \left(\varphi(x, R) - x\varphi_x(x, R)\right), \tag{25}$$

where x solves

$$-y\varphi_x(x, R) = W. \tag{26}$$

Theorem 3 (BC)
Suppose $\nu > 0$, $\beta_2 > 0$, and $\beta_3 > 0$. The solution to the investor's Problem 3 can be written in terms of the dual variable x_t. Specifically, let the dual variable be defined by

$$x_t = \frac{x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)t - \sigma_x^\top Z_t}}{\max(1, \sup_{0 \leq s \leq \min(t, \tau^*)} x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)s - \sigma_x^\top Z_s} / \overline{x})} \tag{27}$$

where x_0 solves

$$-y^0 \varphi_x(x_0, R^0) = W^0, \tag{28}$$

the optimal stopping time

$$\tau^* = (1 - R^0) \inf\{t \geq 0 : x_t \leq \underline{x}\},$$

the dual value function

$$\varphi(x, R) = \begin{cases} -\hat{\eta}\frac{x^b}{b} & \text{if } R = 1 \text{ or } x \leq \underline{x} \\ A_+ x^{\alpha_-} + A_- x^{\alpha_+} - \eta\frac{x^b}{b} + \frac{1}{\beta_2}x & \text{otherwise,} \end{cases} \tag{29}$$

where

$$A_- = \frac{\eta(b - \alpha_-)}{\alpha_+(\alpha_+ - \alpha_-)}\overline{x}^{b-\alpha_+} - \frac{1 - \alpha_-}{\alpha_+(\alpha_+ - \alpha_-)\beta_2}\overline{x}^{1-\alpha_+},$$

$$A_+ = \frac{\eta(\alpha_+ - b)}{\alpha_-(\alpha_+ - \alpha_-)}\overline{x}^{b-\alpha_-} - \frac{\alpha_+ - 1}{\alpha_-(\alpha_+ - \alpha_-)\beta_2}\overline{x}^{1-\alpha_-},$$

the x value at which the financial wealth is zero is

$$\overline{x} = \left(\frac{\left(\frac{\eta-\hat{\eta}}{b}\zeta^{b-\alpha_-} - \frac{\eta}{\alpha_-}\right)(\alpha_+ - b)\beta_2}{(\zeta^{1-\alpha_-} - \frac{1}{\alpha_-})(\alpha_+ - 1)} \right)^{\gamma},$$

the optimal retirement boundary

$$\underline{x} = \zeta\overline{x},$$

where ζ solves $q(\zeta) = 0$, where

$$\begin{aligned} q(\zeta) \quad \equiv \quad & \left(\frac{1-K^{-b}}{b(1+\delta k^{-b})}\zeta^{b-\alpha_-} - \frac{1}{\alpha_-}\right)\left(\zeta^{1-\alpha_+} - \frac{1}{\alpha_+}\right)(\alpha_+ - b)(\alpha_- - 1) \\ - \quad & \left(\frac{1-K^{-b}}{b(1+\delta k^{-b})}\zeta^{b-\alpha_+} - \frac{1}{\alpha_+}\right)\left(\zeta^{1-\alpha_-} - \frac{1}{\alpha_-}\right)(\alpha_- - b)(\alpha_+ - 1), \end{aligned} \tag{30}$$

and

$$\alpha_+ = \frac{\beta_2 - \beta_3 + \frac{1}{2}\beta_1 + \sqrt{(\beta_2 - \beta_3 + \frac{1}{2}\beta_1)^2 + 2\beta_3\beta_1}}{\beta_1}. \tag{31}$$

Then the optimal consumption policy is

$$c_t^* = K^{-bR_t^*} y_t x_t^{-1/\gamma},$$

the optimal trading strategy is

$$\theta_t^* = y_t[\sigma^{-2}(\mu - r)x_t\varphi_{xx}(x_t, R_t^*) - \sigma^{-1}\sigma_y(\gamma x_t\varphi_{xx}(x_t, R_t^*) + \varphi_x(x_t, R_t^*))],$$

the optimal bequest policy is

$$B_t^* = k^{-b} y_t x_t^{-1/\gamma},$$

the optimal retirement policy is

$$R_t^* = \iota\{t \geq \tau^*\},$$

the corresponding retirement wealth threshold is

$$\overline{W}_t = -y_t\varphi_x(\underline{x}, 0),$$

and the optimal wealth is

$$W_t^* = -y_t\varphi_x(x_t, R_t^*). \tag{32}$$

In addition, the value function is

$$v(W, y, R) = y^{1-\gamma}(\varphi(x, R) - x\varphi_x(x, R)) \tag{33}$$

where x solves

$$-y\varphi_x(x, R) = W. \tag{34}$$

Proposition 1 *The present value of the human capital corresponding to cases in Theorems 4, 5 and 6 are respectively*

$$H(y_t,t) = \frac{y_t}{\beta_2}(1-e^{-\beta_2(T-t)})\iota\{t \le T\},$$

$$H(x_t,y_t) = \frac{y_t}{\beta_2}(-\underline{x}^{1-\alpha_-}x_t^{\alpha_- -1}+1)\iota\{x_t \ge \underline{x}\},$$

$$H(x_t,y_t) = \frac{y_t}{\beta_2}(Ax_t^{\alpha_- -1}+Bx_t^{\alpha_+ -1}+1)\iota\{x_t \ge \underline{x}\},$$

where

$$A = \frac{(1-\alpha_+)\underline{x}^{1-\alpha_-}\overline{x}^{\alpha_+-\alpha_-}}{(\alpha_+-1)\overline{x}^{\alpha_+-\alpha_-}-(\alpha_--1)\underline{x}^{\alpha_+-\alpha_-}},$$

$$B = \frac{(\alpha_--1)\underline{x}^{1-\alpha_-}}{(\alpha_+-1)\overline{x}^{\alpha_+-\alpha_-}-(\alpha_--1)\underline{x}^{\alpha_+-\alpha_-}}.$$

Proposition 2 *If $\mu_x < \frac{1}{2}\sigma_x^2$, then the expected time to retirement corresponding to cases for Theorems 5 and 6 are respectively*

$$E[\tau^*|x_t = x] = \frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2-\mu_x}, \forall x_t > \underline{x}$$

and

$$E[\tau^*|x_t = x] = \frac{\underline{x}^m - x^m}{(\frac{1}{2}\sigma_x^2-\mu_x)m\overline{x}^m} + \frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2-\mu_x}, \forall x_t \in [\underline{x},\overline{x}],$$

where

$$m = 1-\frac{2\mu_x}{\sigma_x^2}.$$

V. Imperfectly Correlated Labor Income

Now we consider the case with imperfectly correlated labor income, i.e.,

$$(\forall t \ge 0) \quad \frac{dy_t}{y_t} = \mu_y dt + \sigma_y dZ_t + \hat{\sigma}_y d\hat{Z}_t, \tag{35}$$

where $\hat{Z}_t$ is a one-dimensional Brownian motion independent of Z_t. The primal problem is difficult to solve due to a singular boundary condition at $W_t = 0$. We therefore solve this case also using the dual approach.

