Economic Growth, Business Cycles, and Expected Stock Returns

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Abstract

This article presents a dynamic general equilibrium model which jointly accounts for main asset pricing phenomena of the time series and the cross section. Learning about changes in long-run output growth leads to time-variation in risk premia. Covariance with GDP growth through business cycles determines a countercyclical equity premium. Countercyclical value (size) spreads and value (size) premia arise in compensation for systematic differences in the time-varying exposure of firm fundamentals to business cycle fluctuations. The Fama and French (1993) factors HML and SMB are shown to capture firms' relative exposure to recessionary periods of the economy. The model extends the consumption-based CAPM (CCAPM) to an output-based CAPM (YCAPM).

Key words: asset pricing, general equilibrium, learning, macroeconomic risk, time-varying risk premia

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1 Introduction

The "failure" of the CAPM, more precisely, its inability to account for the cross section of average stock returns, has probably remained one of the main drivers behind research in asset pricing. Since Fama and French (1993), many empirically successful asset pricing models have been proposed. However, much less has been said when it comes to providing economically founded explanations for the asset pricing anomalies detected.

What is the economic intuition behind the Fama and French (1993) factors? What risk factors can possibly explain their existence? What would candidate explanatory factors imply for standard asset pricing models? Should we rethink common definitions of risk?

While aggregate macroeconomic risk is generally accepted to be the source of risk premia in asset markets, so far, theoretical asset pricing models have restricted its definition to the variability of consumption. However, consumption is just one of many macroeconomic time series. Moreover, it happens to be one of the smoothest, making it not the most favorable choice, given the size of risk premia that need to be explained[1].

Concerning the relation of risk premia to alternative macroeoconomic aggregates, Rangvid (2006), for example, provides evidence on the relation between expected returns and output. He shows that the ratio of share price to GDP captures more of the variation over time in expected returns on the aggregate market, than do ratios of price-to-earnings or price-to-dividends. Cooper and Priestley (2007) demonstrate that the output gap is a predictor of returns on stocks as well as bonds. Interestingly, there also exists a link between output and returns in the cross section: Liew and Vassalou (2000) show that the Fama and French (1993) factors forecast GDP growth.

The present article develops an asset pricing model relating expected returns to variations in output growth. The proposed dynamic general equilibrium model can rationalize risk premia in the time series, as well as the cross section, based on firms' exposure to variability in GDP growth. Risk premia obtain as a function of the covariation of firm fundamentals with changes in aggregate output. As fundamentals of small and value firms are more adversely affected by recessionary periods of the

[1] see Hansen and Singleton (1983), Mehra and Prescott (1985), or Grossman, Melino, and Shiller (1987), for example, for the literature on the "equity premium puzzle"

economy[2], this has implications for the cross section. Value and size premia arise in compensation for a comparatively higher exposure of firm fundamentals to negative business cycle shocks. As the model operates in an environment of incomplete information, the agent continuously learns about exposures inferring from firm-level productivity observations. The dynamic structure of the model then allows generating countercyclical risk premia, reflecting firms' time-varying exposure to changes in GDP growth.

The model is constructed to combine a latent two-state Markov Switching Model with i.i.d. shocks to the underlying state variable. The two regimes are defined through GDP growth. The latter can be either high, should the economy be in an expansion or low, in a contraction. The agent however, cannot observe the state of the economy and has to draw inferences from productivity observations at the firm level. Firm productivity is modeled to equate GDP growth[3] supplemented by idiosyncratic shocks, characterized by a mean zero underlying distribution. However, consistent with the above mentioned bias in the reaction of fundamentals of small and value firms to recessions, the mean value of the underlying distribution exhibits a negative bias when defined cross-sectionally.

The agent evaluates firms in order to optimally allocate capital. Observing productivity[4] at the firm level, she simultaneously learns about firms' business cycle exposure, as well as the overall state of the economy, when considering her observations in the aggregate. The agent's belief about the time-varying state of the economy then leads to a countercyclical equity premium.

In the cross section, learning takes two effects. First, learning is at the source of the emergence of value (size) premia, as well as, value (size) spreads, a systematic difference in the average market-to-book ratio of value and growth firms. The size spread is commonly defined as the difference in market-to-book ratios between small and large firms. In the present model, it can also be understood as the difference in average market values of these firms. Second, just as in the time series, time-varying beliefs about the state of the economy lead to countercyclicalities. Hence, the model endogenizes the cyclical behavior of value and size premia, a stylized fact pointed out by Lettau and Ludvigson (2001), and Campbello, Chen, and Zhang (2006), respectively. It also accounts for cyclicalities in value (size) spreads, as observed by Cohen, Polk and Vuolteenaho (2003)[5].

[2]see Gertler and Gilchrist (1994), who provide evidence for small firms; see Xing and Zhang (2005) for value versus growth firms

[3]for the purpose of simplification the model abstracts from labor markets and the economy is closed

[4]In fact, the stucture of the model is such that productivity equals profitability. The latter might be a more plausible variable of observation for some readers.

[5]Cohen, Polk and Vuolteenaho (2003) provide evidence that the value premium is high when

The emergence of empirical regularities in the cross section is driven by agent's optimizing behavior with respect to time. Within the model, the agent disposes of a limited amount of time to evaluate a continuum of firms. She therefore faces a trade off: learning more on one firm means learning less on the others. As learning reduces uncertainty surrounding firm-specific shocks to productivity, it positively affects agent's expected utility from investment. It can then be shown that due to the cross-sectional bias in the reaction of firm fundamentals to recessions, the marginal utility from learning is strictly higher for small and value firms at all times. Hence, it is optimal for the agent to always learn strictly more on these type of firms[6]. Moreover, the optimal learning bias is increasing in recessions. As the resolution of uncertainty through learning negatively affects valuation ratios[7], countercyclical value and size spreads emerge. Concerning the emergence of risk premia in the cross section, the learning bias takes effect as soon as the agent becomes aware of an underlying systematic bias in fundamentals.

Combining countercyclical value (size) spreads and premia, the model can account for stock return predictability in the cross section[8]. Moreover, it provides an explanation for the predictability of GDP growth by the Fama and French size and book-to-market factors, as pointed out by Liew and Vassalou (2000).

Finally, the model extends the common consumption-based definition of risk (CCAPM) to an output-based definition (YCAPM). Within the model, variation in consumption obtains as a consequence of variation in output. The positive correlation of consumption with output makes it a valuable proxy for macroeconomic risk, but not a risk factor by itself. Equity premia are exclusively generated by the covariation of firms' output with changes in GDP growth. A conditional CAPM representation shows that systematic differences in covariation lead to betas that diverge in the cross section, and are countercyclical. Eventually, the model shows how the conditional CAPM combines in one, what the "unconditional" Fama and French (1993) model splits into three different factor exposures.

Related Literature

the value spread is large and vice versa. As the value premium was found to be countercyclical (Lettau and Ludvigson (2001)), the value spread needs to be countercyclical, too. To the best of my knowledge, there is currently no empirical evidence on the time-series behavior of size spreads.

[6]The current paper takes the empirically observed differences in the reaction of firm fundamentals to recessions as given. Müller (2008b) shows that this higher sensitivity is reflected in systematically higher leverage ratios. The latter might be thought of as causing the aforementioned bias.

[7]see Pástor and Veronesi (2003), who show that the market-to-book ratio is increasing in uncertainty about profitability

[8]The model could also account for return predictability at frequencies higher than business cycles. Regime-switches at business-cycle frequencies, could be combined with i.i.d shocks at higher frequencies, for example, where the cross-sectional bias in mean values of the underlying distribution is defined to be time-varying.

This article is related to the literature on asset pricing and learning[9]. It draws on work by Pástor and Veronesi (2003, 2005), but in comparison to these authors differs in motivation, its focus being on the role of learning in relation to expected returns and return predictability. Revealing "learning" as a potential explanatory factor for the emergence of time-varying risk premia, this article is related to recent work by Hansen and Sargent (2007)[10]. While these authors obtain countercyclical uncertainty premia from a combination of model and parameter uncertainty, the present paper makes use of the latter only.

As the model derives asset pricing implications from the real side of the economy, it is also related to the emerging literature on production-based asset pricing[11]. Berk, Green and Naik (1999) were the first to establish a relation between investment decisions and expected returns. They show that as a consequence of optimal investment decisions firms' assets and growth options change in predictable ways with market value, which becomes a proxy for the state variable describing their relative importance. More recent contributions include those of Gomes, Kogan, and Zhang (2003), Carlson et al. (2004), or Gala (2006), for example. The present paper differs from this literature, as it does not explain empirical regularities from firms' investment decisions. Instead, "equity return puzzles" are rationalized as reflecting a higher exposure to systematic risk in form of changes in GDP growth. Therefore, the present article introduces a link to macroeconomics and business cycle theory, which is absent from existing production-based asset pricing models.

Finally, in view of the obtained cyclicalities in risk premia, this paper is related to asset pricing models featuring habit formation. Santos and Veronesi (2005), for example, construct a "multiple endowments" economy, specifying cash flows exogenously. They introduce habit persistence and obtain effects in the cross section through the interaction of a time-varying aggregate risk premium with changes in the duration of an asset's cash flow. However, the authors point to a "cash-flow risk puzzle", i.e. the cross-sectional dispersion in cash flow risk needed to match the cross-sectional properties of stock returns is found to be too large. Making the stochastic discount factor exogenous, driven by investor sentiment, Lettau and Wachter (2005) circumvent this problem. This approach, however, comes at the cost of permitting at best a weak correlation with macroeconomic aggregates and hence, with a general difficulty to provide fundamental explanations of cyclicalities in risk premia.

The paper is now organized as follows. Section 2 presents the production economy. Section 3 establishes its link to financial markets and results for stock valuation and stock return predictability. Section 4 calibrates the model and section 5 concludes.

[9] see Detemple (1986), David (1997), Veronesi (2000) and Brennan and Xia (2001), for example

[10] see also Hansen (2007)

[11] see Cochrane (2006) for a detailed review

2 The Economy

I consider a representative agent economy. While the economy can be thought of as being of infinite horizon, the representative agent's investment horizon is finite and denoted by T. The agent is endowed with initial capital B_0 that needs to be allocated for the time of the investment period.

The economy is populated with a continuum of heterogeneous firms $i \in I$, where the set of firms I is assumed exogeneously fixed. There is a linear technology, producing output, Y_{it}, from capital, B_{it}, for each firm $i \in I$, such that

$$Y_{it} = \rho_t B_{it}. \tag{1}$$

Firm productivity is assumed to follow a mean-reverting Ornstein Uhlenbeck process[12] of the form

$$d\rho_t = \phi \left(\overline{\rho}^{\nu} - \rho_t\right) dt + \sigma dZ_t \tag{2}$$

with standard Brownian motion Z_t, constant speed of mean reversion $\phi \in \mathbb{R}^{++}$ and constant volatility $\sigma \in \mathbb{R}^{++}$. Productivity, ρ_t, depends on the state of the economy ν. The variable creating state dependency is long-run output growth, $\overline{\rho}^{\nu} \in \mathbb{R}^{++}$. It can be either high, $\overline{\rho}^{H}$, should the economy be in an *expansion*, or low, $\overline{\rho}^{L}$, in a *contraction*.

At some point in time, t^*, where $t^* \in [0, T)$, firm-specific shocks to productivity, ζ_i^{ν}, are assumed to spread over all firms in the economy. For the purpose of simplification, t^* is assumed to be singular. However, shocks could occur at arbitrary frequencies within the investment period T. Technology shocks are *idiosyncratic*, that is they are i.i.d and drawn from a single underlying distribution whose mean value μ^{ν} is zero, whatever the actual state of the economy ν. Hence,

$$\zeta_i^{\nu} \sim i.i.d. \;\; N\left[0, \widehat{\sigma}_{t^*}^2\right] \tag{3}$$

for all firms $i \in I$. The time t^* rate of variance, $\widehat{\sigma}_{t^*}^2 \in \mathbb{R}^{++}$, is assumed known by the agent.

Once technology shocks have occurred, the firm-specific process for productivity denotes

[12] Empirical evidence on the mean reversion of firm profitability was provided by Penman (1991), Fama and French (2000), and Pakos (2001), among others. Evidence on the non-stationarity of real output was provided by Nelson and Plosser (1982), Cheung and Chinn (1996), Rapach (2002), and David, Lumsdaine, and Papell (2003), for example.

$$d\rho_{it} = \phi\left(\overline{\rho}^{\nu} + \zeta_i^{\nu} - \rho_{it}\right)dt + \sigma dZ_t + \sigma_1 dZ_{\zeta_i^{\nu},t} \qquad t^* \leq t \leq T \tag{4}$$

with constant variances $\sigma, \sigma_1 \in \mathbb{R}^{++}$. The process now contains two standard Brownian motions Z_t and $Z_{\zeta_i^{\nu},t}$, for all $i \in I$[13]. The first is common to all firms and relates to systematic risk. The second relates to idiosyncratic variations driven by firm-specific technology shocks. As the number of shocks is spanned by a corresponding number of Brownian motions, markets are complete.

Although technology shocks are idiosyncratic, they are assumed to display a *systematic* bias in the cross section. This bias relates exclusively to the bad state of the economy. Empirical evidence shows that economic fundamentals of small and value firms are more negatively affected by recessionary periods of the economy than those of large and growth firms. Xing and Zhang (2005) provide evidence comparing fundamentals of value and growth firms, while similar findings for small and large firms are provided in Gertler and Gilchrist (1994).

Hence, while firms are assumed observationally equivalent to the agent[14], there is a latent cross section defined through systematic differences in the reaction of firm fundamentals to changes in GDP growth. Consequently, firms can be of type "small" or of type "large", they can be of type "value" or of type "growth". Each firm $i \in I$ can be thought of as having been randomly assigned to one of two categories along each dimension at time 0.

In order to reflect the above described bias from firm fundamentals, let me define the unconditional cross-sectional mean, $\mu_{\zeta^{\nu}} \in \mathbb{R}$. It obtains as the aggregate (average) value of the under (3) defined technology shocks of firms $i \in \zeta$, that is $\mu_{\zeta^{\nu}} \equiv \int_{i\in\zeta} \zeta_i^{\nu} di$, where $\zeta = \eta, \psi, s, l$ for $\nu = H, L$. As there are no systematic differences in the reaction of firm fundamentals during expansions, it follows that $\mu_{\zeta^H} = \mu^H \equiv 0$. In recessions, however, the mean values in the cross-section of technology shocks display the following characteristics

$$\mu_{\eta^L} < \mu_{\psi^L} \tag{5}$$

denotes the higher exposure to negative business cycle shocks of value firms, η, compared to growth firms, ψ,

$$\mu_{s^L} < \mu_{l^L} \tag{6}$$

[13]Note that consequently, ρ_{it} replaces ρ_t in (1) for all $t^* \leq t \leq T$.

[14]One could think of assuming the agent to distinguish firms by size. However, as long as she remains oblivious to associated differences in exposures of firm fundamentals to recessions, such an assumption would not bear on subsequent results.

denotes the higher exposure of fundamentals of small firms, s, compared to those of large firms, l.

I take the empirically observed difference in reactions of firm fundamentals as given. Introducing bond markets, Müller (2008b) shows that this difference is reflected in a systematic cross-sectional divergence in leverage ratios. One might in turn consider the difference in leverage ratios to be at the source of the bias in reactions of firm fundamentals. It would seem rather plausible that a firm with little financial slack lacks a buffer in recessionary periods of the economy. Consequently, its earnings, ouput, and dividends should react more adversely to negative business cycle shocks.

I will now turn to the implications of the features of the constructed production economy for firm valuation.

3 Financial Markets

Firms are all-equity financed. Capital B_{it} of firms $i \in I$ is then equal to firms' book value of equity, and firm profitability is given by

$$\rho_{it} = \frac{Y_{it}}{B_{it}}, \tag{7}$$

which is nothing but the instantaneous accounting return on equity, with Y_{it} denoting firm's earnings at time t. Consistent with the definition of a regime-switching production economy as in (4), the level of profitability can be either high, $\overline{\rho}^H$, should the economy be in an expansion, or low, $\overline{\rho}^L$, in a contraction.

Firms $i \in I$ pay out dividends D_{it} for all $t \in [0, T]$. In order to smooth dividends over time, they are payed out as a constant fraction c, $c \in \mathbb{R}^{++}$, of book equity

$$D_{it} = cB_{it}. \tag{8}$$

Capital of firm i then follows the process

$$dB_{it} = (\rho_{it} - c)B_{it}dt. \tag{9}$$

As the economy's consumption good is immediately perishable and non-storable, the assumed dividend policy leads to the following equilibrium restriction on consumption for all $t \in [0, T]$

$$C_t = \int_{i \in I} D_{it} di. \tag{10}$$

For the purpose of simplification, I will assume in the remaining analysis that $c = 0$. Note however, that all results go through for $c \in \mathbb{R}^{++}$, where $c < \rho_{it}$ $\forall i$, $\forall t$, with marginal quantitative impacts[15].

The representative agent with risk aversion $\gamma > 1$, has preferences defined by power utility. She aims to allocate capital across firms such as to maximize expected utility from terminal wealth[16],

$$\max_{B_{it}} \; E_t \left[\frac{W_T^{1-\gamma}}{1-\gamma} \right] \qquad \forall i \in I. \tag{11}$$

The market-clearing condition is given by $W_T = B_T \equiv \int_{i \in I} B_{iT} di$. As firms were assumed observationally equivalent to the agent, her priors on firm fundamentals are unbiased. Consequently, at time 0, it is optimal to equally allocate initial capital B_0 across firms $i \in I$. Capital can be reallocated instantaneously and at no cost.