Let a convex and decreasing function $\varphi(x,R)$ be such that the value function $v(W,y,R) = y^{1-\gamma}(\varphi(x,R)-x\varphi_x(x,R))$, where x solves $-y\varphi_x(x,R) = W$. Then after retirement $\varphi(x,1)$ is the same as the one in the previous section. After straightforward simplification, the HJB equation for $\varphi(x,0)$ becomes

$$\frac{1}{2}\beta_1 x^2\varphi_{xx}(x,0)-(\beta_2-\beta_3)x\varphi_x(x,0)-\beta_3\varphi(x,0)-\frac{1}{2}\hat{\sigma}_y^2\frac{\varphi_x^2(x,0)}{\varphi_{xx}(x,0)}-(1+\delta k^{-b})\frac{x^b}{b}+x=0, \tag{36}$$

where

$$\beta_1=\kappa^\top\kappa+\gamma^2\sigma_y^\top\sigma_y-2\gamma\sigma_y\kappa^\top, \tag{37}$$

$$\beta_2=r+\delta-\mu_y+\sigma_y\kappa^\top+\gamma\hat{\sigma}_y^2, \tag{38}$$

and

$$\beta_3=\rho+\delta+\frac{1}{2}\gamma(1-\gamma)(\sigma_y^\top\sigma_y+\hat{\sigma}_y^2)-(1-\gamma)\mu_y. \tag{39}$$

Note that if the labor income is perfectly correlated with the stock market, i.e., $\hat{\sigma}_y=0$, then this ODE reduces to (61) for the previous section.

We need to solve ODE (36) subject to (62)-(65). Different from the case with perfectly correlated labor income, the HJB ODE (36) is fully nonlinear and an explicit form for the value function seems unavailable. However, this nonlinear ODE with free boundaries can be easily solved numerically.

VI. Conclusion

In this paper, we consider how the concern about the living standard after retirement affects optimal consumption and investment policy throughout an investor's life. We consider both the case with mandatory retirement date and the case with voluntary retirement. We show that in the case of voluntary retirement an investor retires once the ratio of financial wealth to labor income reaches a critical value. This critical wealth-to-income retirement ratio depends on financial market characteristics such as riskfree rate and risk premium on the market and investor personal characteristics such as wage rate, wealth, mortality rate, and risk aversion. We also find that the flexibility for retirement date is valuable and changes significantly the optimal investment and consumption policy, especially if we can borrow against future labor income. Generally speaking, an investor invests more in the stock market. However, the investor may save less or more than the mandatory retirement case depending on the time to the mandatory retirement date.

Appendix

In this Appendix, we collect all the proofs.

PROOF OF THEOREM 1.

The approach of the proof is to use a separating hyperplane to separate preferred consumptions from the feasible consumptions. Feasibility of the claimed optimum follows from direct algebraic substitution. We can verify the budget equation (2) using Itô's lemma, the claimed form of the strategy $(c^*, \theta^*, B^*, W^*)$ in (11), (12), (13), and (10). The no-borrowing-without-repayment constraint (4) follows from positivity of $f(t)$ and x and the definition of W^* in (10).

We start by using the state-price density and martingale properties to derive pricing results, first for labor income, then for consumption and bequest. Define the state price density process ξ by

$$\xi_t \equiv e^{-(r+\delta+\frac{1}{2}\kappa^\top\kappa)t-\kappa^\top Z_t}. \tag{40}$$

This is the usual state-price density but adjusted to condition on living given the mortality rate δ and fairly-priced long and short positions in life insurance.

Now, we will show that $g(t)y_t$ is the market value at t of subsequent labor income and g is defined in (5). (Note that $g(t) = 0$ for any $t \geq T$.) By Itô's lemma, (3), (5), and simple algebra,

$$d\left(g(t)\xi_t y_t\right) = -(1-R_t)\xi_t y_t dt + g(t)\xi_t y_t(\sigma_y - \kappa)^\top dZ_t.$$

Furthermore,

$$E\int_0^t \left|g(s)\xi_s y_s(\sigma_y - \kappa)\right|^2 ds < \infty,$$

since $\xi_s y_s$ is a standard lognormal diffusion and the other factors are bounded and zero for $t > T$. Therefore the local martingale

$$g(t)\xi_t y_t + \int_0^\infty (1-R_s)\xi_s y_s\, ds \tag{41}$$

is a martingale that is constant for $t > T$. In particular, $g(t)y_t = E_t[\int_t^\infty (1-R_s)(\xi_s/\xi_t)y_s ds]$, that is, $g(t)y_t$ is the value at t of subsequent labor income.

Now we turn to valuation of consumption and bequest. Note that the no-borrowing-without-repayment constraint (4) together with $c_t \geq 0$ and $B_t \geq 0$ imply that

$$\xi_t W_t + \int_0^t \xi_s(c_s + \delta B_s - (1-R_s)y_s)\, ds \geq -[g(t)\xi_t y_t + \int_0^t (1-R_s)\xi_s y_s\, ds]. \tag{42}$$

The budget constraint (2), Itô's lemma, and the definition of ξ (40) imply that for any feasible strategy, the left-hand side of (42) has zero drift and is therefore a local martingale for any feasible strategy (c, θ, B). Furthermore, the right-hand side of (42) was found from the argument around (41) to be a martingale. Since any local martingale bounded below by a martingale is a supermartingale,[4] the left-hand-side of (42) is a supermartingale for any feasible strategy.