There is a risk less bond in zero net supply, whose yield is normalized to zero for simplicity. Standard arguments then imply that the state-price density obtains from the pricing equation

$$\pi_t = \lambda^{-1} E_t \left[W_T^{-\gamma} \right], \tag{12}$$

where λ denotes the Lagrange multiplier from the utility maximization problem of the representative agent.

Stocks are defined as contingent claims to be liquidated at the end of the agent's investment period, with a market clearing under $W_T = B_T$. The agent expects markets to be perfectly competitive, thus leaving no abnormal earnings in equilibrium. She therefore assumes $M_{iT} = B_{iT}$ for all $i \in I$. The current market value of firms' stock is then given by the pricing equation

$$M_{it} = E_t \left[\frac{\pi_T B_{iT}}{\pi_t} \right] \; . \tag{13}$$

Recalling the latent Markov structure of the model, I will now turn to the development of agent's beliefs about the actual regime of the economy, ν, when inferring from profitability observations ρ_{it} across firms $i \in I$. These time-varying beliefs will be at the source of cyclicalities in risk premia of the time series and the cross section.

[15] The assumption of $c < \rho_{it}$ $\forall i$, $\forall t$, ensures that the economy does not stagnate.

[16] Note that even in the presence of intermediate consumption, i.e. $c \neq 0$, the investment problem reduces to the maximization of terminal wealth as c was assumed exogeneous.

3.1 The State Price Density

The state variable $\overline{\rho}$, defining whether the economy is in an expansion or contraction, is assumed to follow a two state, continuous-time Markov switching process with transition probability matrix between time t and $t+\triangle$ given by

$$\mathbf{P}(\triangle)=\begin{pmatrix} 1-\lambda\triangle & \lambda\triangle \\ \mu\triangle & 1-\mu\triangle \end{pmatrix} \tag{14}$$

where $\lambda\Delta$ is the probability that during an infinitesimal time interval Δ, mean productivity shifts from the high state, $\overline{\rho}^H$, to the low state, $\overline{\rho}^L$, while $\mu\Delta$ denotes the inverse.

The agent observes productivity, ρ_{it}, across firms $i \in I$. As the firm-specific shocks to productivity, ζ_i^ν, disappear at the market level for $\nu = H, L$, the process of inference when forming beliefs about the overall state of the economy is as defined in (2), for all $t \in [0, T]$.

Lemma 1 *The posterior probability of the good state, $\theta_t = \Pr\left(\overline{\rho} = \overline{\rho}^H | \digamma_t\right)$, where the filtration is generated by $\digamma_t = \{\rho_t : 0 \le t \le T\}$, follows the law of motion*

$$d\theta_t = (\lambda+\mu)(\theta^s - \theta_t)\,dt + h(\theta_t)dZ_{0,t} \tag{15}$$

where

$$dZ_{0,t} = \frac{1}{\sigma}\left(d\rho_t - E_t\left(d\rho_t|\digamma_t\right)\right) \tag{16}$$

$$h(\theta_t) = \left(\frac{\overline{\rho}^H - \overline{\rho}^L}{\sigma}\right)\theta_t(1-\theta_t) \tag{17}$$

$$\theta^s = \frac{\mu}{\mu+\lambda} \tag{18}$$

and $Z_{0,t}$ is a Wiener process with respect to $\digamma_t$, and θ^s the probability of $\overline{\rho}^H$ under the Markov chain stationary distribution.

Proof: See David (1997) Theorem 1, Theorem 9.1 in Liptser and Shiryayev (1977)

By means of the posterior belief distribution, the effects of the Poisson processes implicit in the continuous-time Markov chain are smoothed out. Therefore, although changes of the state variable through time occur in jumps, learning leads to continuous processes.

Lemma 2 *The values 0 and 1 are entrance boundaries for θ, i.e., there is a probability zero that θ equals either of these two values in any finite time.*

Proof: see David (1997)

Given the posterior belief of the good state, θ_t, the dynamics of aggregate productivity as defined in (2) can be reexpressed as follows

Lemma 3 *The process for aggregate productivity, which accounts for the agent's beliefs about the actual state of the economy obtains as*

$$d\rho_t^\theta = \phi\left(\left(\theta_t \overline{\rho}^H + (1-\theta_t)\,\overline{\rho}^L\right) - \rho_t\right)dt + \sigma dZ_{0,t}. \tag{19}$$

Proof: see Appendix

This process serves as the basis for stock valuation. It allows importing into the agent's valuation of firms' equity the fact that the same firm-level profitability observation can lead to very different conclusions on underlying firm value, when made at different stages of the business cycle. Low profitability is usually a bad sign for firm productivity and valuation, but it becomes much less so when observed during a recession, a period when the profitability of the overall economy is low.

Given the agent's belief about the state of the economy, the state-price density obtains from pricing equation (12) where $W_T = B_T = M_T$. The first equality is the market-clearing condition. The second, where $B_T \equiv \int_{i\in I} B_{iT} di$, is a consequence of perfectly competitive capital markets, leaving no abnormal earnings in equilibrium for all $i \in I$.

Proposition 1 *For $t \in [0, t^{**})$, the state-price density is given by*

$$\pi_t = \lambda^{-1} B_t^{-\gamma} \left[\begin{array}{c} E_t[\theta_T] \cdot \exp\left\{\widetilde{A}_0^H (T-t) - \gamma A_1 (T-t)\,\rho_t\right\} \\ +\left(1 - E_t[\theta_T]\right) \cdot \exp\left\{\widetilde{A}_0^L (T-t) - \gamma A_1 (T-t)\,\rho_t\right\} \end{array}\right] \tag{20}$$

where

$$E_t[\theta_T] = \theta^s + (\theta_t - \theta^s) \cdot \exp\left\{-(\lambda+\mu)(T-t)\right\}. \tag{21}$$

$\widetilde{A}_0^\nu(\cdot)$, for $\nu = H, L$, and $A_1(\cdot)$ are given in the Appendix.

Proof: see Appendix

Note that the state-price density obtains as a weighted average of the relevant stochastic discount factor for each regime of the economy. Note also, that the weights are given by $E_t[\theta_T]$, the conditional expectation of the probability of the good state of the economy for time T. This is a consequence of the agent's investment problem. Taking dividend policy c as given, she allocates initial capital B_0, as to maximize expected wealth at the end of her investment period. Hence, the only probability distribution relevant to her investment problem, is that expected for time T. The latter obtains from the forward equation of θ_t.

3.2 Optimal Learning Allocations

The *Separation Theorem*[17] provides the possibility of separating the filtering problem from the study of portfolio allocation within Markovian structures of incomplete information. While this result is not novel, so far, the estimation itself was not associated with considerations of optimization. This changes with the present setup. The agent's investment period being finite, her time to learn is limited. Combined with a continuum of firms to observe, this leads to a trade-off for the agent: learning more on one firm means learning less on the others. I will show in the following, how empirical regularities in the cross section can arise from the optimizing behavior induced by such a trade-off, as soon as there is a bias in underlying firm fundamentals.

I assume that at some point in time t^*, where $t \in [0, T)$, the under (3) defined productivity shocks spread over all firms $i \in I$. This immediately bears on filtration. Observing profitability, ρ_{it}, across firms $i \in I$, the agent now simultaneously develops beliefs about the state of the economy, as well as firms' exposure to it, when trying to infer the value of underlying firm-specific technology shocks.

The posterior belief about firm-specific shocks to productivity, $\widehat{\zeta}_{it}^{\nu}$, obtains for all $i \in I$ from the filtration generated by ρ_t^θ and $\rho_{it}^{\zeta^\nu}$. The process followed by ρ_t^θ is as defined in Lemma 3, the dynamics of $\rho_{it}^{\zeta^\nu}$ are as follows

$$d\rho_{it}^{\zeta^\nu} = \phi(\left(\theta_t \overline{\rho}^H + (1-\theta_t)\,\overline{\rho}^L\right) + \zeta_i^\nu - \rho_{it})dt + \sigma_0 dZ_{0,t} + \sigma_1 dZ_{\zeta_i^\nu,t} \qquad t^* \leq t \leq T \quad (22)$$

with constant variances $\sigma_0, \sigma_1 \in \mathbb{R}^{++}$. The process contains two standard Brownian motions. The first, $Z_{0,t}$, relates to systematic risk. The second, $Z_{\zeta_i^\nu,t}$, differs across firms $i \in I$, driven by firm-specific technology shocks.

Accounting for the above defined filtration, the posterior mean and variance of firm-specific shocks to output growth obtains as follows

Lemma 4 *Suppose that at $t = t^*$ and for $\nu = H, L$ the prior distribution of the idiosyncratic shock to productivity of firm $i \in I$ for $\zeta = \eta, \psi, s, l$ is normal, $\zeta_i^\nu \sim N\left[0, \widehat{\sigma}_{t^*}^2\right]$. Then the posterior distribution of ζ_i^ν at time t, $t^* \leq t \leq T$, conditional on $\mathcal{F}_t = \left\{\left(\rho_t^\theta, \rho_{it}^{\zeta^\nu}\right) : t^* \leq t \leq T\right\}$ is also normal, $\zeta_i^\nu|_{\mathcal{F}_t} \sim N\left[\widehat{\zeta}_{it}^\nu, \widehat{\sigma}_{it}^2\right]$, where the posterior mean $\widehat{\zeta}_{it}^\nu$ follows the process*

$$d\widehat{\zeta}_{it}^\nu = \widehat{\sigma}_{it}^2 \left(\frac{\phi}{\sigma_1} - \frac{\phi\sigma_0}{\sigma\sigma_1}\right) d\widetilde{Z}_{\zeta_i^\nu,t} \quad (23)$$

[17] see Dothan and Feldman (1986) or Gennotte (1986), for example. More recently, Feldman (2005) establishes a more general version in form of a state space representation theorem.

where $\widetilde{Z}_{\zeta_i^\nu,t}$ *is a standard Brownian motion given in the Appendix.*
The posterior variance $\widehat{\sigma}_{it}^2$ *is given by*

$$\widehat{\sigma}_{it}^2 = \left[\widehat{\sigma}_{t^*}^{-2} + \left(\frac{\phi}{\sigma_1} - \frac{\phi \sigma_0}{\sigma \sigma_1} \right)^2 \cdot (\tau_{it} - t^*) \right]^{-1} \tag{24}$$

where $\tau_{it} \in [0, t]$ *denotes the aggregate time spent observing firm* i *between* 0 *and* t.

Proof: see Appendix

The posterior variance, $\widehat{\sigma}_{it}^2$, of firm $i \in I$ is a direct function of τ_{it}, the aggregate time having been spent *learning* about firm i between 0 and t. The higher the value associated with τ_{it}, i.e. the more time has been spent learning about firm i, the lower the uncertainty surrounding expectations about the value of the underlying shock to productivity. As the value function $V\left(B_{it}, \rho_t, \widehat{\zeta}_{it}, \widehat{\sigma}_{it}^2, T-t\right)$ of firms $i \in I$[18] is decreasing in uncertainty of the posterior belief distribution, the reduction of uncertainty positively affects agent's expected utility from investment.

Due to the finite horizon of the investment problem, the agent will have to decide how to optimally allocate the time at her disposition across firms $i \in I$ of the economy. Hence, filtration hides a maximization problem with respect to time, which can be denoted as follows

$$\max_{\tau_{it} \in [0, T-t]} E_t \left[\frac{B_T^{1-\gamma}}{1-\gamma} \right] \qquad \forall\, i \in I, \tag{25}$$

where τ_{it} is the aggregate time to be optimally allocated towards firm $i \in I$, between time t and T. The maximization problem is subject to the constraint $\int_{i \in I} \int_t^T \tau_{is} di ds \leq T - t$ for all $t \in [0, T]$, i.e. aggregate time spent learning cannot exceed the time being at the agent's disposal.

At every instant t, the agent therefore decides on the optimal amount of time to be allocated towards each firm $i \in I$, given that she has time $T - t$ at her disposition. The solution to this optimization problem implies an optimal instantaneous learning ratio between any two firms of the economy, which I denote by $\frac{\tau_{it}^*}{\tau_{jt}^*}$, where $i \neq j$, with $i, j \in I$. This learning ratio obtains as a consequence of the fact that marginal utilities have to equate for an equilibrium to obtain. The agent therefore allocates

[18] see equation (84) in the appendix

her time such that expected marginal utilities from learning, i.e. $\partial V(\cdot)/\partial \tau_{it}$, equal across all firms $i \in I$, at every instant $t \in [0, T]$ [19].

As long as no technology shocks have occurred, i.e. $t \in [0, t^*)$, the problem is trivial. As capital allocation is unbiased, the exact same amount of time is optimally spent learning across firms $i \in I$ of the economy. For $t \in [t^*, T]$, however, firm fundamentals exhibit a cross-sectional bias, as observed by Gertler and Gilchrist (1994), and Xing and Zhang (2005), respectively. As a consequence, and in immediate response, learning allocations will reflect this bias. In fact, as soon as $t = t^*$, marginal utilities do no longer equal when time is allocated equally across firms $i \in I$. Instead, marginal utilities from learning are now strictly higher for small and value firms, *ceteris paribus*. This result would seem intuitive. In fact, small and value firms are riskier as they are affected more strongly by negative business cycle shocks. Learning about such firms should therefore be more beneficial, as the same amount of time invested will protect the agent from more negative "surprises" on average, than when allocated towards large and growth firms. In equilibrium, the risk-averse agent therefore learns strictly more on small and value firms[20].

For the sake of space, subsequent results are presented for value and growth firms only. They are, however, equally valid comparing small and large firms. Note that the optimal learning ratio in the cross section obtains by aggregating $\frac{\tau_{it}^*}{\tau_{jt}^*}$ across $i \in \eta, s$ and $j \in \psi, l$, respectively.

Proposition 2 *The optimal learning ratio between value and growth firms $\forall\ t \in [t^*, T]$, is given by*

$$\begin{aligned} \left(\frac{\tau_t^{\eta}}{\tau_t^{\psi}}\right)^* &= E_t\left[\theta_T\right] \cdot \left(\frac{\tau_t^{\eta^H}}{\tau_t^{\psi^H}}\right)^* + \left(1 - E_t\left[\theta_T\right]\right) \cdot \left(\frac{\tau_t^{\eta^L}}{\tau_t^{\psi^L}}\right)^* \\ &= E_t\left[\theta_T\right] + \left(1 - E_t\left[\theta_T\right]\right) \cdot \chi_t\left(\mu_{\psi^L} - \mu_{\eta^L}\right) \end{aligned} \tag{26}$$

[19] Example: The agent disposes of 4 days in an economy of 4 firms. Assuming that these firms exhibit the same profitability, it is optimal for the agent to allocate the same amount of time towards each firm. This means 1 day in total for each firm, and implies the allocation of the same infinitesimal unit of time towards each firm at very instant t. The latter is, of course, a theoretical implication, based on the presumption that time is inifnitely divisible. In practice, the agent could be thought of as starting by studying for half an hour each firm's company report, before revisiting the decision problem again.

[20] Generally speaking, the result that risk aversion would drive us towards learning more about comparatively riskier "underlyings", seems like a rather close reflection of reality. Taking a look at a typical daily newspaper or TV journal, for example, it would seem that the "worse" the event, the more we get informed about it.

where

$$\chi_t\left(\mu_{\psi^L}-\mu_{\eta^L}\right)>1 \tag{27}$$

and $\frac{\partial \chi_t(\cdot)}{\partial\left(\mu_{\psi^L}-\mu_{\eta^L}\right)}>0$. $\left(\tau_t^{\zeta}\right)^* \equiv \int_{i\in\zeta}\tau_{it}^* di$ *is the time* t *aggregate time to be optimally spent learning about firms* $i\in\zeta$, $\zeta=\eta,\psi$ *between* 0 *and* $T-t^*$.

Proof: see Appendix

Corollary 1 *The optimal learning ratio* $\left(\frac{\tau_t^{\eta}}{\tau_t^{\psi}}\right)^*$ *is strictly larger than one, and increasing (decreasing) in contractions (expansions)* $\forall\ t\in[t^*,T]$.

Accounting for Lemma 2, and the fact that $\chi_t(\cdot)>1$, the learning ratio is strictly larger than one for all $t\in[t^*,T]$, i.e. conditional on a bias from fundamentals, the agent learns strictly more on value firms. Moreover, $\chi_t(\cdot)<<\infty$, implying that even though some firms are fundamentally less risky, it is always optimal to learn about these firms as well. Also, $\chi_t(\cdot)$ is strictly increasing in $\mu_{\psi^L}-\mu_{\eta^L}$, the difference in cross-sectional means of productivity shocks in a recession. The learning bias towards value firms is therefore an increasing function of their comparatively higher exposure to recessionary periods of the economy. Finally, the cross-sectional learning bias is *countercyclical.*

Can learning ever come to an end? No. The investment horizon, T, being finite, the maximum amount of time that could theoretically be allocated towards a specific firm $i\in I$ is T. This would mean that the agent observes only one single firm during her entire investment period. Such an allocation of time does not constitute an equilibrium, however, it will serve as a "worse case" example. Replacing τ_{it} by T in Lemma 4, uncertainty of beliefs on mean productivity is reduced, but certainly not eliminated. Moving from $T=T_{n=0}$ to T_n, where $n\longrightarrow\infty$, through an infinitely repeated "game", yields the same conclusion, as each investment period remains finite and characterized by a new draw of firm-specific technology shocks. Moreover, assuming t^* to be singular is the lower limit, chosen for the purpose of simplification. Shocks could be drawn at arbitrary frequencies, keeping the agent learning throughout. Finally, the bias in cross-sectional means could be time-varying, introducing another source of uncertainty making it impossible for the learning process to ever come to an end.