For the claimed optimum strategy, it follows that the left-hand side of (42) is actually a martingale, since it is a local martingale and the integrand with respect to dZ_t has lognormal terms that can be shown to be bounded in L^2 over finite time intervals.

Recalling that $\xi_t W_t \geq 0$ for any $t \geq T$ by (4), since the left-hand-side of (42) is a local martingale, we can take $t \uparrow \infty$ to obtain the pricing result

$$E[\int_0^\infty \xi_t (c_t + \delta B_t - (1-R_t)y_t) dt] \leq W^0 \tag{43}$$

for any feasible strategy, with equality for our candidate optimum. To link pricing to utilities, we look at the support functions to felicity of consumption and bequest. Since any concave function lies below the tangent line at any point, we have that

$$\frac{(K^R c)^{1-\gamma}}{1-\gamma} \leq \frac{(K^R c^*)^{1-\gamma}}{1-\gamma} + (K^R)^{1-\gamma} c^{*-\gamma}(c - c^*) \tag{44}$$

and

$$\frac{(kB)^{1-\gamma}}{1-\gamma} \leq \frac{(kB^*)^{1-\gamma}}{1-\gamma} + k^{1-\gamma} B^{*-\gamma}(B - B^*). \tag{45}$$

Therefore, for any admissible strategy (c, θ, B),

$$\begin{aligned}
&E[\int_0^\infty e^{-(\rho+\delta)t} \left(\frac{(K^{R_t} c_t)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t)^{1-\gamma}}{1-\gamma} \right) dt] \\
&\leq \quad E[\int_0^\infty e^{-(\rho+\delta)t} \left(\frac{(K^{R_t} c_t^*)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t^*)^{1-\gamma}}{1-\gamma} \right) dt] \\
&\quad + \frac{x_0}{y_0^\gamma} E[\int_0^\infty \xi_t (c_t + \delta B_t - (1-R_t)y_t - c_t^* - \delta B_t^* + (1-R_t)y_t) dt] \\
&\leq \quad E[\int_0^\infty e^{-(\rho+\delta)t} \left(\frac{(K^{R_t} c_t^*)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t^*)^{1-\gamma}}{1-\gamma} \right) dt],
\end{aligned} \tag{46}$$

where the first inequality follows from the support equations (44) and (45) and the form of the candidate optimum strategies c^*, B^* and W^* in (11), (13) and (10), and the second inequality follows from pricing (43) for all strategies and equality for the candidate optimum. This says that the candidate optimum dominates all other feasible strategies. We showed previously that the candidate optimum is feasible, so it must indeed be optimal.♣

PROOF OF THEOREM 2:

[4] If a local martingale is bounded below by a martingale, the local martingale minus the martingale is a local martingale that is bounded below and is therefore a supermartingale. The local martingale is the martingale plus the difference that is a supermartingale, and is therefore a supermartingale.

If $R^0 = 1$, then Problem 1 and Problem 2 are identical. Therefore the optimality of the candidate strategy follows from Theorem 1. We will take this as given and assume w.l.o.g. from now on that $R^0 = 0$. It is straightforward to verify that $W_t^*, c_t^*, \theta_t^*, R_t^*$ satisfy the budget constraint (2). In addition, it can be directly verified that $\varphi_x(x,0) \geq \frac{1}{\beta_2}$ and $\varphi_x(x,1) < 0$ and thus $W_t^* \geq -(1-R_t^*)\frac{1}{\beta_2}y_t$ for all $t \geq 0$.

Direct differentiation shows that, for all $x > 0$, we have $\varphi_{xx}(x,0) > 0$, $\varphi_{xx}(x,1) > 0$, and $\varphi(x,0) \geq \varphi(x,1)$, with equality for $x \leq \underline{x}$. Define $\overline{W} = -y\varphi_x(\underline{x},0)$. Then for all $W \geq 0$.

$$v(W,y,0) \geq v(W,y,1), \tag{47}$$

with equality for $W \geq \overline{W}$.[5]

By Doob's optional sampling theorem, without loss of generality, we can restrict to the set of admissible policies that implement the optimal policy stated in Theorem 1 after retirement. Accordingly, define

$$\begin{aligned} M_t &= \int_0^t e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + v(W_s,y_s,1)dR_s\right] \\ &\quad +(1-R_t)e^{-(\rho+\delta)t}v(W_t,y_t,0). \end{aligned} \tag{48}$$

Applying the generalized Itô's lemma, we have

$$\begin{aligned} M_t &= M_0 + \int_0^t (1-R_s)\left[e^{-(\rho+\delta)s}\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right) + E\left[d\left(e^{-(\rho+\delta)s}v(W_s,y_s,0)\right)\right]\right]ds \\ &\quad + \int_0^t e^{-(\rho+\delta)s}(v(W_s,y_s,1)-v(W_s,y_s,0))dR_s \\ &\quad + \int_0^t (1-R_s)e^{-(\rho+\delta)s}(v_W(W_s,y_s,0)\theta_s^\top\sigma + y_s v_y(W_s,y_s,0)\sigma_y^\top)dZ_s. \end{aligned} \tag{49}$$

By (25), (26), (22), some algebra shows that the first integral is always nonpositive for all admissible policy (c,B,θ,R) and is equal to zero for the candidate policy (c^*,B^*,θ^*,R^*). By (47), the third term in (49) is always nonpositive for all admissible retirement policy R_t and equal to zero for the candidate policy R_t^*. In addition, using the expressions for the candidate θ_t^*, (25), (26), and (22), we can show that the last integral is a martingale because both y_t and x_t are geometric Brownian motions. This shows that M_t is a local supermartingale for all admissible policy and a martingale for the candidate policy.

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[5]This can be shown as follows: Let x and x^R be such that $-y\varphi_x(x,0) = W$ and $-y\varphi_x(x^R,1) = W$. Then we have

$$\varphi(x,0)-\varphi(x^R,1) \geq \varphi(x,1)-\varphi(x^R,1) \geq \varphi_x(x^R,1)(x-x^R) = x\varphi_x(x,0) - x^R\varphi_x(x^R,1),$$

where the first inequality follows from $\varphi(x,0) \geq \varphi(x,1)$ and the second inequality from the convexity of $\varphi(x,1)$. After rearranging, we obtain (47).

Since $(1-\gamma)v(W,y,0) \geq 0$,[6] for the candidate policy, we have

$$\begin{aligned} 0 &\leq \lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}(1-\gamma)v(W_t,y_t,0)] \\ &= \lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}y_t^{1-\gamma}(1-\gamma)(\varphi(x_t,0)-x_t\varphi_x(x_t,0))] \\ &\leq \lim_{t\to\infty} E[e^{-(\rho+\delta)t}y_t^{1-\gamma}((1-\gamma)A_+(1-\alpha_-)x^{\alpha_-}+\hat{\eta}x_t^b)] \\ &\leq \lim_{t\to\infty} E[L_1 e^{-(\rho+\delta)t}y_t^{1-\gamma}+\hat{\eta}e^{-(\rho+\delta)t/\gamma}\xi_t^b] \\ &= 0, \end{aligned} \tag{50}$$

where the second inequality follows from the fact that $R_t = 1$ if $t \geq \tau^*$, the third inequality follows from $A_+ > 0$, $\alpha_- < 0$, $x_t > \underline{x}\ \forall t < \tau^*$, and the last equality in (50) follows from the conditions that $\beta_3 > 0$ and $\nu > 0$.

Therefore, for the candidate policy, taking expectation and taking limit as $t \to \infty$ in (49), we obtain $M_0 = \lim_{t\to\infty} E[M_t]$, i.e.,

$$v(W^0,y^0,0) = E\int_0^\infty e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{(c_s^*)^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s^*)^{1-\gamma}}{1-\gamma}\right)ds + v(W_s^*,y_s,1)dR_s^*\right]. \tag{51}$$

If $\gamma < 1$, then M_t is always nonnegative and thus a supermartingale. This implies that $M_0 \geq E[M_t]$, i.e.,

$$\begin{aligned} & v(W^0,y^0,0) \\ \geq\ & E\int_0^t e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + v(W_s,y_s,1)dR_s\right] \\ & +E[(1-R_t)e^{-(\rho+\delta)t}v(W_t,y_t,0)] \\ \geq\ & E\int_0^t e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + v(W_s,y_s,1)dR_s\right], \end{aligned} \tag{52}$$

where the last inequality holds because $v(W,y,0) \geq 0$ in this case. Taking limit as $t \to \infty$, we conclude that the candidate policy (c^*,B^*,θ^*,R^*) is optimal when $\gamma < 1$.

If $\gamma > 1$, consider an investor who has an initial endowment of $(W^0,(1+\varepsilon)y^0)$ $(\varepsilon > 0)$ but follows the same strategy (c,B,θ,R) for an investor who has an initial endowment of (W^0,y^0) until retirement and follows the optimal strategy given the implied wealth afterwards. Let the implied wealth process be W_t^ε, which converges to W_t as $\varepsilon \to 0$. By (49), there exists a series of stopping times $\tau_n \to \infty$ such that

$$v(W^0,(1+\varepsilon)y^0,0)$$

[6] This can be shown as follows: $v_W(W,y,0) = y^{-\gamma}x > 0$ and thus $v(W,y,0)$ increases in W. If $\gamma < 1$, then $v(\underline{W},y,0) \geq 0$ because $v(W,y,0) \geq v(W,y,1) \geq 0$. If $\gamma > 1$, then $v(W,y,0) < 0$ because $v(W,y,0) \leq v(\overline{W},y,0) = v(\overline{W},y,1) < 0$.

$$\begin{aligned} &\geq \; E\int_0^{\tau_n\wedge t} e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds+v(W_s^{\varepsilon},(1+\varepsilon)y_s,1)dR_s\right] \\ &\quad +E[(1-R_{\tau_n\wedge t})e^{-(\rho+\delta)(\tau_n\wedge t)}v(W_{\tau_n\wedge t}^{\varepsilon},(1+\varepsilon)y_{\tau_n\wedge t},0)]. \end{aligned} \tag{53}$$

Since the integrand in the integral of (53) is always negative, this integral is monotonically decreasing in time. In addition,

$$\begin{aligned} 0 &\geq (1-R_t)e^{-(\rho+\delta)t}v(W_t^{\varepsilon},(1+\varepsilon)y_t,0) \\ &\geq e^{-(\rho+\delta)t}v(-\frac{y_t}{\beta_2},(1+\varepsilon)y_t,0) \\ &\geq v(-\frac{1}{\beta_2},1+\varepsilon,0)\sup_{t\in[0,\infty)}\left(e^{-(\rho+\delta)t}{y_t}^{1-\gamma}\right), \end{aligned} \tag{54}$$

where the second inequality follows from the borrowing constraint given an initial income of y^0 and $v(W,y,0)$ increasing in W and the last inequality follows from the form of $v(W,y,0)$ and $v(-\frac{1}{\beta_2},1+\varepsilon,0)<0$. In addition, it can be shown that $v(-\frac{1}{\beta_2},1+\varepsilon,0)>-\infty$. The supremum in (54) is in L^1, as can be shown by standard methods under the assumption of $\beta_3>0$. One way to prove this is to set $x=y$ in equation 1.8.8 of Harrison [1985] and take the limit as $t\uparrow\infty$ (which converges because the distribution function is the expectation of the indicator function of a shrinking set of states). This allows us to compute exactly the density for the log of the residual term and also the finite expectation of the residual terms.