Given the optimal allocation of time, another question worth asking is whether the determined equilibrium bias is reflected in capital markets. Are we effectively learning more about small and value firms? Bhushan (1989) examines major determinants of the number of analysts following a firm. He finds a positive relation between the latter and firms' return variability. Also is the number of analysts positively related to the squared correlation between firms' return and the market return. Both these findings are supportive for the above results, as small and value stocks will turn out to be those whose returns exhibit higher variability, as well as a higher correlation with market returns. On the other hand, the number of analysts following a company is found to grow in its size. This would seem to contradict above predictions. However, taking into account that the size effect has disappeared, the contradiction is less pronounced. Finally, Bhushan provides indicative evidence that industries with lower market-to-book ratios are followed by a larger number of analysts, which is what one would expect given above results.

3.2.1 Learning and Valuation Ratios

Having established a systematic divergence in learning allocations in the cross section, I will now turn to determine its effects on firm valuation. The reduction of uncertainty through learning bears on valuation ratios.

Proposition 3 *The market-to-book ratio of firm i is decreasing in τ_{it}, the time spent learning about it, $\forall i \in I$, $t \in [0, T]$.*

Proof: see Appendix

The above result was established by Pástor and Veronesi (2003), who derive a negative relation between the resolution of firm-specific uncertainty and the level of valuation ratios. The latter is a result of the convex nature of compounding. Higher uncertainty is equivalent to an increase in both the probability that an individual firm's future growth rate of book equity will be persistently low, or persistently high. Because of risk aversion, the impact of the latter on valuations will be strictly higher, which in turn increases expected future book values and ultimately today's market-to-book ratios. As learning reduces idiosyncratic variations surrounding expectations on firm fundamentals, it will lead to a decrease in valuation ratios.

Consequently, the equilibrium learning bias towards value firms translates into a systematic difference in market-to-book ratios between value and growth firms: the *value spread*. The latter is expressed as the difference in aggregate valuation ratios, where the aggregate (average) market-to-book ratio of firms $i \in I$ of type ζ, where $\zeta = \eta, \psi$, is denoted by $\frac{M_t^\zeta}{B_t} \equiv \int_{i \in \zeta} \frac{M_{it}}{B_{it}} di$.

Proposition 4 *For $t \in [t^*, t^{**})$, the value spread obtains as*

$$\frac{M_t^\psi}{B_t} - \frac{M_t^\eta}{B_t} = \left(1 - \theta^s - (\theta_t - \theta^s) \cdot \exp\left\{-(\lambda + \mu)(T - t)\right\}\right) \cdot \frac{\exp\left\{A_0^L (T-t) + (1-\gamma) A_1 (T-t) \rho_t\right\}}{E_t[\theta_T] \cdot \pi_t^H + (1 - E_t[\theta_T]) \cdot \pi_t^L} \cdot \xi_t^{\psi - \eta} \tag{28}$$

where

$$\xi_t^{\psi-\eta} = \exp\left\{\frac{1}{2}(1-\gamma)^2 A_2 (T-t)^2 \widehat{\sigma}\left(\left(\tau_t^{\psi^L}\right)^*\right)^2\right\} - \exp\left\{\frac{1}{2}(1-\gamma)^2 A_2 (T-t)^2 \widehat{\sigma}\left(\left(\tau_t^{\eta^L}\right)^*\right)^2\right\}, \quad (29)$$

$$\pi_t^\nu = \exp\left\{\widetilde{A}_0^\nu (T-t) - \gamma A_1 (T-t)\rho_t\right\} \text{ for } \nu = H, L \quad (30)$$

and

$$\frac{M_t^\psi}{B_t} - \frac{M_t^\eta}{B_t} > 0. \quad (31)$$

$A_0^\nu(\cdot)$, $\widetilde{A}_0^\nu(\cdot)$, $A_1(\cdot)$, $A_2(\cdot)$ *are given in the Appendix, and* $\left(\tau_t^{\zeta^L}\right)^* \equiv \int_{i\in\zeta}\left(\tau_{it}^L\right)^* di$.

Proof: see Appendix

Similarly, the learning bias towards small firms will translate into a *size spread.* For the sake of space, I will restrict myself to the value spread in the subsequent analysis.

As can be seen from Proposition 4, the value spread is a direct function of the equilibrium learning bias through $\left(\tau_t^{\zeta^L}\right)^*$, the aggregate optimal time allocated towards firms $i \in \zeta$ in the low state, L. As a consequence, the cross-sectional difference in mean values of firm-specific productivity shocks causes a divergence in market-to-book ratios to the extent that it is reflected in a systematic divergence in learning allocations in equilibrium. Note also, that $\partial\left(\frac{M_t^\eta}{B_t} - \frac{M_t^\psi}{B_t}\right)/\partial\rho_t > 0$, for all γ, ϕ, τ. As GDP growth, ρ_t, is increasing in troughs and decreasing at peaks, the value spread is negatively correlated with business cycles and exhibits *countercyclical* dynamics, a stylized fact pointed out by Cohen, Polk and Vuolteenaho (2003) [21].

The agent thus learns about output growth, but, so far, is still assumed unaware of its systematic bias in the cross section. I will now turn to study the effects of an awareness of systematic differences in the reaction of firm fundamentals to recessionary periods of the economy.

3.3 Learning about Systematic Risks in the Cross-Section

I assume that at some point in time t^{**}, where $t^{**} \in (t^*, T)$, the agent becomes aware that output and earnings of small and value firms are systematically more adversely affected by recessions. Her believed means are $\widehat{\mu}_{\psi^L}$ and $\widehat{\mu}_{\eta^L}$, where $\widehat{\mu}_{\psi^L} > \widehat{\mu}_{\eta^L}$. For

[21] see Footnote 5

simplicity, I will assume that believed means equal actual ones μ_{ψ^L} and μ_{η^L}. Note however, that this does not imply that the agent eliminated uncertainty, which was shown to be impossible. Again, for the sake of space, results are presented for value and growth firms only.

For subsequent results to hold, I will have to make an assumption on the relation between the differential in cross-sectional means of productivity shocks in state L, and the corresponding difference in optimal learning times of value firms, $\left(\tau_t^{\eta^L}\right)^*$, and growth firms, $\left(\tau_t^{\psi^L}\right)^*$.

Assumption 1: *The difference in mean values of the underlying distribution of technology shocks between value and growth firms in state L, fulfills the following condition* $\forall t \in [t^{**}, T]$

$$\mu_{\psi^L} - \mu_{\eta^L} < \frac{1}{2}(1-\gamma) A_2 (T-t) \left[\widehat{\sigma} \left(\left(\tau_t^{\eta^L}\right)^* \right)^2 - \widehat{\sigma} \left(\left(\tau_t^{\psi^L}\right)^* \right)^2 \right]. \tag{32}$$

The newly arising cross-sectional divergence in beliefs, $\mu_{\psi^L} - \mu_{\eta^L}$, will bear on return expectations. Expected excess returns of firm $i \in I$ are a measure of its systematic risk, obtaining from the covariance of returns, R_{it}, with the state-price density, π_t, via the standard pricing equation $E_t[R_{it}] = -cov_t\left(\frac{d\pi_t}{\pi_t}, dR_{it}\right)$. Recall that for the present investment problem, the state-price density denotes $\pi_t = \lambda^{-1} E_t\left[W_T^{-\gamma}\right]$, where $W_T = B_T$. Whereas so far, expected terminal wealth did not differ in the cross section, i.e. $\int_{i\in\eta} E_t\left[B_{iT}^{-\gamma}\right] di = \int_{i\in\psi} E_t\left[B_{iT}^{-\gamma}\right] di$, as firm-specific technology shocks, $\widehat{\zeta}_{it}$, washed out when aggregated, this can no longer be the case. The awareness of a systematic negative bias in productivity shocks towards value firms, will decrease expectations of terminal wealth for all $i \in \eta$, *ceteris paribus.* The resulting divergence of expectations in the cross section then leads to a divergence of state-price densities as denoted by $\pi_{\zeta,t}$, for all $t \in [t^{**}, T]$.

The cross-sectional divergence of state-price densities reflects the comparatively higher systematic risk of value firms, due to their higher downside risk from fundamentals. Consequently, it translates into an expected return differential, the *value premium.* Similarly, a *size premium* obtains as the difference between expected returns of small and large firms.

Proposition 5 *For $t \in [t^{**}, T]$, the value premium is given by*

$$\begin{aligned} E_t[R_t^\eta] - E_t[R_t^\psi] &= \left(\sigma_{\pi_\eta,\rho,t} \cdot \sigma_{R^\eta,\rho,t} - \sigma_{\pi_\psi,\rho,t} \cdot \sigma_{R^\psi,\rho,t}\right) \\ &\quad + \left(\sigma_{\pi_\eta,\eta,t} \cdot \sigma_{R^\eta,\eta,t} - \sigma_{\pi_\psi,\psi,t} \cdot \sigma_{R^\psi,\psi,t}\right) \end{aligned} \tag{33}$$

where for $\zeta = \eta, \psi$

$$\sigma_{\pi_\zeta,\rho,t} = \left(\gamma \frac{1 - e^{-\phi(T-t)}}{\phi} - S_{\pi_\zeta,\rho,t}\right)\sigma, \tag{34}$$

$$\sigma_{\pi_\zeta,\zeta,t} = S_{\pi_\zeta,\zeta,t}\widehat{\sigma}\left(\left(\tau_t^\zeta\right)^*\right)^2 \left(\frac{\phi}{\sigma_1} - \frac{\phi\sigma_0}{\sigma\sigma_1}\right) \tag{35}$$

and

$$E_t[R_t^\eta] - E_t[R_t^\psi] > 0. \tag{36}$$

$\sigma_{R^\zeta,\rho,t}$, $S_{\pi_\zeta,\rho,t}$ *and* $\sigma_{R^\zeta,\zeta,t}$, $S_{\pi_\zeta,\zeta,t}$ *are given in the Appendix.*

Proof: see Appendix

The value premium obtains as a function of both, the cross-sectional means of productivity shocks in a recession, μ_{ψ^L} and μ_{η^L}, as well as the corresponding optimal learning times $\left(\tau_t^{\psi^L}\right)^*$ and $\left(\tau_t^{\eta^L}\right)^*$. It is important to note, however, that only given Assumption 1 will there be a value premium. This has implications for the role of learning in explaining the cross section of stock returns. Assuming the bias in firm fundamentals to be directly observable, for example, $\widehat{\sigma}\left(\left(\tau_t^{\zeta^L}\right)^*\right)^2$ equals zero for $\zeta = \eta, \psi$, as there is no uncertainty left to be reduced. At the same time, as $\mu_{\psi^L} - \mu_{\eta^L} > 0$, complete information violates Assumption 1. Such a situation would then lead to a *growth* premium.

Hence, in the present model setup, learning proofs necessary for a value premium to obtain. Moreover, calibrations show that the size of the value premium is highly sensitive to the difference in learning times, $\tau_t^{\eta^L}$ and $\tau_t^{\psi^L}$, while the actual divergence in mean values, μ_{ψ^L} and μ_{η^L}, is of minor importance. This is interesting when compared to findings by Santos and Veronesi (2006), who point to growth premia arising in setups featuring habit formation. The authors show that cash flow risk alone is insufficient to counteract such premia. It would therefore seem that learning might play a potentially important role for the explanation of asset pricing anomalies in the cross section.

Corollary 2 *The value (size) premium is countercyclical* $\forall t \in [t^{**}, T)$.

Proof: see Appendix

The countercyclicality of value and size premia was pointed out by Lettau and Ludvigson (2001), and Campello, Chen, and Zhang (2006), respectively. Because of its countercyclical nature, the value (size) premium then preserves individual assets' return dynamics through business cycles and consequently, the time series properties of their aggregate, the market portfolio. In the present model, the time-series behavior of the cross-section is driven by two sources. First, agent's time-varying beliefs about the state of economy lead to cyclicalities in risk premia. Second, these movements are amplified by a time-varying countercyclical learning ratio. Hence, one would expect the magnitude of induced cyclicalities to be larger for the cross-section than for the time-series, which is confirmed when calibrating the model.

Changes in return expectations for $t \in [t^{**}, T]$, will bear on *portfolio allocations*. Priors being initially unbiased, capital B_0 was equally allocated across firms $i \in I$. As the agent is updating her beliefs, she optimally reallocates capital across firms. However, as long as $t < t^{**}$ these reallocations will not be in a systematic fashion, so that $\int_{i \in \zeta} B_{it} di \equiv B_t^{\zeta} = B$ for $t \in [0, t^{**})$. Once, however, that return expectations diverge in the cross section, misvaluations appear. In fact, (small) value firms become overvalued, as the market value of their equity does not account for their comparatively higher downside risk from fundamentals. (Large) growth firms, on the other hand, are undervalued. Consequently, at $t = t^{**}$, a cross-sectional bias in capital allocation emerges, such that $B_t^{\eta} < B_t^{\psi}$ and $B_t^{s} < B_t^{l}$ for $t \in [t^{**}, T]$. The bias in capital allocation will however only counteract part of the cross-sectional divergence in market values. In fact, in equilibrium, it exactly offsets the difference in cross-sectional market values arising from expected return differentials, such that the magnitude of value (size) spreads remains unchanged.

Looking at results in conjunction, the return dynamics of the market portfolio - the equity premium - obtain as a function of changes in aggregate firm productivity, or GDP growth. Cross-sectional differences in expected returns, as well as their dynamics over time, are determined by time-varying systematic differences in the covariation of firms' output with GDP growth.

The model therefore explains the empirical success of the Fama-French three factor model (1993) as a consequence of its ability to have depicted factor-mimicking portfolios that capture firms' time-varying exposure to business cycle fluctuations. This exposure is split into variations with overall GDP growth, as captured by the market portfolio, and additional, firm-specific exposure to recessionary periods of the economy. The latter is captured by the factors HML and SMB, as the output of small and value firms displays a systematically stronger covariation with GDP growth for these periods of the business cycle.

Additionally, results are obtained from a framework demonstrating that book-to-market and size are associated with persistent differences in profitability, as found by Fama and French (1995). The relation was established to exist only conditional on book-to-market ratios being high or low, a finding supported by above results. Also, within the model, stock prices forecast the reversion of earnings growth, after ranking firms on size and book-to-market, just as observed by Fama and French, who find market, size and book-to-market factors in earnings like those in returns.

I will now turn to the implications of the obtained results for standard asset pricing models.

3.4 Conditional CAPM

The single risk factor of the economy are variations in GDP growth. These define the business cycle by means of regime switches in the overall mean value of firm productivity. Moreover, changes in productivity during recessions differ systematically in the cross section through differences in mean values of the underlying distribution of technology shocks. Hence, returns arise in compensation for firms' relative exposure to macroeconomic risk. The conditional beta representation captures this sensitivity.

Proposition 6 *For $t \in [t^{**}, T]$, the cross sectional beta for firms $i \in \zeta$ denotes*

$$\beta_t^{\zeta} \equiv \frac{cov_t\left(dR_t^{\zeta}, dR_t^m\right)}{var_t\left(dR_t^m\right)} = \frac{\sigma_{R^{\zeta},\rho,t}}{\sigma_{R,t}^m} = \frac{\frac{1-e^{-\phi(T-t)}}{\phi} + S_{R^{\zeta},\rho,t} - S_{\pi_{\zeta},\rho,t}}{\frac{1-e^{-\phi(T-t)}}{\phi} + S_{R,t}^m - S_{\pi,t}^m} \tag{37}$$

where $S_{R^{\zeta},\rho,t}$, $S_{\pi_{\zeta},\rho,t}$ and $S_{R,t}^m$, $S_{\pi,t}^m$ are given in the Appendix.

Proof: see Appendix

Concerning the above expression, two things should be mentioned. First, as soon as variations in output of firms $i \in \zeta$ are stronger than those of overall GDP, i.e. $\sigma_{R^{\zeta},\rho,t} > \sigma_{R,t}^m$, the resulting beta is strictly larger than one. As higher variability is induced by stronger downside risk from fundamentals, higher betas duly induce higher expected returns. Second, the higher the risk from fundamentals of firms $i \in \zeta$, the higher their beta *ceteris paribus*. Consequently, as $\sigma_{R^{\eta},\rho,t} > \sigma_{R^{\psi},\rho,t}$, a systematic cross sectional divergence in betas between (small) value, i.e. high beta, and (large) growth, i.e. low beta, firms arises.

Corollary 3 *The cross sectional differential in betas, $\beta_t^{\eta} - \beta_t^{\psi}$, is countercyclical $\forall t \in [t^{**}, T]$.*

Proof: see Appendix.