Therefore, taking $n\to\infty$ in (53), by the monotone convergence theorem for the first term and the dominated convergence theorem for the second term, we have

$$\begin{aligned} &v(W^0,(1+\varepsilon)y^0,0) \\ &\geq \; E\int_0^{t} e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds+v(W_s^{\varepsilon},(1+\varepsilon)y_s,1)dR_s\right] \\ &\quad +E[(1-R_t)e^{-(\rho+\delta)t}v(W_t^{\varepsilon},(1+\varepsilon)y_t,0)]. \end{aligned} \tag{55}$$

Moreover, since $\beta_3>0$, we have $\lim_{t\to\infty}E[e^{-(\rho+\delta)t}{y_t}^{1-\gamma}]=0$. Therefore, taking limit as $t\to\infty$ in (55), by (54) we obtain

$$\begin{aligned} &v(W^0,(1+\varepsilon)y^0,0) \\ &\geq \; E\int_0^{\infty} e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds+v(W_s^{\varepsilon},(1+\varepsilon)y_s,1)dR_s\right] \end{aligned} \tag{56}$$

Finally, taking limit as $\varepsilon\to 0$, we have

$$v(W^0,y^0,0)\geq E\int_0^{\infty} e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds+v(W_s,y_s,1)dR_s\right]. \tag{57}$$

Therefore the arbitrary alternative strategy is dominated by our candidate optimum. This completes the proof of Theorem 2. ♣

Before proving Theorem 3, we list some results in the following lemma that are used to in the proof. Let

$$\psi(x) \equiv A_+ x^{\alpha_-} + A_- x^{\alpha_+} - \eta \frac{x^b}{b} + \frac{1}{\beta_2} x$$

and

$$\hat{\psi}(x) \equiv -\hat{\eta} \frac{x^b}{b},$$

where A_+ and A_- are as defined in Theorem 3.

Lemma 1 *Suppose* $\nu > 0$, $\beta_2 > 0$, *and* $\beta_3 > 0$. *Suppose there exists a solution* $\zeta \in (0,1)$ *to equation (30). Then*

(i). $\hat{\psi}(x)$ *is strictly convex and strictly decreasing for* $x \geq 0$*;*

(ii). $\forall x \leq \bar{x}$ *we have* $\psi(x) \geq \hat{\psi}(x)$, $\forall x \in [\underline{x}, \bar{x}]$ *we have* $\psi_x(x) \geq \hat{\psi}_x(x)$ *and*

$$\underline{x} < \left(\frac{1 - K^{-b}}{b} \right)^{\gamma}. \tag{58}$$

(iii).

$$A_- < 0, \ \ A_+ > 0, \ \ \bar{x} > \left(\frac{(1 - \alpha_-)(1 + \delta k^{-b})}{b - \alpha_-} \right)^{\gamma}.$$

(iv). $\psi(x)$ *is strictly convex and strictly decreasing for all* $x < \bar{x}$*.*

PROOF OF LEMMA 1: (i). $\gamma > 0$ implies that $b = 1 - 1/\gamma < 1$. Then since $\nu > 0$, direct differentiation shows that φ^R is strictly convex and strictly decreasing for $x > 0$.

(ii). First, since $\nu > 0$, $\beta_2 > 0$, and $\beta_3 > 0$, it is straightforward to show that

$$\alpha_+ > 1 > b > \alpha_-, \ \ \alpha_- < 0. \tag{59}$$

Define

$$h(x) \equiv \psi(x) - \hat{\psi}(x).$$

It can be easily verified that

$$\frac{1}{2} \beta_1 x^2 \hat{\psi}_{xx}(x) - (\beta_2 - \beta_3) x \hat{\psi}_x(x) - \beta_3 \hat{\psi}(x) - (K^{-b} + \delta k^{-b}) \frac{x^b}{b} = 0, \tag{60}$$

and

$$\frac{1}{2} \beta_1 x^2 \psi_{xx}(x) - (\beta_2 - \beta_3) x \psi_x(x) - \beta_3 \psi(x) - (1 + \delta k^{-b}) \frac{x^b}{b} + x = 0, \tag{61}$$

with

$$\psi(\underline{x}) = \hat{\psi}(\underline{x}), \tag{62}$$

$$\psi_x(\underline{x}) = \hat{\psi}_x(\underline{x}), \tag{63}$$

$$\psi_x(\overline{x}) = 0 \tag{64}$$

and

$$\psi_{xx}(\overline{x}) = 0. \tag{65}$$

Then by (60) and (61), $h(x)$ must satisfy

$$\frac{1}{2}\beta_1 x^2 h'' - (\beta_2 - \beta_3) x h' - \beta_3 h = \frac{1-K^{-b}}{b} x^b - x. \tag{66}$$

By (62)-(64) and the fact that $\hat{\psi}(x)$ is monotonically decreasing for $x > 0$, we have

$$h(\underline{x}) = 0, \;\; h'(\underline{x}) = 0, \;\; h'(\overline{x}) > 0. \tag{67}$$

Differentiating (66) once, we obtain

$$\frac{1}{2}\beta_1 x^2 h''' + (\beta_1 - \beta_2 + \beta_3) x h'' - \beta_2 h' = (1-K^{-b}) x^{b-1} - 1. \tag{68}$$

We consider two possible cases.

Case 1: $(1-K^{-b})\underline{x}^{b-1} - 1 < 0$. In this case, the RHS of equation (68) is negative. Since $\beta_2 > 0$, $h'(x)$ cannot have any interior nonpositive minimum. To see this, suppose $\hat{x} \in (\underline{x}, \overline{x})$ achieves an interior minimum with $h'(\hat{x}) \leq 0$. Then we would have $h'''(\hat{x}) > 0$ and $h''(\hat{x}) = 0$, which implies that the LHS is positive. A contradiction. Since $h'(\underline{x}) = 0$, $h'(\overline{x}) > 0$, we must have $h'(x) > 0$ for all $x \in (\underline{x}, \overline{x}]$ because otherwise there would be an interior nonnegative minimum. Then the fact that $h(\underline{x}) = 0$ implies that $h(x) > 0$ for all $x \in (\underline{x}, \overline{x}]$. Since $h'(x) > 0$ for all $x \in (\underline{x}, \overline{x}]$, we must have $h''(\underline{x}) > 0$ by (67). Then (66), (67) and $h''(\underline{x}) > 0$ imply that