Because of these dynamics in betas over time, a CAPM in unconditional form must "fail". Note however, that time-variation in the cross-sectional risk *exposure* accounts for only part of the overall time-variation in the value premium. The remaining portion is due to a time-varying *price of risk*[22]. This finding is in agreement with empirical evidence presented by Daniel and Titman (1997) who show that the firm characteristic "size" or "book-to-market" is still related to expected returns after controlling for the stock's loading with respect to the mimicking portfolio.

[22] As can be seen in section 4, only a very small portion of the value premium will be due to differences in risk exposures.

3.5 YCAPM

The economy's single risk factor are variations in output (Y) growth. Expected returns over and above the risk-free rate are determined by the covariation of firm output with changes in GDP growth. I introduce a YCAPM which differs from the consumption CAPM (CCAPM) in that the latter defines risk as a function of covariation with consumption growth. The introduction of consumption into the present setup would immediately give way to consumption risk premia[23]. However, these would reflect only part of the overall risk in the economy. Moreover, as c was assumed exogeneous and constant, variability in consumption would obtain as a direct function of variability in output. The latter then remains the single risk factor of the economy.

Proposition 7 *For $t \in [0, T]$, the market (equity) premium obtains as*

$$E_t[R_t^m] \equiv -cov_t\left(\frac{d\pi_t}{\pi_t}, dR_t^m\right) = \sigma_{\pi,t}^m \cdot \sigma_{R,t}^m \tag{38}$$

where

$$\sigma_{\pi,t}^m = \left(\gamma\frac{1-e^{-\phi(T-t)}}{\phi} - S_{\pi,t}^m\right) \cdot \sigma \tag{39}$$

and

$$\sigma_{R,t}^m = \left(\frac{1-e^{-\phi(T-t)}}{\phi} + S_{R,t}^m - S_{\pi,t}^m\right) \cdot \sigma. \tag{40}$$

$S_{\pi,t}^m$ and $S_{R,t}^m$ are as denoted in the Appendix.

Proof: see Appendix

Note, that as risk premia are determined by covariation with output growth, the corresponding volatility that is found to be priced at the market level is σ. The obtained definition of risk thus allows for the introduction of higher variability and therefore the potential for higher levels of risk premia, than those commonly obtainable from consumption-based frameworks. Moreover, the speed of mean reversion of productivity, ϕ, becomes a priced risk factor. Generally speaking, the lower the speed of mean reversion, the higher the required risk premium, *ceteris paribus*. Calibrations show that scaling risks from changes in output by ϕ, allows matching the equity premium with low levels of risk aversion.

Corollary 4 *The equity premium is countercyclical $\forall t \in [0, T]$.*

Proof: see Appendix

[23] Recall that it suffices to assume $c \neq 0$ in (9), for example.

Again, agent's time-varying beliefs about the state of the economy, lead to cyclicalities in risk premia. Countercyclicality in the time-series behavior of the market premium was pointed out by Campbell and Cochrane (1999), for example.

4 Calibration

The model is calibrated using the following parameter values for the mean-reverting process followed by productivity[24] [25]: $\overline{\rho}^H = 0.011643$, $\overline{\rho}^L = -0.003577$, $\mu = 0.0951$, $\lambda = 0.245$, and $\sigma = 0.0328611$. The first four parameter values are maximum likelihood estimates of a two-state Markov-Switching Model applied to US real GDP data for the period of 1952 to 1984. The annual volatility of real GDP growth, σ, obtains from quarterly NIPA data for the period of 1967 to 2002. Volatility parameters associated with idiosyncratic shocks to firm productivity, $\widehat{\sigma}_{t^*}$, σ_0, and σ_1, are assumed to equal 0.07, following Pástor and Veronesi (2005).

Data for the first and second moments of asset returns in the time series and cross section, as well as valuation ratios, obtain from the merged CRSP-COMPUSTAT database for the period of 1948 to 2001. CAPM β's (Table IV) obtain running time-series regressions of excess returns on respective M/B-decile portfolios. The latter are constructed following the standard procedure of Fama and French (1992). Returns on portfolios are from July of year t to June of year $t+1$. Data for the "growth" portfolio are based on averages of small and large high-M/B portfolios, data for the "value" portfolio are based on averages of small and large low-M/B portfolios. Results for the zero-investment portfolio HML obtain from their differential.

The remaining parameters of the model are the coefficient of relative risk aversion γ, the speed of mean reversion of productivity (or GDP growth) ϕ, and the average investment period τ. The latter is chosen to take values between 4 and 7, as Atkins and Dyl (1997) report an average holding period of firms' common stock of 4.01 years for NYSE quoted firms over the period of 1975 to 1989, and of 6.99 years for Nasdaq quoted firms over the period of 1983 to 1991. The speed of mean reversion ϕ is chosen to vary between 0.1 and 0.4. Risk aversion γ adjusts in order to match the data. Results in Table II to Table V are obtained assuming θ to equal 0.5, i.e. the agent is unbiased in her beliefs about the actual state of the economy.

As can be seen from Table II, concerning the *time series,* a coefficient of relative risk aversion of 4 (combined with $\tau = 6$ and $\phi = 0.1$) allows obtaining a high equity premium of 8.8 percent, and annual equity return volatility of 14.83 percent.

[24] see equation (4)

[25] Note that all results obtain in closed form, except for the derivative $\frac{\partial \theta_t}{\partial \rho_t}$, which is approximated via simulation.

Consequently, the Sharpe ratio displays a reasonable 0.59. The model is thus able to replicate main moments of the time series, maintaining a low coefficient of risk aversion. While the expected return on the market portfolio is slightly higher than observed in the data, volatility is slightly lower. Varying parameters τ and ϕ shows that volatility decreases in the speed of mean reversion of productivity (GDP growth).

Table III displays results for ratios from fundamentals of the time-series. Again, a low coefficient of relative risk aversion of 4 (combined with $\tau = 7$ and $\phi = 0.15$), allows matching observed data.

Turning to the *cross section*, given that mean productivity of the bad state, $\overline{\rho}^L$, obtained as -0.003577 from the Markov-Switching regression, cross-sectional mean values of firm-specific productivity shocks for state L, were chosen to equal -0.004 and -0.003 for μ_{η^L} and μ_{ψ^L}, respectively. The corresponding cross-sectional divergence in optimal learning times is chosen as 10/1 for value firms in bad times. Concerning results from calibration as displayed in Table V, two things should be mentioned. First, compared to the time series, the coefficient of relative risk aversion has to be increased in order to match observed empirical moments of the cross section. Setting γ to 10, for example, leads to a value premium of 3.43 percent, with an average expected return on value firms of 9.4 percent, and an average expected return on growth firms of 5.97 percent. Second, the coefficient of relative risk aversion needed to replicate the value spread is strictly lower than the one necessary for the value premium to be matched. Hence, the model comes with a tradeoff for the cross section: value spreads and premia cannot be matched simultaneously. However, independently, the model replicates their size, as well as time variation in both empirical regularities.

Note also, that the magnitude of the value premium proofs sensitive to the choice of learning ratio, while the actual divergence in cross-sectional means of the distribution of technology shocks has much less of an impact *ceteris paribus*. Figure 5 shows that for given parameter values, increasing the optimal learning ratio in state L, i.e. $\left(\tau_t^{\eta^L}/\tau_t^{\psi^L}\right)^*$, from 0 to 10, for example, induces a change in the size of the value premium of approximately 3.4 percent. On the other hand, changing the differential in cross-sectional means from 0 to 0.02 percent, while keeping the learning ratio at a constant 10, leads to variations of approximately 0.075 percent, only.

It is important to realize that in the present model, differences in equity premia across firms arise as a function of both differences in *risk exposures*, as well as differences in *risk prices*. Going back to the above mentioned value premium of 3.43 percent, for example, it can be seen from Table V, that it is associated with an unconditional beta[26] of 1.01 for value firms, and 0.99 for growth firms. The market price

[26] Note that I assume an unbiased agent, i.e. $\theta = 0.5$, for all measures that are "unconditional". Another possibility would have been to take an average of calibrated parameter values for the period 1948-2001, for example.

of risk implied by parameter values $\gamma = 10$, $\phi = 0.35$, and $\tau = 6$ is 6.8 percent. It is then straightforward to verify that only 2.62 percent of the differential in expected returns of value firms and the market return is compensation for risk exposure, while the remaining part of the difference is explained by a higher price of risk for these firms. For growth firms, the expected return is lower than the market premium, and 8.19 percent of this difference is due to risk exposure[27].

The model endogenizes time variation in risk premia, as well as valuation ratios. Comparing Figure 1 and Figure 4, it is apparent that time variation in the equity premium is relatively small, while the one generated for the value premium is comparatively important. Recall that for both premia time variation is induced by learning. However, while learning affects the time series through time-varying beliefs about the state of the economy only, in the cross section, it takes a second effect through time variation in the learning ratio. In the present model context, learning therefore not only accounts for most of the absolute size of the value premium, it also gives rise to important time variation across business cycles. The same holds true for the value spread. One is therefore inclined to conclude that learning might be a particularly important explanatory factor for regularities in the cross section.

Note that the induced movements in empirical regularities in the time series and the cross section coincide with NBER recessions. While on average, displayed variations are at slightly higher frequencies than business cycles, premia peak in all recessions[28].

Comparing the present model to models generating time-variation in risk premia through habit formation, it seems noteworthy that in the present context relative risk aversion is comparatively low and constant. Campbell and Cochrane (1999), for example, report levels of risk aversion of 60 and higher to match the equity premium. Moreover, relative risk aversion implied by models of habit formation is countercyclical. This in turn presumes that economic conditions impact agent's attitude towards risk.

Finally, in view of the *interest rate risk puzzle*, it is worth studying the model's implications for the risk-free rate, which so far had been normalized to zero for simplicity. Application of Itô's lemma to the state-price density of Proposition 1 lead to an implied risk-free rate that denotes

[27] I would like to thank Lars Hansen for suggesting to decouple risk exposure and risk price.

[28] note also that the value spread as well as the Sharpe Ratio are countercyclical

$$r_{f,t} = \phi\left(\gamma\frac{1-e^{-\phi(T-t)}}{\phi} - S_{\pi,\rho,t}\right)\left[\left(\theta_t\overline{\rho}^H + (1-\theta_t)\,\overline{\rho}^L\right) - \rho_t\right] \quad (41)$$
$$-\gamma\left(\frac{1-e^{-\phi(T-t)}}{\phi}\right)\left(\frac{\gamma}{2}\frac{1-e^{-\phi(T-t)}}{\phi} - S_{\pi,\rho,t}\right)\sigma^2 - S_{\pi,t_1,t} - S_{\pi,t_2,t}$$

where $S_{\pi,\rho,t}$, $S_{\pi,t_1,t}$, and $S_{\pi,t_2,t}$, are as denoted in the Appendix[29]. For the period of 1948-2001, calibrations yield a long-run risk-free rate of -1.06%, and an annual volatility of 4.79%, in comparison to an actual average rate of the 3-month Treasury Bill of 1.44%, and an average standard deviation of 3.08%. This result leads to two conclusions with respect to the interest rate risk puzzle. First, the model comes close to inducing observed interest rate *volatility*, while matching the equity premium. Second, in terms of the *level* of the risk-free rate, the puzzle seems slightly reversed. While consumption-based models are known to imply far too high risk-free rates when matching the equity premium, the present model instead implies a negative long-run rate.

5 Conclusion

The proposed general equilibrium model relates variations in GDP growth to risk premia in equity markets. It establishes variability in output as a priced risk factor, and reveals the potential significance of learning for the understanding of asset pricing anomalies. The model justifies equity premia of the time series and the cross section qualitatively, as well as quantitatively. Risk premia rationalize as they arise in compensation for exposure to non-diversifiable macroeconomic risk, their cyclical behavior obtains endogenously thanks to the inference structure of the model. The Fama and French (1993) factors are given economic content in representing the factor-mimicking portfolios of firms' time-varying relative exposure to recessionary periods of the economy. Finally, the model shows how the conditional CAPM combines in one, what the "unconditional" Fama and French model splits into three different factor exposures.

[29] see the second part to the Proof of Proposition 1.

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6 Appendix

6.1 Proof of Lemma 3

Aggregate productivity follows the process

$$d\rho_t = \phi\left(\overline{\rho} - \rho_t\right)dt + \sigma dZ_t. \tag{42}$$

Replacing into $dZ_{0,t}$ from Lemma 1, I obtain

$$dZ_{0,t} = \frac{1}{\sigma}(\phi\left(\overline{\rho} - E[\overline{\rho}|\mathcal{F}_t]\right)dt) + \sigma dZ_t$$

which - when plugged back into the initial expression - yields

$$d\rho_t^\theta \equiv d\rho_t = \phi(E[\overline{\rho}|\mathcal{F}_t] - \rho_t)dt + \sigma dZ_{0,t} \tag{43}$$

Q.E.D.

6.2 Proof of Lemma 4

The process followed by the agent's posterior belief about firm-specific shock $\widehat{\zeta}_{it}^{\nu}$, $\forall\ i \in I$, where $\zeta = \eta, \psi, s, l$ for $\nu = H, L$, is conditional on the filtration $\mathcal{F}_t = \left\{\left(\rho_t^\theta, \rho_{it}^{\zeta^\nu}\right) : t^* \leq t \leq T\right\}$ with processes

$$d\rho_t^\theta = \phi(\left(\theta_t\overline{\rho}^H + (1-\theta_t)\,\overline{\rho}^L\right) - \rho_t)dt + \sigma dZ_{0,t} \tag{44}$$

and

$$d\rho_{it}^{\zeta^\nu} = \phi(\left(\theta_t\overline{\rho}^H + (1-\theta_t)\,\overline{\rho}^L\right) + \zeta_i^\nu - \rho_{it})dt + \sigma_0 dZ_{0,t} + \sigma_1 dZ_{\zeta_i^\nu,t}. \tag{45}$$

Applying results from Liptser and Shiryaev (1977), the process for $\widehat{\zeta}_{it}^{\nu} = E_t\left[\zeta_i^\nu\right]$ can be expressed as

$$d\widehat{\zeta}_{it}^{\nu} = \widehat{\sigma}_{it}^2\mathbf{C}'\left(\textstyle\sum{}'\right)^{-1}d\widetilde{\mathbf{Z}}_t \tag{46}$$

where $\mathbf{C} = (0,\phi)'$, $\widetilde{\mathbf{Z}}_t = \left(\widetilde{Z}_{0,t}, \widetilde{Z}_{\zeta_i^\nu,t}\right)'$ and

$$\sum = \begin{pmatrix} \sigma & 0 \\ \sigma_0 & \sigma_1 \end{pmatrix},$$

where the process followed by $\widetilde{\mathbf{Z}}_t$ is given by

$$d\widetilde{\mathbf{Z}}_t = \sum{}^{-1}\left(\begin{array}{c} d\rho_t^\theta \\ d\rho_{it}^{\zeta^\nu}\end{array} - E_t\left[\begin{array}{c} d\rho_t^\theta \\ d\rho_{it}^{\zeta^\nu}\end{array}\right]\right).$$

The process followed by the posterior variance denotes

$$\frac{d\widehat{\sigma}_{it}^2}{dt} = -\left(\widehat{\sigma}_{it}^2\right)^2 \mathbf{C}' \left(\sum \sum{}'\right)^{-1} \mathbf{C}. \tag{47}$$

Simple algebra then yields

$$\mathbf{C}' \left(\sum{}'\right)^{-1} = \left(0, \frac{\phi}{\sigma_1} - \frac{\phi\sigma_0}{\sigma\sigma_1}\right) \tag{48}$$

which when replaced into (46) yields

$$d\widehat{\zeta}_{it}^{\nu} = \widehat{\sigma}_{it}^2 \left(\frac{\phi}{\sigma_1} - \frac{\phi\sigma_0}{\sigma\sigma_1}\right) d\widetilde{Z}_{\zeta_i^\nu, t}. \tag{49}$$

Next, I obtain

$$g \equiv \mathbf{C}' \left(\sum \sum{}'\right)^{-1} \mathbf{C} = \left(\frac{\phi}{\sigma_1} - \frac{\phi\sigma_0}{\sigma\sigma_1}\right)^2 \tag{50}$$

which when replaced into (47) yields

$$\frac{d\widehat{\sigma}_{it}^2}{dt} = -\left(\widehat{\sigma}_{it}^2\right)^2 \cdot g, \tag{51}$$

a Riccati differential equation with solution

$$\widehat{\sigma}_{it}^2 = \left[\widehat{\sigma}_{t^*}^{-2} + g \cdot (\tau_{it} - t^*)\right]^{-1} \tag{52}$$

where $\tau_{it} \in [0, t]$ denotes the aggregate time spent observing firm i between 0 and t.

Q.E.D.

6.3 Lemma A1

For $t \in [t^*, T]$, conditional on regime $\nu = H, L$, the original processes can be written as follows,

$$db_{it} = \rho_t^\nu dt \tag{53}$$

$$d\rho_t^\nu = \phi \left(\overline{\rho}^\nu + \widehat{\zeta}_{it}^\nu - \rho_t\right) dt + \sigma d\widetilde{Z}_{0,t} \tag{54}$$

$$d\widehat{\zeta}_{it}^{\nu} = \widehat{\sigma}_{it}^2 \left(\frac{\phi}{\sigma_1} - \frac{\phi\sigma_0}{\sigma\sigma_1}\right) d\widetilde{Z}_{\zeta_i^\nu, t} \tag{55}$$

$$d\widehat{\sigma}_{it}^2 = -\left(\widehat{\sigma}_{it}^2\right)^2 \cdot g \cdot dt \tag{56}$$

where $b_{it} \equiv \ln[B_{it}]$ $\forall$ $i \in I$, and $\zeta = \eta, \psi, s, l$.

The conditional value function obtains as

$$V\left(B_{it}, \rho_t^{\nu}, \widehat{\zeta}_{it}^{\nu}, \widehat{\sigma}_{it}^2, T-t\right) = E_t\left[\frac{\left(B_{iT}^{\nu}\right)^{1-\gamma}}{1-\gamma} \mid \nu = H, L\right] \quad (57)$$
$$= \frac{B_{it}^{1-\gamma}}{1-\gamma} e^{A_0^{\nu}(T-t)+(1-\gamma)A_1(T-t)\rho_t+(1-\gamma)A_2(T-t)\widehat{\zeta}_{it}^{\nu}+\frac{1}{2}(1-\gamma)^2 A_2(T-t)^2\widehat{\sigma}_{it}^2}$$

where

$$A_0^{\nu}(T-t) = (1-\gamma)\,\overline{\rho}^{\nu} A_2(T-t) + \frac{\sigma^2}{2}\frac{(1-\gamma)^2}{\phi^2}\left\{(T-t) + \frac{1-e^{-2\phi(T-t)}}{2\phi} - 2A_1(T-t)\right\} \quad (58)$$

$$A_1(T-t) = \frac{1-e^{-\phi(T-t)}}{\phi} \qquad A_2(T-t) = (T-t) - A_1(T-t). \quad (59)$$

6.3.1 Proof of Lemma A1

Given that

$$V\left(b_{it}, \rho_t^{\nu}, \widehat{\zeta}_{it}^{\nu}, \widehat{\sigma}_{it}^2, t\right) = \frac{1}{1-\gamma} E\left[e^{(1-\gamma)\int_t^T b_{is} ds} \mid \digamma_t\right]$$

one is looking for a function $f\left(\rho_t^{\nu}, \widehat{\zeta}_{it}^{\nu}, \widehat{\sigma}_{it}^2, t\right)$, such that

I) $f\left(\rho_t^{\nu}, \widehat{\zeta}_{it}^{\nu}, \widehat{\sigma}_{it}^2, t\right) \cdot e^{(1-\gamma)\int_0^t b_{is} ds} = X_{it}$
is a martingale, i.e. $E\left[X_{iT} | \digamma_t\right] = X_{it}$,
and
II) $f\left(\rho_T^{\nu}, \widehat{\zeta}_{iT}^{\nu}, \widehat{\sigma}_{iT}^2, T\right) = 1$ $\forall \rho_T^{\nu}, \widehat{\zeta}_{iT}^{\nu}, \widehat{\sigma}_{iT}^2$.

Hence, $X_{iT} = e^{(1-\gamma)\int_0^T b_{is} ds}$ and $X_{it} = E\left[e^{(1-\gamma)\int_t^T b_{is} ds} | \digamma_t\right]$.

Applying Itô's Lemma, assuming $\widetilde{Z}_{0,t}$ to be uncorrelated with $\widetilde{Z}_{\zeta_i^{\nu},t} \forall i \in I$ for each regime $\nu = H, L$, one obtains that

$$df\left(\rho_t^{\nu}, \widehat{\zeta}_{it}^{\nu}, \widehat{\sigma}_{it}^2, t\right) = \left\{\begin{array}{c} f_{\rho}\phi\left(\overline{\rho}^{\nu} + \widehat{\zeta}_{it}^{\nu} - \rho_t\right) - f_{\sigma^2}\left(\widehat{\sigma}_{it}^2\right)^2 g \\ +f_t + \frac{1}{2} f_{\rho\rho}\sigma^2 + \frac{1}{2} f_{\zeta\zeta}\left(\widehat{\sigma}_{it}^2\right)^2 g \end{array}\right\} dt$$

$$+f_\rho \sigma d\widetilde{Z}_{0,t} + f_\zeta \widehat{\sigma}_{it}^2 \left(\frac{\phi}{\sigma_1} - \frac{\phi\sigma_0}{\sigma\sigma_1} \right) d\widetilde{Z}_{\zeta_i^\nu,t}$$

where f_x and f_{xx} denote the respective first and second order partial derivatives, where $x = \rho, \zeta, \sigma^2, t$.

Knowing that $E[X_{iT}|\digamma_t] = X_{it}$ with X_{it} as defined under I), one can write that

$$f_\rho \phi \left(\overline{\rho}^\nu + \widehat{\zeta}_{it}^\nu - \rho_t \right) - f_{\sigma^2} \left(\widehat{\sigma}_{it}^2 \right)^2 g + f_t + \frac{1}{2} f_{\rho\rho} \sigma^2 + \frac{1}{2} f_{\zeta\zeta} \left(\widehat{\sigma}_{it}^2 \right)^2 g = 0.$$

I then conjecture a solution of the form

$$f\left(\rho_t^\nu, \widehat{\zeta}_{it}^\nu, \widehat{\sigma}_{it}^2, t \right) = e^{(1-\gamma)\left[a(t) + c(t)\rho_t + d(t)\widehat{\zeta}_{it}^\nu, + h(t)\widehat{\sigma}_{it}^2 \right]} \qquad \forall \rho_t^\nu, \widehat{\zeta}_{it}^\nu, \widehat{\sigma}_{it}^2.$$

Applying the method of undetermined coefficients and rearranging terms, yields the solution as denoted in (57), which in turn can be verified to fulfill the above PDE.
Q.E.D.

6.4 Proof of Proposition 1

As the Proof of Lemma A1, replacing $1-\gamma$ by $-\gamma$, as $\pi_t = \lambda^{-1} E_t \left[B_T^{-\gamma} \right]$. As productivity shocks ζ_i^ν are idiosyncratic, their market-level aggregate equals zero, i.e. $\int_{i \in I} \zeta_i^\nu di = 0$ for $\nu = H, L$. The economy as a whole therefore grows at rate ρ_t, such as defined in (2). Hence, I obtain that

$$\pi_t^\nu \equiv E_t \left[\left(\frac{B_T^\nu}{B_t} \right)^{-\gamma} \mid \nu = H, L \right] = e^{\widetilde{A}_0^\nu (T-t) - \gamma A_1 (T-t) \rho_t} \tag{60}$$

where

$$\widetilde{A}_0^\nu (T-t) = -\gamma \overline{\rho}^\nu A_2 (T-t) + \frac{\sigma^2}{2} \frac{\gamma^2}{\phi^2} \left\{ \tau + \frac{1 - e^{-2\phi(T-t)}}{2\phi} - 2A_1 (T-t) \right\}.$$

The state-price density is given by

$$\pi_t = \lambda^{-1} E_t \left[B_T^{-\gamma} \right] = \lambda^{-1} \left\{ E_t [\theta_T] \cdot E_t \left[\left(B_T^H \right)^{-\gamma} \right] + (1 - E_t [\theta_T]) \cdot E_t \left[\left(B_T^L \right)^{-\gamma} \right] \right\} \tag{61}$$

In order to obtain $E[\theta_T \mid \digamma_t]$, I integrate $d\theta_t$ from Lemma 1, obtaining

$$\theta_T = \theta_t + \int_t^T (\lambda + \mu)(\theta^s - \theta_u) du + \int_t^T h(\theta_u) dZ_{0,u}. \tag{62}$$

It follows that

$$E\left[\theta_T \mid \digamma_t\right] = \theta_t + \int_t^T (\lambda+\mu)\left(\theta^s - E\left[\theta_u \mid \digamma_t\right]\right) du. \tag{63}$$

Assuming that t is fixed, and denoting $\psi_T = E\left[\theta_T \mid \digamma_t\right]$, where $\psi_t = \theta_t$, then

$$\psi_T = \theta_t + \int_t^T (\lambda+\mu)\left(\theta^s - \psi_u\right) du, \tag{64}$$

$$\frac{\partial \psi_T}{\partial T} = (\lambda+\mu)\left(\theta^s - \psi_T\right), \tag{65}$$

and it follows that

$$\psi_T = E\left[\theta_T \mid \digamma_t\right] = \theta^s + \left(\theta_t - \theta^s\right) e^{-(\lambda+\mu)(T-t)}. \tag{66}$$

The state-price density then obtains as

$$\pi_t = \lambda^{-1} B_t^{-\gamma}\left[\left(\theta^s + \left(\theta_t - \theta^s\right) e^{-(\lambda+\mu)(T-t)}\right)\cdot \pi_t^H + \left(1 - \left(\theta^s + \left(\theta_t - \theta^s\right) e^{-(\lambda+\mu)(T-t)}\right)\right)\cdot \pi_t^L\right] \tag{67}$$

Q.E.D.

The risk-free rate r_f obtains from applying Itô's lemma to π_t, and is given by

$$\begin{aligned} r_{f,t} \;=\; & \phi\left(\gamma\frac{1-e^{-\phi(T-t)}}{\phi} - S_{\pi,\rho,t}\right)\left[\left(\theta_t\overline{\rho}^H + \left(1-\theta_t\right)\overline{\rho}^L\right) - \rho_t\right] \\ & -S_{\pi,t_1,t} - S_{\pi,t_2,t} - \gamma\left(\frac{1-e^{-\phi(T-t)}}{\phi}\right)\left(\frac{\gamma}{2}\frac{1-e^{-\phi(T-t)}}{\phi} - S_{\pi,\rho,t}\right)\sigma^2 \end{aligned} \tag{68}$$

where

$$S_{\pi,\rho,t} = \frac{\frac{\partial E_t[\theta_T]}{\partial \rho_t}\left(\pi_t^H - \pi_t^L\right)}{E_t\left[\theta_T\right]\cdot \pi_t^H + \left(1 - E_t\left[\theta_T\right]\right)\cdot \pi_t^L}, \tag{69}$$

$$S_{\pi,t_1,t} = \frac{\frac{\partial E_t[\theta_T]}{\partial t}\left(\pi_t^H - \pi_t^L\right)}{E_t\left[\theta_T\right]\cdot \pi_t^H + \left(1 - E_t\left[\theta_T\right]\right)\cdot \pi_t^L}, \tag{70}$$

and

$$S_{\pi,t_2,t} = \frac{E_t\left[\theta_T\right]\frac{\partial \pi_t^H}{\partial t} + \left(1 - E_t\left[\theta_T\right]\right)\frac{\partial \pi_t^L}{\partial t}}{E_t\left[\theta_T\right]\cdot \pi_t^H + \left(1 - E_t\left[\theta_T\right]\right)\cdot \pi_t^L}. \tag{71}$$

$\frac{\partial E_t[\theta_T]}{\partial \rho_t} = \frac{\partial \theta_t}{\partial \rho_t} e^{-(\lambda+\mu)(T-t)}$, as can be verified from (66), $\frac{\partial E_t[\theta_T]}{\partial t}$, and $\frac{\partial \pi_t^\nu}{\partial t}$ for $\nu = H, L$, obtain as easily, and are neglected for the sake of space. Note, that as r_f was assumed normalized to 0, there will be no drift term when applying Itô's lemma to π_t in subsequent Proofs.

Q.E.D.

6.5 Proof of Proposition 2

For $t \in [0, t^*)$, firm productivity does not differ across firms $i \in I$. Consequently, it is optimal for the agent to equally allocate her time across the continuum of firms. Subsequent derivations are then for $t \in [t^*, T]$.

Let me denote

$$V\left(B_{it}, \rho_t^{\nu}, \widehat{\zeta}_{it}^{\nu}, \widehat{\sigma}_{it}^2, T-t\right) = \frac{B_{it}^{1-\gamma}}{1-\gamma} \cdot F\left(\rho_t^{\nu}, \widehat{\zeta}_{it}^{\nu}, \widehat{\sigma}_{it}^2, T-t\right). \tag{72}$$

The time t optimal amount of time $(\tau_{it}^{\nu})^* \in [0, T-t^*]$, to be spent observing firm i in regime ν, needs to satisfy the following optimality condition for $\nu = H, L$

$$\frac{\partial}{\partial \tau_{it}^{\nu}} \cdot F\left(\rho_t^{\nu}, \widehat{\zeta}_{it}^{\nu}, \widehat{\sigma}\left(\tau_{it}^{\nu}\right)^2, T-t\right) = 0. \tag{73}$$

Recalling the expression for $\widehat{\sigma}_{it}^2$ from (52), the marginal utility from learning about firm $i \in I$ during regime ν obtains as

$$\frac{\partial F_i(\cdot)}{\partial \tau_{it}^{\nu}} = -\frac{1}{2}(1-\gamma)^2 A_2 (T-t)^2 g \cdot \left[(\widehat{\sigma}_{t^*})^{-2} + g \cdot (\tau_{it}^{\nu} - t)\right]^{-2} \cdot F_i(\cdot) \tag{74}$$

where

$$F_i(\cdot) \equiv F\left(\rho_t^{\nu}, \widehat{\zeta}_{it}^{\nu}, \widehat{\sigma}\left(\tau_{it}^{\nu}\right)^2, T-t\right) \quad \forall i \in I. \tag{75}$$

The equilibrium condition for optimal learning allocations is then given by

$$\frac{\partial F_i(\cdot)}{\partial \tau_{it}^{\nu}} = \frac{\partial F_j(\cdot)}{\partial \tau_{jt}^{\nu}} \qquad \forall i \neq j \text{ with } i, j \in I, \tag{76}$$

which simplifies to

$$\frac{F\left(\rho_t^{\nu}, \widehat{\zeta}_{it}^{\nu}, \widehat{\sigma}\left(\left(\tau_{it}^{\nu}\right)^*\right)^2, T-t\right)}{F\left(\rho_t^{\nu}, \widehat{\zeta}_{jt}^{\nu}, \widehat{\sigma}\left(\left(\tau_{jt}^{\nu}\right)^*\right)^2, T-t\right)} = \frac{\left[(\widehat{\sigma}_{t^*})^{-2} + g \cdot (\tau_{it}^{\nu})^*\right]^2}{\left[(\widehat{\sigma}_{t^*})^{-2} + g \cdot \left(\tau_{jt}^{\nu}\right)^*\right]^2} \qquad \forall i \neq j \text{ with } i, j \in I \tag{77}$$

where $(\tau_{it}^{\nu})^* \in [0, T-t^*]$ denotes the time t optimal time to learn about firm $i \in I$ for $t \in [t^*, T]$ for $\nu = H, L$.

For the sake of space, I will restrict myself to the comparison of value firms, $i \in \eta$, and growth firms, $i \in \psi$. In order to obtain an expression for the aggregate (average) ratio of optimal learning times of both types of firms, I need to aggregate both sides of (77) across firms of each type. For the RHS it suffices to replace τ_{it}^{ν} by $\tau_t^{\psi^{\nu}} \equiv \int_{i\in\psi} \tau_{it}^{\nu} di$

and τ_{jt}^{ν} by $\tau_t^{\eta^\nu} \equiv \int_{j\in\eta} \tau_{jt}^{\nu} dj$. The LHS is slightly more complex. I will apply a law of large numbers in strong form[30] by means of the Glivenko-Cantelli Theorem. It follows from the latter that the cross-sectional distribution of the i.i.d. shocks to firm productivity equals its stationary distribution. Accounting for (3), (5), and (6), the stationary distribution for ζ and $\nu = H, L$, is given by

$$f\left(\zeta^\nu; \mu_{\zeta^\nu}, \widehat{\sigma}\left(\tau_t^{\zeta^\nu}\right)\right) = \frac{1}{\widehat{\sigma}\left(\tau_t^{\zeta^\nu}\right)\sqrt{2\Pi}} \exp\left(-\frac{\left(\zeta^\nu - \mu_{\zeta^\nu}\right)^2}{2\widehat{\sigma}\left(\tau_t^{\zeta^\nu}\right)^2}\right). \tag{78}$$

Denoting $F^\zeta(\cdot) \equiv \int_{i\in\zeta} F(\cdot)\, di$, I can then write that

$$\begin{aligned} F^\zeta(\cdot) &= \int_{i\in\zeta} e^{A_0^\nu(T-t)+(1-\gamma)A_1(T-t)\rho_t+(1-\gamma)A_2(T-t)\widehat{\zeta}_{it}^\nu+\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\left(\tau_{it}^\nu\right)^*\right)^2} di \\ &= e^{A_0^\nu(T-t)+(1-\gamma)A_1(T-t)\rho_t} \int_{i\in\zeta} e^{(1-\gamma)A_2(T-t)\widehat{\zeta}_{it}^\nu+\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\left(\tau_{it}^\nu\right)^*\right)^2} di, \end{aligned} \tag{79}$$

which can be reexpressed as

$$F^\zeta(\cdot) = e^{A_0^\nu(T-t)+(1-\gamma)A_1(T-t)\rho_t} \int_{i\in\zeta} E_t\left[e^{(1-\gamma)A_2(T-t)\widehat{\zeta}_{it}^\nu}\right] di. \tag{80}$$

Applying the Glivenko-Cantelli Theorem, I can write that

$$\begin{aligned} F^\zeta(\cdot) &= e^{A_0^\nu(T-t)+(1-\gamma)A_1(T-t)\rho_t} \int_0^\infty E_t\left[e^{(1-\gamma)A_2(T-t)\widehat{\zeta}_t^\nu}\right] \cdot f\left(\zeta^\nu; \mu_{\zeta^\nu}, \widehat{\sigma}\left(\tau_t^{\zeta^\nu}\right)\right) d\zeta_t^\nu \\ &= e^{A_0^\nu(T-t)+(1-\gamma)A_1(T-t)\rho_t} \cdot E\left[E_t\left[e^{(1-\gamma)A_2(T-t)\widehat{\zeta}_t^\nu}\right]\right] \end{aligned} \tag{81}$$

which by the law of iterated expectations yields

$$F^\zeta(\cdot) = e^{A_0^\nu(T-t)+(1-\gamma)A_1(T-t)\rho_t+(1-\gamma)A_2(T-t)\mu_{\zeta^\nu}+\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\tau_t^{\zeta^\nu}\right)^2}. \tag{82}$$

As $\mu_{\eta^L} < \mu_{\psi^L}$, it follows that a priori, that is when $\tau_t^{\psi^\nu} = \tau_t^{\eta^\nu}$, for $\nu = H, L$, as $\gamma > 1$ and $A_2(T-t) > 0$,

$$\frac{F^{\psi^L}(\cdot)}{F^{\eta^L}(\cdot)} = e^{(1-\gamma)A_2(T-t)\left(\mu_{\psi^L}-\mu_{\eta^L}\right)} < 1. \tag{83}$$

[30] see Judd (1985) for a proof of existence for a continuum of i.i.d. random variables

As $\frac{\partial\left[\widehat{\sigma}_{t^*}^{-2}+g\cdot\left(\tau_{it}^{\nu}-t^*\right)\right]^{-1}}{\partial\tau_{it}^{\nu}} < 0$, it follows that $\frac{\partial F\left(\rho_t^{\nu},\widehat{\zeta}_{it}^{\nu},\widehat{\sigma}\left(\tau_{it}^{\nu}\right)^2,T-t\right)}{\partial\tau_{it}^{\nu}} < 0, \forall i \in I, \nu = H, L$. Condition (77) then implies that $\left(\tau_t^{\eta^L}\right)^* > \left(\tau_t^{\psi^L}\right)^* \ \forall\ t \in [t^*, T]$ for an equilibrium to obtain. As $\mu_{\eta^H} = \mu_{\psi^H}$, $F^{\psi^H}(\cdot)/F^{\eta^H}(\cdot) = 1$ and hence $\left(\tau_t^{\eta^H}\right)^* = \left(\tau_t^{\psi^H}\right)^* \forall t \in [t^*, T]$.