$$\underline{x} < \left(\frac{1-K^{-b}}{b}\right)^{\gamma},$$

Case 2: $(1-K^{-b})\underline{x}^{b-1} - 1 \geq 0$. In this case, we must have $0 < b < 1$ because $K > 1$ and therefore $\underline{x} \leq \left(1-K^{-b}\right)^{\gamma} < \left(\frac{1-K^{-b}}{b}\right)^{\gamma}$. This implies that $h''(\underline{x}) > 0$ by (66) and (67). Therefore there exists a $\varepsilon > 0$ such that $h'(x) > 0$ for all $x \in (\underline{x}, \underline{x} + \varepsilon]$ because $h'(\underline{x}) = 0$. The RHS of equation (68) is monotonically decreasing in x. Let x^* be such that the RHS is 0. Then for all $x \leq x^*$, the RHS is nonnegative and thus $h'(x)$ cannot have any interior nonnegative (local) maximum in $[\underline{x}, x^*]$ for similar reasons to those in Case 1. Thus there cannot exist any $\hat{x} \in (\underline{x} + \varepsilon, x^*)$ such that $h'(\hat{x}) \leq 0$. If $x^* < \overline{x}$, then for all $x \in (x^*, \overline{x}]$, the RHS is nonpositive and thus $h'(x)$ cannot have any interior nonpositive (local) minimum in $(x^*, \overline{x}]$. Thus there cannot exist any $\hat{x} \in (x^*, \overline{x}]$ such that $h'(\hat{x}) \leq 0$. Therefore, there cannot exist any $\hat{x} \in (\underline{x}, \overline{x})$ such that $h'(\hat{x}) \leq 0$ and thus we have $h'(x) > 0$ and $h(x) > 0$ for all $x \in (\underline{x}, \overline{x}]$.

Now we show for both cases, $h(x) > 0$ for all $x < \underline{x}$. (58) implies that the RHS of (66) is positive for $x < \underline{x}$ and h cannot achieve an interior positive maximum for $x < \underline{x}$. On the other hand, $h''(\underline{x}) > 0$ and $h'(\underline{x}) = 0$ imply that there exists an $\varepsilon > 0$ such that

$$\forall x \in [\underline{x} - \varepsilon, \underline{x}], \;\; h'(x) < 0. \tag{69}$$

Thus $\forall x \in [\underline{x} - \varepsilon, \underline{x})$, $h(x) > 0$. Therefore $\forall x < \underline{x}$, $h(x) > 0$. Otherwise h would achieve an interior positive maximum in $(0, \underline{x})$.

(iii). It can be shown that

$$A_+ = \frac{(\eta - \hat{\eta})(\alpha_+ - b)}{b(\alpha_+ - \alpha_-)} \underline{x}^{b-\alpha_-} - \frac{(\alpha_+ - 1)}{(\alpha_+ - \alpha_-)\beta_2} \underline{x}^{1-\alpha_-}$$

and

$$\eta = \frac{(\alpha_+ - 1)(1 - \alpha_-)(1 + \delta k^{-b})}{(\alpha_+ - b)(b - \alpha_-)\beta_2}. \tag{70}$$

(58) then implies that $B > 0$. Since we also have

$$A_+ = \frac{\eta(\alpha_+ - b)}{\alpha_-(\alpha_+ - \alpha_-)} \overline{x}^{b-\alpha_-} - \frac{\alpha_+ - 1}{\alpha_-(\alpha_+ - \alpha_-)\beta_2} \overline{x}^{1-\alpha_-} > 0,$$

$\overline{x}$ must satisfy

$$\overline{x} > \left(\frac{\eta(\alpha_+ - b)\beta_2}{\alpha_+ - 1} \right)^{\gamma} = \left(\frac{(1 - \alpha_-)(1 + \delta k^{-b})}{b - \alpha_-} \right)^{\gamma}.$$

Since

$$\frac{\alpha_+ - b}{\alpha_+ - b} > \frac{b - \alpha_-}{1 - \alpha_-},$$

we have

$$\overline{x} > \left(\frac{\eta(b - \alpha_-)\beta_2}{1 - \alpha_-} \right)^{\gamma},$$

which implies that $A_- < 0$. (iv). $A_- < 0$, $A_+ > 0$ and (59) imply that

$$A_-\alpha_+(\alpha_+ - 1)(\alpha_+ - b)x^{\alpha_+ - b - 1} + A_+\alpha_-(\alpha_- - 1)(\alpha_- - b)x^{\alpha_- - b - 1} < 0,$$

which in turn implies that $\forall x < \overline{x}$,

$$\begin{aligned}
\psi_{xx}(x) &= (A_-\alpha_+(\alpha_+ - 1)x^{\alpha_+ - b} + A_+\alpha_-(\alpha_- - 1)x^{\alpha_- - b} - \eta(b-1))x^{b-2} \\
&> (A_-\alpha_+(\alpha_+ - 1)\overline{x}^{\alpha_+ - b} + A_+\alpha_-(\alpha_- - 1)\overline{x}^{\alpha_- - b} - \eta(b-1))x^{b-2} = 0,
\end{aligned}$$

where the last equality follows from $\psi_{xx}(\overline{x}) = 0$. Thus $\psi(x)$ is strictly convex $\forall x < \overline{x}$. Since $\psi_x(\overline{x}) = 0$ and $\forall x < \overline{x}$, $\psi_{xx}(x) > 0$, we must also have $\forall x < \overline{x}$, $\psi_x(x) < 0$. ♣

PROOF OF THEOREM 3:

If $R^0 = 1$, then Problem 1 and Problem 3 are identical. Therefore the optimality of the candidate strategy follows from Theorem 1. We will take this as given and assume w.l.o.g. from now on that $R^0 = 0$. It is straightforward to verify that $W_t^*, c_t^*, \theta_t^*, R_t^*$ satisfy the budget constraint (2). In addition, since $\varphi(x,0)$ and $\varphi(x,1)$ are strictly decreasing, $W_t^* \geq 0$ for all $t \geq 0$.

Similar to the proof of Theorem 2, it can be shown that

$$v(W,y,0) \geq v(W,y,1), \tag{71}$$

with equality for $W \geq \overline{W}$.

Define

$$\begin{aligned} M_t &= \int_0^t e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + v(W_s,y_s,1)dR_s\right] \\ &\quad +(1-R_t)e^{-(\rho+\delta)t}v(W_t,y_t,0). \end{aligned} \tag{72}$$

Applying the generalized Itô's lemma, we have

$$\begin{aligned} M_t &= M_0 + \int_0^t (1-R_s)\left\{e^{-(\rho+\delta)s}\left(\frac{c_s^{1-\gamma}}{1-\gamma}+\delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)+E\left[d\left(e^{-(\rho+\delta)s}v(W_s,y_s,0)\right)\right]\right\}ds \\ &\quad + \int_0^t e^{-(\rho+\delta)s}(v(W_s,y_s,1)-v(W_s,y_s,0))dR_s \\ &\quad + \int_0^t (1-R_s)e^{-(\rho+\delta)s}(v_W(W_s,y_s,0)\theta_s^\top\sigma + y_s v_y(W_s,y_s,0)\sigma_y^\top)dZ_s, \end{aligned} \tag{73}$$

By (33), (34), (29), and the fact that $\varphi(x,0)$ satisfies (60)-(65), we obtain that the first integral is always nonpositive for all admissible policy (c,B,θ,R) and is equal to zero for the candidate policy (c^*,B^*,θ^*,R^*). By (71), the third term in (73) is always nonpositive for all admissible retirement policy R_t and equal to zero for the candidate policy R_t^*. In addition, using the expressions for the candidate θ_t^*, (33), (34), and (29), it is easy to see that under the candidate policy, the stochastic integral is a martingale because y_t is a geometric Brownian motion, x_t is bounded between $\underline{x}$ and $\bar{x}$ before retirement, and x_t is also a geometric Brownian motion after retirement. This shows that M_t is a super local martingale for all admissible policy and a martingale for the candidate policy.

Since $(1-\gamma)v(W,y,0) \geq 0$ for any admissible policy,[7] we have

$$\begin{aligned} 0 &\leq \lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}(1-\gamma)v(W_t,y_t,0)] \\ &= \lim_{t\to\infty} E[(1-R_t)e^{-(\rho+\delta)t}y_t^{1-\gamma}(1-\gamma)(\varphi(x_t,0)-x_t\varphi_x(x_t,0))] \\ &\leq \lim_{t\to\infty} E[L_1 e^{-(\rho+\delta)t}y_t^{1-\gamma}] \\ &= 0, \end{aligned} \tag{74}$$

[7] This can be shown as in Theorem 2.

where the second inequality follows from the fact that if $t < \tau^*$, then x_t, $\varphi(x_t)$, and $\varphi_x(x_t)$ are all bounded, while $R_t = 1$ for $t > \tau^*$. The last equality in (74) follows from the conditions that $\beta_3 > 0$.

Therefore, for the candidate policy, taking expectation and taking limit as $t \to \infty$ in (73), we obtain $M_0 = \lim_{t\to\infty} E[M_t]$, i.e.,

$$v(W^0,y^0,0) = E\int_0^\infty e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{(c_s^*)^{1-\gamma}}{1-\gamma}ds + \delta\frac{(kB_s^*)^{1-\gamma}}{1-\gamma}\right)ds + v(W_s^*,y_s,1)dR_s^*\right]. \tag{75}$$

If $\gamma < 1$, then M_t is always nonnegative and thus a supermartingale. This implies that $M_0 \geq E[M_t]$, i.e.,

$$\begin{aligned} & v(W^0,y^0,0) \\ & \geq E\int_0^t e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + v(W_s,y_s,1)dR_s\right] \\ & \quad + E[(1-R_t)e^{-(\rho+\delta)t}v(W_t,y_t,0)] \\ & \geq E\int_0^t e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + v(W_s,y_s,1)dR_s\right], \end{aligned} \tag{76}$$

where the last inequality holds because $v(W,y,0) \geq 0$ in this case. Taking limit as $t \to \infty$, we conclude that the candidate policy (c^*,B^*,θ^*,R^*) is optimal when $\gamma < 1$.

If $\gamma > 1$, the integrand and all the terms in the definition (72) of M_t are negative. We know M is a local supermartingale and we will use the fact that is is bounded in L^1 in the interesting cases to show it is a supermartingale.

Start with the integral term in M. Let $\tilde{u}$ be the limit as $t \uparrow \infty$ of the integral in the (72); $E[\tilde{u}]$ is the expected utility for the strategy. Any useful strategy has finite expected utility (not $-\infty$) or else it would be clearly dominated by our candidate optimum. Therefore, we will assume without loss of generality that $0 \geq E[\tilde{u}] > -\infty$ is the expected utility from the strategy; since the integrand is negative, convergence of the integral to $\tilde{u}$ is monotone decreasing and $\tilde{u}$ is a uniform L^1 lower bound on the integral as a process.

The "residual" (or transversality condition) term in M can also be bounded in L^1. We have that

$$\begin{aligned} 0 & \geq (1-R_t)e^{-(\rho+\delta)t}v(W_t,y_t,0) \\ & \geq e^{-(\rho+\delta)t}v(0,y_t,0) \\ & = v(0,1,0)e^{-(\rho+\delta)t}{y_t}^{1-\gamma} \\ & \geq v(0,1,0)\sup_{t\in[0,\infty)}\left(e^{-(\rho+\delta)t}{y_t}^{1-\gamma}\right) \end{aligned} \tag{77}$$

where the second inequality follows from v negative and increasing in W, the equality follows from the form of v as defined by (33) and (34), and the last inequality follows from $v(0,1,0) < 0$. The supremum in (77) is in L^1, as can be shown by standard methods under the assumption of $\beta_3 > 0$. One way to prove this is to set $x = y$ in equation 1.8.8 of Harrison [1985] and take the limit as $t \uparrow \infty$ (which converges because the distribution function is the expectation of the indicator function of a shrinking set of states). This allows us to compute exactly the density for the log of the residual term and also the finite expectation of the residual terms.

So, in the case $\gamma > 1$, for any strategy with bounded utility the process M_t is a local supermartingale bounded in L^1 between 0 and the sum of $\tilde{u}$ and the right-hand-side of (77). Therefore M_t is a supermartingale and we have that

$$v(W^0, y^0, 0) \geq E\int_0^\infty e^{-(\rho+\delta)s}\left[(1-R_s)\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \delta\frac{(kB_s)^{1-\gamma}}{1-\gamma}\right)ds + v(W_s, y_s, 1)dR_s\right]$$

where we have taken the limit as $t \uparrow \infty$ and used the fact from (74) that the residual term vanishes in this limit. Therefore the arbitrary alternative strategy is dominated by our candidate optimum. This completes the proof of Theorem 3. ♣

Lemma 2 *Suppose* $\nu > 0$, $\beta_2 > 0$, *and* $\beta_3 > 0$. *Then there exists a unique solution* $\zeta^* \in [0,1]$ *to equation (30) and*

$$\zeta^* < \overline{\zeta} = Min(\left(\frac{1-K^{-b}}{b(1+\delta k^{-b})}\right)^\gamma, 1).$$