Given the conditional value function of (57), recalling that initial capital allocation is unbiased for $\nu = H, L$, the unconditional value function can be expressed as

$$
\begin{aligned}
& V\left(B_{it}, \rho_t, \widehat{\zeta}_{it}, \widehat{\sigma}_{it}^2, T-t\right) \qquad (84)\\
= \ & \frac{B_{it}^{1-\gamma}}{1-\gamma}\left[E_t\left[\theta_T\right]\cdot F\left(\rho_t^H, \widehat{\zeta}_{it}^H, \widehat{\sigma}_{it}^2, T-t\right) + \left(1-E_t\left[\theta_T\right]\right)\cdot F\left(\rho_t^L, \widehat{\zeta}_{it}^L, \widehat{\sigma}_{it}^2, T-t\right)\right].
\end{aligned}
$$

I can then express the optimal learning ratio between value and growth firms as

$$
\begin{aligned}
\left(\frac{\tau_t^{\eta}}{\tau_t^{\psi}}\right)^* &= E_t\left[\theta_T\right]\left(\frac{\tau_t^{\eta^H}}{\tau_t^{\psi^H}}\right)^* + \left(1-E_t\left[\theta_T\right]\right)\left(\frac{\tau_t^{\eta^L}}{\tau_t^{\psi^L}}\right)^* \qquad (85)\\
&= E_t\left[\theta_T\right] + \left(1-E_t\left[\theta_T\right]\right)\cdot\chi_t\left(\mu_{\psi^L}-\mu_{\eta^L}\right)
\end{aligned}
$$

where $\chi_t(\cdot) > 1\ \forall\ t \in [t^*, T]$. Note also that $\frac{\partial\chi_t(\cdot)}{\partial\left(\mu_{\psi^L}-\mu_{\eta^L}\right)} > 0$.
Q.E.D.

6.6 Proof of Proposition 4

The market value of firm $i \in I$ can be expressed as $M_{it} = E_t\left[B_{iT}\pi_T\right]/\pi_t = \lambda^{-1}E_t\left[B_{iT}^{1-\gamma}\right]/\pi_t$, where $E_t\left[B_{iT}^{1-\gamma}\right] = E_t\left[\theta_T\right]\cdot E_t\left[\left(B_{iT}^H\right)^{1-\gamma}\right] + \left(1-E_t\left[\theta_T\right]\right)\cdot E_t\left[\left(B_{iT}^L\right)^{1-\gamma}\right]$. The market-to-book ratio of firms $i \in I$ then obtains as

$$
\begin{aligned}
\frac{M_{it}}{B_{it}} = \ & E_t[\theta_T]\cdot\frac{F\left(\rho_t^H, \widehat{\zeta}_{it}^H, \widehat{\sigma}\left(\left(\tau_{it}^H\right)^*\right)^2, T-t\right)}{E_t[\theta_T]\cdot\pi_t^H + \left(1-E_t[\theta_T]\right)\cdot\pi_t^L}\\
& + \left(1-E_t[\theta_T]\right)\cdot\frac{F\left(\rho_t^L, \widehat{\zeta}_{it}^L, \widehat{\sigma}\left(\left(\tau_{it}^L\right)^*\right)^2, T-t\right)}{E_t[\theta_T]\cdot\pi_t^H + \left(1-E_t[\theta_T]\right)\cdot\pi_t^L}, \qquad (86)
\end{aligned}
$$

where

$$
\begin{aligned}
& F\left(\rho_t^{\nu}, \widehat{\zeta}_{it}^{\nu}, \widehat{\sigma}\left(\left(\tau_{it}^{\nu}\right)^*\right)^2, T-t\right)\\
= \ & e^{A_0^{\nu}(T-t)+(1-\gamma)A_1(T-t)\rho_t+(1-\gamma)A_2(T-t)\widehat{\zeta}_{it}^{\nu}+\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\left(\tau_{it}^{\nu}\right)^*\right)^2}. \qquad (87)
\end{aligned}
$$

I then aggregate market-to-book ratios across firms i of type ζ, defining $\frac{M_t^\zeta}{B_t} \equiv \int_{i\in\zeta} \frac{M_{it}}{B_{it}} di$, where $\int_{i\in\zeta} B_{it} di = B_t^\zeta = B_t$, given that capital allocation is unbiased. As in the Proof of Proposition 2, I will make use of the Glivenko-Cantelli Theorem. Note however, that the stationary distribution in the present application differs slightly from (78). The reason is that true mean values are replaced by the agent's unbiased prior expectations on idiosyncratic technology shocks. Hence, $\mu_{\zeta^\nu} = 0$ for $\nu = H, L$. The stationary distribution is then given by

$$f\left(\zeta^\nu; 0, \widehat{\sigma}\left(\left(\tau_t^{\zeta^\nu}\right)^*\right)\right) = \frac{1}{\widehat{\sigma}\left(\left(\tau_t^{\zeta^\nu}\right)^*\right)\sqrt{2\Pi}} \exp\left(-\frac{\left(\zeta^\nu\right)^2}{2\widehat{\sigma}\left(\left(\tau_t^{\zeta^\nu}\right)^*\right)^2}\right). \tag{88}$$

I can therefore now write that

$$\begin{aligned}\frac{M_t^\zeta}{B_t} &= E_t[\theta_T] \cdot \left(E_t[\theta_T] \cdot \pi_t^H + (1 - E_t[\theta_T]) \cdot \pi_t^L\right)^{-1} \\ &\cdot \int_{i\in\zeta} e^{A_0^H(T-t)+(1-\gamma)A_1(T-t)\rho_t+(1-\gamma)A_2(T-t)\widehat{\zeta}_{it}^H+\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\left(\tau_{it}^H\right)^*\right)^2} di \\ &+ (1 - E_t[\theta_T]) \cdot \left(E_t[\theta_T] \cdot \pi_t^H + (1 - E_t[\theta_T]) \cdot \pi_t^L\right)^{-1} \\ &\cdot \int_{i\in\zeta} e^{A_0^L(T-t)+(1-\gamma)A_1(T-t)\rho_t+(1-\gamma)A_2(T-t)\widehat{\zeta}_{it}^L+\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\left(\tau_{it}^L\right)^*\right)^2} di \end{aligned} \tag{89}$$

which I reexpress as

$$\begin{aligned}\frac{M_t^\zeta}{B_t} &= \left(E_t[\theta_T] \cdot \pi_t^H + (1 - E_t[\theta_T]) \cdot \pi_t^L\right)^{-1} \times \\ &\left[\begin{array}{c} E_t[\theta_T] \cdot e^{A_0^H(T-t)+(1-\gamma)A_1(T-t)\rho_t} \int_{i\in\zeta} E_t\left[e^{A_2(T-t)\widehat{\zeta}_{it}^H}\right] di \\ + (1 - E_t[\theta_T]) \cdot e_t^{A_0^L(T-t)+(1-\gamma)A_1(T-t)\rho_t} \int_{i\in\zeta} E_t\left[e^{A_2(T-t)\widehat{\zeta}_{it}^L}\right] di \end{array}\right]. \end{aligned} \tag{90}$$

Applying the Glivenko-Cantelli Theorem, I obtain

$$\begin{aligned}\frac{M_t^\zeta}{B_t} &= \left(E_t[\theta_T] \cdot \pi_t^H + (1 - E_t[\theta_T]) \cdot \pi_t^L\right)^{-1} \times \\ &\left[\begin{array}{c} E_t[\theta_T] \cdot e^{A_0^H(T-t)+(1-\gamma)A_1(T-t)\rho_t} \\ \cdot \int_0^\infty E_t\left[e^{A_2(T-t)\widehat{\zeta}_t^H}\right] \cdot f\left(\zeta^H; 0, \widehat{\sigma}\left(\left(\tau_t^{\zeta^H}\right)^*\right)\right) d\zeta^H \\ + (1 - E_t[\theta_T]) \cdot e^{A_0^L(T-t)+(1-\gamma)A_1(T-t)\rho_t} \\ \cdot \int_0^\infty E_t\left[e^{A_2(T-t)\widehat{\zeta}_t^L}\right] \cdot f\left(\zeta^L; 0, \widehat{\sigma}\left(\left(\tau_t^{\zeta^L}\right)^*\right)\right) d\zeta^L \end{array}\right]\end{aligned}$$

$$
\begin{aligned}
= & \left(E_t[\theta_T]\cdot\pi_t^H+(1-E_t[\theta_T])\cdot\pi_t^L\right)^{-1}\times \\
& \left[\begin{array}{c} E_t\left[\theta_T\right]\cdot e^{A_0^H(T-t)+(1-\gamma)A_1(T-t)\rho_t}\cdot E\left[E_t\left[e^{A_2(T-t)\widehat{\zeta}_t^H}\right]\right] \\ +\left(1-E_t\left[\theta_T\right]\right)\cdot e^{A_0^L(T-t)+(1-\gamma)A_1(T-t)\rho_t}\cdot E\left[E_t\left[e^{A_2(T-t)\widehat{\zeta}_t^L}\right]\right]\end{array}\right]. \qquad (91)
\end{aligned}
$$

Applying the law of iterated expectations, accounting for the under (88) defined stationary distribution $f\left(\zeta^{\nu};0,\widehat{\sigma}\left(\left(\tau_t^{\zeta^H}\right)^*\right)\right)$ and simplifying, I obtain that

$$
\begin{aligned}
\frac{M_t^{\zeta}}{B_t} = & E_t\left[\theta_T\right]\cdot\frac{e^{A_0^H(T-t)+(1-\gamma)A_1(T-t)\rho_t+\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\left(\tau_t^{\zeta^H}\right)^*\right)^2}}{E_t\left[\theta_T\right]\cdot\pi_t^H+\left(1-E_t\left[\theta_T\right]\right)\cdot\pi_t^L} \\
& +\left(1-E_t\left[\theta_T\right]\right)\cdot\frac{e^{A_0^L(T-t)+(1-\gamma)A_1(T-t)\rho_t+\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\left(\tau_t^{\zeta^L}\right)^*\right)^2}}{E_t\left[\theta_T\right]\cdot\pi_t^H+\left(1-E_t\left[\theta_T\right]\right)\cdot\pi_t^L} \qquad (92)
\end{aligned}
$$

As optimal learning allocations between value and growth firms only differ during bad times, that is $\left(\tau_t^{\eta^L}\right)^*>\left(\tau_t^{\psi^L}\right)^*$ $\forall t\in[t^*,T]$, the value spread obtains as

$$
\begin{aligned}
\frac{M_t^{\psi}}{B_t}-\frac{M_t^{\eta}}{B_t} = & \frac{(1-E_t[\theta_T])}{E_t[\theta_T]\cdot\pi_t^H+(1-E_t[\theta_T])\cdot\pi_t^L}\cdot e^{A_0^L(T-t)+(1-\gamma)A_1(T-t)\rho_t} \qquad (93) \\
& \cdot\left(e^{\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\left(\tau_t^{\psi^L}\right)^*\right)^2}-e^{\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\left(\tau_t^{\eta^L}\right)^*\right)^2}\right).
\end{aligned}
$$

It follows that as $\pi_t^H-\pi_t^L<0$, $\partial\frac{M_t^{\psi}}{B_t}/\partial\rho_t-\partial\frac{M_t^{\eta}}{B_t}/\partial\rho_t>0$, $\forall\gamma,\phi,\tau$. As GDP growth, ρ_t, is decreasing at peaks and increasing in troughs, the value spread is negatively correlated with business cycles.

Q.E.D.

6.7 Proof of Proposition 3

As $\frac{\partial\widehat{\sigma}\left(\tau_{it}^{\nu}\right)^2}{\partial\tau_{it}^{\nu}}/<0\,\forall\,i\in I$, $\nu=H,L$, the result follows given that $\partial\left(\frac{M_{it}}{B_{it}}\right)/\partial\widehat{\sigma}\left(\tau_{it}^{\nu}\right)^2>0$ $\forall i\in I$, $\nu=H,L$.

Q.E.D.

6.8 Proof of Proposition 5

At $t = t^{**}$, where $t \in (t^*, T)$, the agent is assumed to become aware of the difference in cross-sectional means of the underlying distribution of technology shocks during recessions. Consequently, for $\nu = H$, the stationary distribution remains as denoted in (88), whereas for $\nu = L$, it is the one of (78).

In order to obtain an expression for the value premium, I now turn to derive return expectations for firms $i \in \zeta$. These obtain from the standard pricing equation $\mu_t^\zeta \equiv -cov_t\left(\frac{d\pi_{\zeta,t}}{\pi_{\zeta,t}}, dR_t^\zeta\right)$, where R_t^ζ denotes the return on firms $i \in \zeta$, and $\pi_{\zeta,t}$ their state price density. Given the agent's knowledge of μ_{ζ^ν}, the latter, is given by

$$\pi_{\zeta,t} = \lambda^{-1} B_{\zeta,t}^{-\gamma} \cdot \left\{E_t[\theta_T] \cdot \pi_{\zeta,t}^H + (1 - E_t[\theta_T]) \cdot \pi_{\zeta,t}^L\right\} \tag{94}$$

where

$$\pi_{\zeta,t}^\nu = e^{\widetilde{A}_0^\nu (T-t) - \gamma A_1 (T-t)\rho_t - \gamma A_2 (T-t)\mu_{\zeta^\nu}} \quad \text{for } \nu = H, L \tag{95}$$

and $B_{\zeta,t} \equiv \int_{i\in\zeta} B_{it} di$.