PROOF OF LEMMA 2: Since $\nu > 0$, $\beta_2 > 0$, and $\beta_3 > 0$,

$$\alpha_+ > 1 > b > \alpha_-, \ \ \alpha_- < 0.$$

Next, since $\zeta^{b-\alpha_+}$ dominates $\zeta^{1-\alpha_+}$ as $\zeta \to 0$, we have

$$\lim_{\zeta\to 0} q(\zeta) = \lim_{\zeta\to 0} -\frac{1-K^{-b}}{b(1+\delta k^{-b})}(\alpha_- - b)(\alpha_+ - 1)\zeta^{b-\alpha_+} = +\infty.$$

Next, it is easy to verify that

$$q(1) = -\frac{(\alpha_+ - 1)(\alpha_- - 1)(\alpha_+ - \alpha_-)(K^{-b} + \delta k^{-b})}{\alpha_+\alpha_-(1+\delta k^{-b})} < 0.$$

Now suppose $\hat{\zeta} = \left(\frac{1-K^{-b}}{b(1+\delta k^{-b})}\right)^\gamma < 1$. Then we have $\frac{1-K^{-b}}{b(1+\delta k^{-b})}\hat{\zeta}^{b-\alpha_+} - \frac{1}{\alpha_+} = \hat{\zeta}^{1-\alpha_+} - \frac{1}{\alpha_+}$, $\frac{1-K^{-b}}{b(1+\delta k^{-b})}\hat{\zeta}^{b-\alpha_-} - \frac{1}{\alpha_-} = \hat{\zeta}^{1-\alpha_-} - \frac{1}{\alpha_-}$, and $\hat{\zeta}^{1-\alpha_+} > 1 > \frac{1}{\alpha_+}$. It follows that

$$q(\hat{\zeta}) = -\frac{1}{\gamma}(\hat{\zeta}^{1-\alpha_+} - \frac{1}{\alpha_+})(\hat{\zeta}^{1-\alpha_-} - \frac{1}{\alpha_-})(\alpha_+ - \alpha_-) < 0.$$

Then by continuity of q, there exists a solution $\zeta^* \in (0, \overline{\zeta})$ such that $q(\zeta^*) = 0$. Suppose there exists another solution $\hat{\zeta} \in [0,1]$ such that $q(\hat{\zeta}) = 0$. Let $v(W,y,0)$ and $\overline{W}$ be the value function and boundary respectively corresponding to ζ^* and $\hat{v}(W,y,0)$ and $\hat{\overline{W}}$ be the value function and boundary respectively corresponding to $\hat{\zeta}$. Without loss of generality, suppose $\overline{W} > \hat{\overline{W}}$. Since $\hat{\overline{W}}$ is the retirement boundary, the value function corresponding to $\hat{\zeta}$ for $\overline{W} > W > \hat{\overline{W}}$ is equal to $v(W,y,1)$. However, Lemma 1 implies that $v(W,y,0) > v(W,y,1)$ for all $W < \overline{W}$. This implies that $\hat{\overline{W}}$ cannot be the optimal retirement boundary which contradicts Theorem 4. Therefore the solution to equation (30) is unique. ♡

PROOF OF PROPOSITION 1. Given the optimal policies of the investor, the present value of the investor's human capital at t is

$$H(x_t, y_t, t) = \xi_t^{-1} E[\int_t^{T_R} \xi_s y_s ds],$$

where $T_R = T$ for the fixed retirement date case and $T_R = \tau^*$ for the discretionary retirement case.

(i) for the fixed retirement date we have the HJB equation

$$H_t + \frac{1}{2}\sigma_x^2 x^2 H_{xx} + \mu_x x H_x + \frac{1}{2}\sigma_y^2 y^2 H_{yy} + \mu_y y H_y + \sigma_y^\top \sigma_x xy H_{xy} - (x\sigma_x^\top H_x + y\sigma_y^\top H_y)\kappa - (r+\delta)H + y = 0, \tag{78}$$

subject to

$$H(x_T, y_T, T) = 0.$$

It can be verified that the first H function solves this PDE. In addition, the diffusion term $\int_0^T (x_t H_x \sigma_x + y_t H_y \sigma_y - \xi_t \kappa^\top) dB_s$ is a martingale since both y_t and x_t are geometric Brownian motions.

(ii) for the cases with discretionary retirement, we have the same HJB equation (78). It can be shown that the general solution for h is

$$H(x_t, y_t, t) = \frac{w}{r + \delta - \mu_y + \sigma_y^\top \kappa} y_t (C_1 x_t^{\alpha_- - 1} + C_2 x_t^{\alpha_+ - 1} + 1)\iota\{x_t \geq \underline{x}\},$$

where C_1 and C_2 are integration constants to be determined. Since at the retirement date the present value of future human capital is zero, we have the boundary condition

$$H(\underline{x}, y, t) = 0. \tag{79}$$

If there is no borrowing constraint, then we must have $C_2 = 0$. The boundary condition (79) implies the second function for H. If there is a borrowing constraint, then by applying Itô's lemma to H, we have another boundary condition

$$H_x(\bar{x}, y, t) = 0. \tag{80}$$

Conditions (79) and (80) then implies $C_1 = A$ and $C_1 = B$. ♣

PROOF OF PROPOSITION 2. Recall that

$$dx_t = \mu_x x_t + \sigma_x x_t dB_t.$$

Let

$$g(x) = E[\tau^*|x_t = x].$$

Then by Itô's lemma, g must satisfy

$$\frac{1}{2}\sigma_x x^2 g_{xx} + \mu_x x f_x + 1 = 0,$$

subject to

$$g(\underline{x}) = 0. \tag{81}$$

The solution is

$$g(x) = C_1(x^{1-2\mu_x/\sigma_x^2} - \underline{x}^{1-2\mu_x/\sigma_x^2}) - \frac{\log(x/\underline{x})}{\mu_x - \frac{1}{2}\sigma_x^2}.$$

In the absence of borrowing constraint, we must have $C_1 = 0$ for the diffusion term in the dynamics of $g(x_t)$ to be a martingale, which gives the first expression in the proposition. In the presence of borrowing constraint, we have a second boundary condition

$$g'(\overline{x}) = 0,$$

which yields the second expression. ♣

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