Applying Itô's Lemma, the following dynamics for $\pi_{\zeta,t}$ obtain

$$\frac{d\pi_{\zeta,t}}{\pi_{\zeta,t}} = -\sigma_{\pi_\zeta,\rho,t} d\widetilde{Z}_{0,t} - \sigma_{\pi_\zeta,\zeta,t} d\widetilde{Z}_{\zeta,t} \tag{96}$$

where

$$\sigma_{\pi_\zeta,\rho,t} = \left(\gamma A_1 (T-t) - S_{\pi_\zeta,\rho,t}\right)\sigma \tag{97}$$

with

$$S_{\pi_\zeta,\rho,t} = \frac{\frac{\partial E_t[\theta_T]}{\partial \rho_t}\left(\pi_{\zeta,t}^H - \pi_{\zeta,t}^L\right)}{E_t[\theta_T] \cdot \pi_{\zeta,t}^H + (1 - E_t[\theta_T]) \cdot \pi_{\zeta,t}^L}, \tag{98}$$

and

$$\sigma_{\pi_\zeta,\zeta,t} = S_{\pi_\zeta,\zeta,t} \widehat{\sigma}\left(\left(\tau_t^\zeta\right)^*\right)^2 \left(\frac{\phi}{\sigma_1} - \frac{\phi\sigma_0}{\sigma\sigma_1}\right) \tag{99}$$

where

$$S_{\pi_\zeta,\zeta,t} = \frac{(1 - E_t[\theta_T])\,\gamma A_2 (T-t) \cdot \pi_{\zeta,t}^L}{E_t[\theta_T] \cdot \pi_{\zeta,t}^H + (1 - E_t[\theta_T]) \cdot \pi_{\zeta,t}^L}. \tag{100}$$

Applying now Itô's Lemma to $M_t^\zeta = E_t\left[B_T^\zeta \pi_{\zeta,T}\right]/\pi_{\zeta,t}$, the return process for firms $i \in \zeta$ obtains as follows

$$dR_t^\zeta \equiv \frac{dM_t^\zeta}{M_t^\zeta} = \mu_t^\zeta dt + \sigma_{R^\zeta,\rho,t} d\widetilde{Z}_{0,t} + \sigma_{R^\zeta,\zeta,t} d\widetilde{Z}_{\zeta,t} \tag{101}$$

where

$$\mu_t^{\zeta} = \sigma_{\pi_\zeta,\rho,t} \cdot \sigma_{R^\zeta,\rho,t} + \sigma_{\pi_\zeta,\zeta,t} \cdot \sigma_{R^\zeta,\zeta,t} \tag{102}$$

with

$$\sigma_{R^\zeta,\rho,t} = \left(A_1 (T-t) + S_{R^\zeta,\rho,t} - S_{\pi_\zeta,\rho,t}\right) \sigma, \tag{103}$$

where

$$\begin{aligned} & S_{R^\zeta,\rho,t} \\ = & \frac{\frac{\partial E_t[\theta_T]}{\partial \rho_t} \left(F\left(\rho_t^H, 0, \widehat{\sigma}\left(\left(\tau_t^{\zeta^H}\right)^*\right)^2, T-t\right) - F\left(\rho_t^L, \mu_{\zeta^L}, \widehat{\sigma}\left(\left(\tau_t^{\zeta^L}\right)^*\right)^2, T-t\right)\right)}{E_t[\theta_T] \cdot F\left(\rho_t^H, 0, \widehat{\sigma}\left(\left(\tau_t^{\zeta^H}\right)^*\right)^2, T-t\right) + (1-E_t[\theta_T]) \cdot F\left(\rho_t^L, \mu_{\zeta^L}, \widehat{\sigma}\left(\left(\tau_t^{\zeta^L}\right)^*\right)^2, T-t\right)} \end{aligned} \tag{104}$$

and

$$\sigma_{R^\zeta,\zeta,t} = \left(S_{R^\zeta,\zeta,t} + S_{\pi_\zeta,\zeta,t}\right) \widehat{\sigma}\left(\left(\tau_t^{\zeta}\right)^*\right)^2 \left(\frac{\phi}{\sigma_1} - \frac{\phi\sigma_0}{\sigma\sigma_1}\right) \tag{105}$$

where

$$\begin{aligned} & S_{R^\zeta,\zeta,t} \\ = & \frac{(1-\gamma)(1-E_t[\theta_T]) A_2 (T-t) \cdot F\left(\rho_t^L, \mu_{\zeta^L}, \widehat{\sigma}\left(\left(\tau_t^{\zeta^L}\right)^*\right)^2, T-t\right)}{E_t[\theta_T] \cdot F\left(\rho_t^H, 0, \widehat{\sigma}\left(\left(\tau_t^{\zeta^H}\right)^*\right)^2, T-t\right) + (1-E_t[\theta_T]) \cdot F\left(\rho_t^L, \mu_{\zeta^L}, \widehat{\sigma}\left(\left(\tau_t^{\zeta^L}\right)^*\right)^2, T-t\right)}. \end{aligned} \tag{106}$$

where

$$\begin{aligned} F^{\zeta^H}(\cdot) &\equiv F\left(\rho_t^H, 0, \widehat{\sigma}\left(\left(\tau_t^{\zeta^H}\right)^*\right)^2, T-t\right) \\ &= e^{A_0^H(T-t)+(1-\gamma)A_1(T-t)\rho_t+\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\left(\tau_t^{\zeta^H}\right)^*\right)^2} \end{aligned} \tag{107}$$

and

$$\begin{aligned} F^{\zeta^L}(\cdot) &\equiv F\left(\rho_t^L, \mu_{\zeta^L}, \widehat{\sigma}\left(\left(\tau_t^{\zeta^L}\right)^*\right)^2, T-t\right) \\ &= e^{A_0^L(T-t)+(1-\gamma)A_1(T-t)\rho_t+(1-\gamma)A_2(T-t)\mu_{\zeta^L}+\frac{1}{2}(1-\gamma)^2A_2(T-t)^2\widehat{\sigma}\left(\left(\tau_t^{\zeta^L}\right)^*\right)^2}. \end{aligned} \tag{108}$$

For $t \in [t^{**}, T]$, the value premium, i.e. the difference between expected returns of value and growth firms, is then given by

$$\begin{aligned} E_t[R_t^{\eta}] - E_t[R_t^{\psi}] &= \mu_t^{\eta} - \mu_t^{\psi} \\ &= \left(\sigma_{\pi_\eta,\rho,t} \cdot \sigma_{R^\eta,\rho,t} - \sigma_{\pi_\psi,\rho,t} \cdot \sigma_{R^\psi,\rho,t}\right) \\ &\quad + \left(\sigma_{\pi_\eta,\eta,t} \cdot \sigma_{R^\eta,\eta,t} - \sigma_{\pi_\psi,\psi,t} \cdot \sigma_{R^\psi,\psi,t}\right). \end{aligned} \tag{109}$$

For the value premium to be strictly positive, I make the following assumption, $\forall t \in [t^{**}, T]$,

$$\mu_{\psi^L} - \mu_{\eta^L} < \frac{1}{2}(1-\gamma) A_2 (T-t) \left[\widehat{\sigma}\left(\left(\tau_t^{\eta^L}\right)^*\right)^2 - \widehat{\sigma}\left(\left(\tau_t^{\psi^L}\right)^*\right)^2\right]. \tag{110}$$

Q.E.D.

Recall from (63), that

$$\psi_T \equiv E\left[\theta_T \mid F_t\right] = \theta^s + (\theta_t - \theta^s) e^{-(\lambda+\mu)(T-t)}. \tag{111}$$

In order for the derivative $\frac{\partial E_t[\theta_T]}{\partial \rho_t}$, as figuring in (98), for example, to exist, I need to proof the existence of $\frac{\partial \theta_t}{\partial \rho_t}$. Recalling from Lemma 3 that

$$d\rho_t = \phi(\left(\theta_t \overline{\rho}^H + (1-\theta_t)\overline{\rho}^L\right) - \rho_t)dt + \sigma dZ_{0,t},$$

the process for θ_t can be expressed as follows

$$\begin{aligned} d\theta_t &= (\lambda+\mu)(\theta^s - \theta_t)dt + h(\theta_t) dZ_{0,t} \\ &= (\lambda+\mu)(\theta^s - \theta_t)dt + h(\theta_t)\frac{1}{\sigma}\left(d\rho_t - \phi\left[\left(\theta_t\overline{\rho}^H + (1-\theta_t)\overline{\rho}^L\right) - \rho_t\right]dt\right) \\ &= \left((\lambda+\mu)(\theta^s - \theta_t) - h(\theta_t)\frac{\phi}{\sigma}\left(\theta_t\overline{\rho}^H + (1-\theta_t)\overline{\rho}^L\right)\right)dt + h(\theta_t)\frac{1}{\sigma}e^{-\phi t} d\left(e^{\phi t}\rho_t\right). \end{aligned}$$

Hence, $\frac{\partial \theta_t}{\partial \rho_t}$ exists, and I can write that

$$\frac{\partial \psi_T}{\partial \rho_t} = \frac{\partial \theta_t}{\partial \rho_t} \cdot e^{-(\lambda+\mu)(T-t)}. \tag{112}$$

Moreover, as $h(\theta_t) > 0$, θ_t is strictly increasing in ρ_t.
Q.E.D

6.9 Proof of Corollary 2

The value premium is given by

$$\begin{aligned}\mu_t^\eta - \mu_t^\psi \equiv & \left(\sigma_{\pi_\eta,\rho,t} \cdot \sigma_{R^\eta,\rho,t} - \sigma_{\pi_\psi,\rho,t} \cdot \sigma_{R^\psi,\rho,t}\right) \\ & + \left(\sigma_{\pi_\eta,\eta,t} \cdot \sigma_{R^\eta,\eta,t} - \sigma_{\pi_\psi,\psi,t} \cdot \sigma_{R^\psi,\psi,t}\right).\end{aligned} \tag{113}$$

We have that $\sigma_{\pi_\eta,\rho,t} > \sigma_{\pi_\psi,\rho,t}$ and given that $\mu_t^\eta - \mu_t^\psi > 0$ (see assumption in the Proof of Proposition 5), it follows that $\sigma_{R^\eta,\rho,t} > \sigma_{R^\psi,\rho,t}$. Functional analysis shows that given estimated parameter values (Table I), $\frac{\partial \sigma_{\pi_\zeta,\rho,t}}{\partial \rho_t} > 0$, and $\frac{\partial \sigma_{\pi_\zeta,\zeta,t}}{\partial \rho_t} > 0$, $\forall \phi, \gamma, \tau$. Moreover, for values of ϕ, γ, τ, that give rise to a value (size) premium, i.e. $\mu_t^\eta - \mu_t^\psi > 0$, $\mu_t^s - \mu_t^l > 0$, it follows that for $\zeta = \eta, s$, $\frac{\partial \sigma_{R^\zeta,\rho,t}}{\partial \rho_t} > 0$, and $\frac{\partial \sigma_{R^\zeta,\zeta,t}}{\partial \rho_t} < 0$, and for $\zeta = \psi, l$, $\frac{\partial \sigma_{R^\zeta,\rho,t}}{\partial \rho_t} < 0$, and $\frac{\partial \sigma_{R^\zeta,\zeta,t}}{\partial \rho_t} > 0$. It then follows that for values of ϕ, γ, τ, that give rise to a value (size) premium, $\frac{\partial \mu_t^\zeta}{\partial \rho_t} < 0$, for $\zeta = \psi, l$, and $\frac{\partial \mu_t^\zeta}{\partial \rho_t} > 0$, for $\zeta = \eta, s$. Hence, $\partial \left(\mu_t^\eta - \mu_t^\psi\right) / \partial \rho_t > 0$ for values of ϕ, γ, τ, where $\mu_t^\eta - \mu_t^\psi > 0$. As GDP growth, ρ_t, is increasing in troughs and decreasing at peaks, the value premium is negatively correlated with business cycles. The same holds for the size premium.

Q.E.D

6.10 Proof of Proposition 6

As technology shocks are idiosyncratic at the market level, the state price density, π_t^m, for all $t \in [0, T]$, denotes

$$\pi_t^m = \lambda^{-1} B_t^{-\gamma} \left[E_t[\theta_T] \cdot \pi_t^H + (1 - E_t[\theta_T]) \cdot \pi_t^L\right] \tag{114}$$

where

$$\pi_t^\nu = E_t \left[\left(\frac{B_T}{B_t} \right)^{-\gamma} | \nu = H, L \right] = e^{\widetilde{A}_0^\nu (T-t) - \gamma A_1 (T-t) \rho_t} \text{ for } \nu = H, L. \tag{115}$$

Hence, $\pi_t^m = \pi_t$, and applying Itô's Lemma, the dynamics for π_t obtain as follows

$$\frac{d\pi_t}{\pi_t} = -\sigma_{\pi,t}^m d\widetilde{Z}_{0,t}, \tag{116}$$

where

$$\sigma_{\pi,t}^m = \left(\gamma A_1 (T-t) - S_{\pi,t}^m\right) \sigma \tag{117}$$

with

$$S_{\pi,t}^{m} = \frac{\frac{\partial E_t[\theta_T]}{\partial \rho_t}\left(\pi_t^H - \pi_t^L\right)}{E_t[\theta_T]\cdot \pi_t^H + (1-E_t[\theta_T])\cdot \pi_t^L}. \tag{118}$$

Applying Itô's Lemma to $M_t^m = E_t\left[B_T\pi_T\right]/\pi_t$, where $M_t^m \equiv \int_{i\in I} M_{it}di$ is the aggregate market value, the return process of the total wealth portfolio obtains as

$$dR_t^m \equiv \frac{dM_t^m}{M_t^m} = \mu_t^m dt + \sigma_{R,t}^m d\widetilde{Z}_{0,t} \tag{119}$$

where

$$\mu_t^m = \sigma_{\pi,t}^m \cdot \sigma_{R,t}^m, \tag{120}$$

and

$$\sigma_{R,t}^m = \left(A_1\left(T-t\right) + S_{R,t}^m - S_{\pi,t}^m\right)\sigma \tag{121}$$

where

$$S_{R,t}^m = \frac{\frac{\partial E_t[\theta_T]}{\partial \rho_t}\left(F\left(\rho_t^H,0,0,T-t\right) - F\left(\rho_t^L,0,0,T-t\right)\right)}{E_t[\theta_T]\cdot F\left(\rho_t^H,0,0,T-t\right) + (1-E_t[\theta_T])\cdot F\left(\rho_t^L,0,0,T-t\right)} \tag{122}$$

with

$$F\left(\rho_t^\nu,0,0,T-t\right) = e^{A_0^\nu(T-t)+(1-\gamma)A_1(T-t)\rho_t} \quad \text{for } \nu = H, L. \tag{123}$$

Recall that the return process for firms $i \in \zeta$ is given by

$$dR_t^\zeta = \mu_t^\zeta dt + \sigma_{R^\zeta,\rho,t} d\widetilde{Z}_{0,t} + \sigma_{R^\zeta,\zeta,t} d\widetilde{Z}_{\zeta,t}. \tag{124}$$

The cross-sectional beta then obtains as

$$\beta_t^\zeta \equiv \frac{cov_t\left(dR_t^\zeta, dR_t^m\right)}{var_t\left(dR_t^m\right)} = \frac{\sigma_{R,t}^m \cdot \sigma_{R^\zeta,\rho,t}}{\left(\sigma_{R,t}^m\right)^2} \tag{125}$$

$$= \frac{\sigma_{R^\zeta,\rho,t}}{\sigma_{R,t}^m} = \frac{A_1(T-t) + S_{R^\zeta,\rho,t} - S_{\pi_\zeta,\rho,t}}{A_1\left(T-t\right) + S_{R,t}^m - S_{\pi,t}^m} \tag{126}$$

where $\sigma_{R^\zeta,\rho,t}$ stems from (103) and $\sigma_{R,t}^m$ is as denoted in (121).
Q.E.D.

6.11 Proof of Corollary 3

Functional analysis shows that for estimated parameter values (Table I), $\frac{\partial \beta_t^{\zeta}}{\partial \rho_t} > 0$, for $\zeta = \eta, s$, and $\frac{\partial \beta_t^{\zeta}}{\partial \rho_t} < 0$, for $\zeta = \psi, l, \forall \phi, \tau, \gamma$. Consequently, cross-sectional differentials in betas are increasing in GDP growth, i.e. $\frac{\partial \beta_t^{\eta}}{\partial \rho_t} - \frac{\partial \beta_t^{\psi}}{\partial \rho_t} > 0$, $\frac{\partial \beta_t^{s}}{\partial \rho_t} - \frac{\partial \beta_t^{l}}{\partial \rho_t} > 0$, $\forall \phi, \tau, \gamma$. As GDP growth is increasing during troughs and decreasing at peaks, differentials in betas are negatively correlated with business cycles.

Q.E.D.

6.12 Proof of Proposition 7

The market premium denotes $E_t \left[R_t^m \right] \equiv -cov_t \left(\frac{d\pi_t}{\pi_t}, dR_t^m \right)$. It is given by μ_t^m in (120) of the Proof of Proposition 6.

Q.E.D.

6.13 Proof of Corollary 4

Tedious algebra shows that $\frac{\partial \sigma_{\pi,t}^m}{\partial \rho_t} = S_{\pi,t}^m \cdot \sigma > 0, \forall \gamma, \phi, \tau$, where $S_{\pi,t}^m$ is as denoted in (118). $\frac{\partial \sigma_{R,t}^m}{\partial \rho_t} = -(S_{R,t}^m)^2 + (S_{\pi,t}^m)^2 > 0, \forall \gamma, \phi, \tau$, as $(S_{R,t}^m)^2 < (S_{\pi,t}^m)^2 \; \forall \gamma, \phi, \tau$, where $S_{R,t}^m$ is as denoted in (122). As $\sigma_{\pi,t}^m$, $\sigma_{R,t}^m > 0$, it follows that $\frac{\partial \mu_t^m}{\partial \rho_t} > 0 \; \forall \gamma, \phi, \tau$. As GDP growth is increasing in troughs and decreasing at peaks, the equity premium is negatively correlated with business cycles.

Q.E.D.

Table I

Parameter Values

The parameter values are those of the process in equation (19). Parameters $\overline{\rho}^H$, $\overline{\rho}^L$, μ and λ are the maximum likelihood estimates of a two-state Markov-Switching Model applied to US real GDP data (Hamilton (1989))[31]. The annual volatility of real GDP growth is obtained from quarterly NIPA data for the period of 1967:1-2002:3.

Variable		Variable	
$\overline{\rho}^H$	0.011643	σ	0.0328611
$\overline{\rho}^L$	-0.003577	$\widehat{\sigma}_{t^*}$	0.07
μ	0.0951	σ_0	0.07
λ	0.245	σ_1	0.07

[31] The Hamilton algorithm provides a discrete time analogue to the non-linear continuous-time filter of Liptser and Shiryayev (1997), Theorem 9.1.

Table II

Asset Pricing Implications - Time Series

Panel A shows data for the market portfolio for the period 1948-2001 as obtained from the CRSP-COMPUSTAT database. $E(R^{em})$ denotes the annual equity premium, and $\sigma(R^m)$ the annual volatility of the market return. The Sharpe Ratio obtains as the ratio of annualized market excess returns to the standard deviation of the returns on the market porfolio. Panel B shows results from calibration. Parameter configuration is that of Table I, i.e. $\overline{\rho}^H = 0.011643$, $\overline{\rho}^L = -0.003577$, $\mu = 0.0951$, $\lambda = 0.245$, $\sigma = 0.0328611$, $\widehat{\sigma}_{t^*} = \sigma_0 = \sigma_1 = 0.07$. Panel B1 to B4 show results for investment horizons between 4 and 7 years[32]. Within all Panels, results are presented for speeds of mean reversion ϕ of 0.1, 0.3, and 0.4. The coefficient of relative risk aversion γ is chosen to take values of 4 or 8. All results are obtained assuming $\theta = 0.5$, i.e. the agent is unbiased in her beliefs about the regime of the economy.

Panel A: Data		
$E(R^{em})$	$\sigma(R^m)$	Sharpe
7.71	16.25	0.47

Panel B: Model									
γ	ϕ	$E(R^{em})$	$\sigma(R^m)$	Sharpe	γ	ϕ	$E(R^{em})$	$\sigma(R^m)$	Sharpe
		Panel B1: $\tau = 4$					Panel B2: $\tau = 5$		
4	0.1	4.70	10.84	0.43	4	0.1	6.69	12.93	0.52
4	0.3	2.35	7.67	0.31	4	0.3	2.90	8.52	0.34
8	0.4	3.45	6.57	0.53	8	0.4	4.05	7.11	0.57
		Panel B3: $\tau = 6$					Panel B4: $\tau = 7$		
4	0.1	8.80	14.83	0.59	4	0.1	10.95	16.55	0.66
4	0.3	3.35	9.15	0.37	4	0.3	3.70	9.62	0.38
8	0.4	4.48	7.50	0.60	8	0.4	4.77	7.72	0.62

[32] Atkins and Dyl (1997) report an average holding period of firms' common stock of 4.01 years for NYSE quoted firms over the period of 1975 to 1989, and of 6.99 years for Nasdaq quoted firms over the period of 1983 to 1991.

Table III

Asset Pricing Implications - Ratios from Fundamentals

Ratios from fundamentals are for the market portfolio. M/B denotes the average market-to-book ratio and P/D the average price-dividend ratio. The Sharpe Ratio obtains as the ratio of annualized market excess returns to the standard deviation of the returns on the market porfolio. Quarterly data for market equity, dividends and returns are obtained from the CRSP-COMPUSTAT database for the sample period 1948-2001. Model results are obtained using parameter values of Table I, i.e. $\overline{\rho}^H = 0.011643$, $\overline{\rho}^L = -0.003577$, $\mu = 0.0951$, $\lambda = 0.245$, $\sigma = 0.0328611$, $\widehat{\sigma}_{t^*} = \sigma_0 = \sigma_1 = 0.07$. The coefficient of relative risk aversion γ is 4, the speed of mean reversion ϕ takes the value of 0.15, and the average investment horizon τ is assumed to be of 7 years. All results are obtained assuming $\theta = 0.5$, i.e. the agent is unbiased in her beliefs about the regime of the economy.

Data		Model
Variable	Estimate	$\gamma = 4, \phi = 0.15, \tau = 7$
M/B	1.69	1.64
P/D	25.30	24.55
Sharpe	0.53	0.56

Table IV

Asset Pricing Implications - Cross Section

Panel A shows data for the average of small and large high-M/B portfolios (Growth) and the average of small and large low-M/B portfolios (Value), as well as their differential, the zero-investment portfolio HML. $E(R^{em})$ denotes the annualized average excess return, M/B the average market-to-book ratio. The Sharpe Ratio is the ratio of annualized market excess returns to the standard deviation of the returns on the market porfolio. The sample period is 1948-2001. Data for excess returns, market and book equity are obtained from CRSP-COMPUSTAT. Portfolios were constructed following the standard procedure of Fama and French (1992). Returns on portfolios are from July of year t to June of year $t+1$. CAPM β's obtain from running time-series regressions of excess returns on the respective M/B-decile portfolios.

Panel A: Data

Portfolio	Value	Growth	HML
$E(R^{em})$	11.58	7.43	4.15
M/B	0.67	3.24	-2.57
Sharpe	0.63	0.42	0.21
CAPM β	0.94	1.05	-0.12

Table V

Asset Pricing Implications - Cross Section

Panel B1 to B4 show results from calibration for different parameter values of γ, ϕ, τ. $E(R^{em})$ denotes the annualized average excess return, and M/B the average market-to-book ratio. The Sharpe Ratio is the ratio of annualized market excess returns to the standard deviation of the returns on the market porfolio. The YCAPM βs obtain from Proposition 6. The chosen learning ratio is 10/1 for value firms during recessions. Parameter values for cross-sectional means μ_{η^L} and μ_{ψ^L} are chosen to equal -0.004 and -0.003, respectively, given that $\mu_{\eta^L} < \mu_{\psi^L}$, and $\overline{\rho}^L = -0.003577$. All results are obtained assuming $\theta = 0.5$, i.e. the agent is unbiased in her beliefs about the actual state of the economy.

Panel B: Model

Panel B1: $\gamma = 9, \phi = 0.4, \tau = 5$				Panel B2: $\gamma = 9, \phi = 0.4, \tau = 6$			
Portfolio	Value	Growth	HML	Portfolio	Value	Growth	HML
$E(R^{em})$	5.62	4.66	0.96	$E(R^{em})$	7.32	4.43	2.89
M/B	1.53	2.97	-1.44	M/B	2.04	6.84	-4.80
Sharpe	0.39	0.23	0.16	Sharpe	0.33	0.11	0.22
YCAPM β	1.01	0.98	0.03	YCAPM β	1.01	0.98	0.03

Panel B3: $\gamma = 10, \phi = 0.3, \tau = 5$				Panel B4: $\gamma = 10, \phi = 0.35, \tau = 6$			
Portfolio	Value	Growth	HML	Portfolio	Value	Growth	HML
$E(R^{em})$	8.17	7.52	0.65	$E(R^{em})$	9.40	5.97	3.43
M/B	1.81	2.90	-1.09	M/B	2.59	8.76	-6.17
Sharpe	0.62	0.51	0.11	Sharpe	0.46	0.26	0.20
YCAPM β	1.01	0.99	0.02	YCAPM β	1.01	0.99	0.02

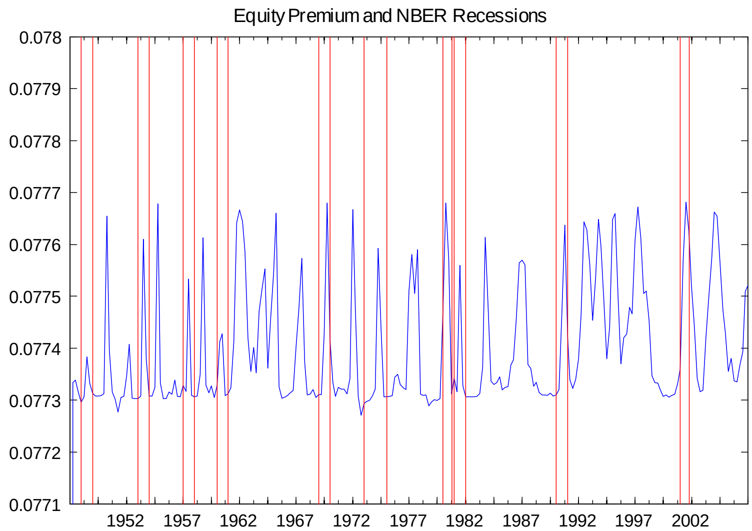

Figure 1: This figure shows the calibrated annual market equity premium over the period of 1947 to 2007. Red bars indicate the beginning and the end dates of NBER recessions. Parameter configuration is that of Table I, i.e. $\overline{\rho}^H = 0.011643$, $\overline{\rho}^L = -0.003577$, $\mu = 0.0951$, $\lambda = 0.245$, $\sigma = 0.0328611$, $\widehat{\sigma}_{t^*} = \sigma_0 = \sigma_1 = 0.07$. Parameters γ, ϕ, and τ are chosen to equal 4, 0.1, and 5.5, respectively. Variations are induced by time-varying beliefs about the state of the economy as reflected by θ_t. The latter obtains as a filtered probability, applying a Hamilton Markov-Switching Model to US quarterly real GDP data (as provided by U.S. BEA).

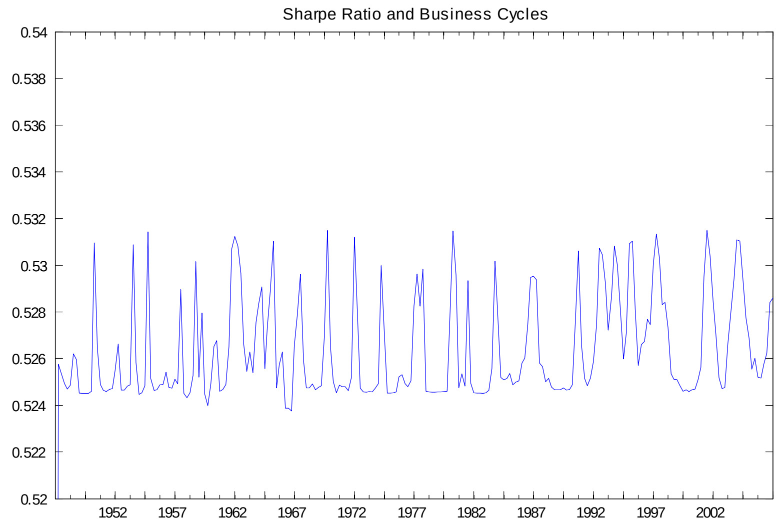

Figure 2: This figure shows the calibrated Sharpe Ratio for the period of 1947 to 2007. Parameter configuration is that of Table I, i.e. $\overline{\rho}^H = 0.011643$, $\overline{\rho}^L = -0.003577$, $\mu = 0.0951$, $\lambda = 0.245$, $\sigma = 0.0328611$, $\widehat{\sigma}_{t^*} = \sigma_0 = \sigma_1 = 0.07$. Parameters γ, ϕ, and τ are chosen to equal 8, 0.4, and 4, respectively. Variations are induced by time-varying beliefs about the state of the economy as reflected by θ_t. The latter obtains as a filtered probability, applying a Hamilton Markov-Switching Model to quarterly real GDP data (as provided by U.S. BEA)

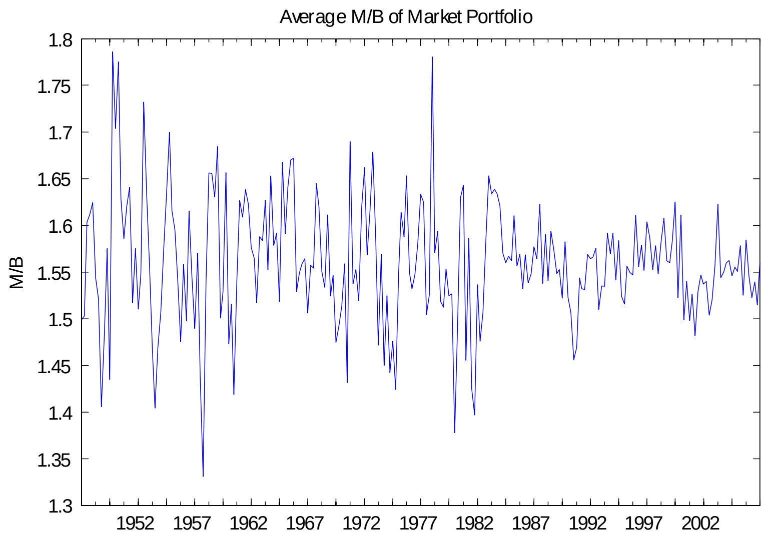

Figure 3: This figure displays the calibrated average M/B ratio for the period of 1947 to 2007. Parameter configuration is that of Table I, i.e. $\overline{\rho}^H = 0.011643$, $\overline{\rho}^L = -0.003577$, $\mu = 0.0951$, $\lambda = 0.245$, $\sigma = 0.0328611$, $\widehat{\sigma}_{t^*} = \sigma_0 = \sigma_1 = 0.07$. Parameters γ, ϕ, and τ are chosen to equal 4, 0.15, and 7, respectively. Variations are induced by time-varying beliefs about the state of the economy as reflected by θ_t. The latter obtains as a filtered probability, applying a Hamilton Markov-Switching Model to quarterly real GDP data (as provided by U.S. BEA)

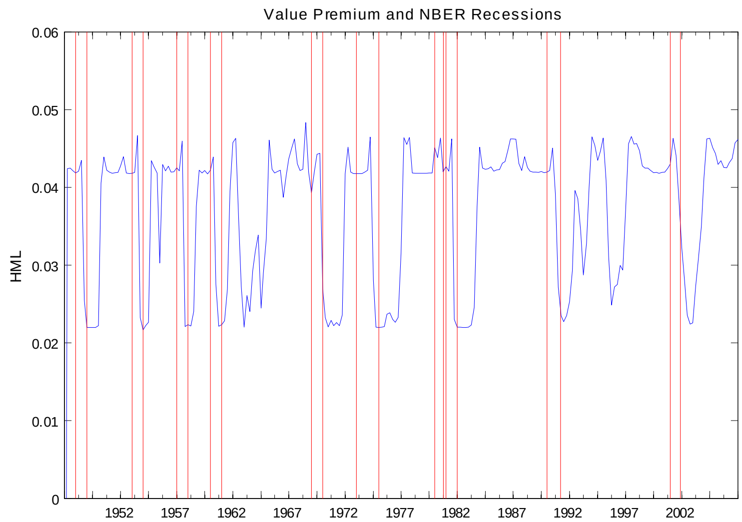

Figure 4: This figure shows the time-varying calibrated value premium for the period of 1947 to 2007. The value premium is the negative difference in annual excess returns of the average of small and large high-M/B portfolios (Growth), and the average of small and large low-M/B portfolios (Value). Red bars indicate the beginning and the end dates of NBER recessions. Parameter configuration is that of Table I, i.e. $\overline{\rho}^H = 0.011643$, $\overline{\rho}^L = -0.003577$, $\mu = 0.0951$, $\lambda = 0.245$, $\sigma = 0.0328611$, $\widehat{\sigma}_{t^*} = \sigma_0 = \sigma_1 = 0.07$. Parameters γ, ϕ, and τ are chosen to equal 10, 0.35, and 6, respectively. Believed cross-sectional means μ_{η^L} and μ_{ψ^L} equal -0.004 and -0.003, respectively. Time-variation in the cross-section of returns is due to two effects induced by learning. First, beliefs about the state of the economy as reflected by θ_t are time-varying. The latter obtains as a filtered probability, applying a Hamilton Markov-Switching Model to quarterly real GDP data (as provided by U.S. BEA). Second, the cross-section is affected by the time-varying learning ratio. The chosen learning ratio is 10/1 for value firms during recessions. Time-variation of the overall ratio is governed by θ_t.

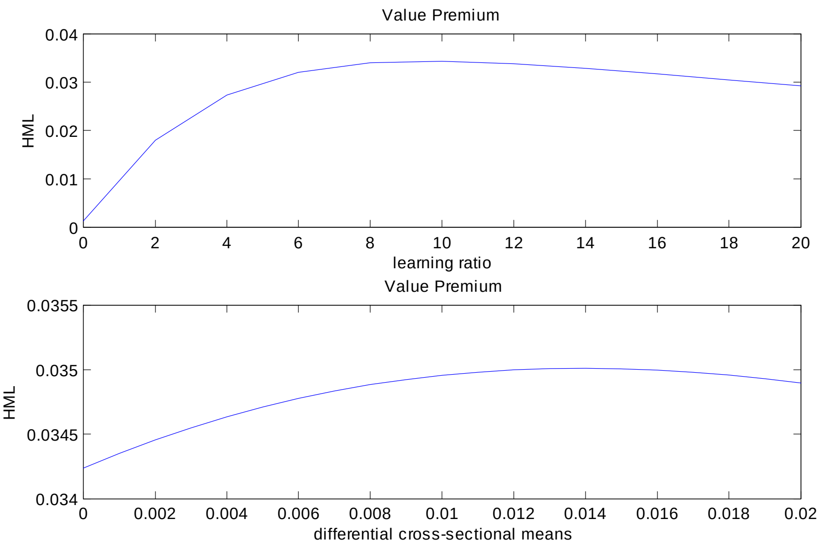

Figure 5: Figure 5A shows the value premium as a function of the optimal learning ratio of value versus growth firms during bad times, i.e. $\left(\tau_t^{\eta^L}/\tau_t^{\psi^L}\right)^*$. Figure 5B shows the value premium as a function of the divergence in cross-sectional mean values of the underlying distribution of the i.i.d. shocks to productivity during bad times, i.e. $\mu_{\psi^L} - \mu_{\eta^L}$. Value premia obtain using the following parameter values: $\overline{\rho}^H = 0.011643$, $\overline{\rho}^L = -0.003577$, $\mu = 0.0951$, $\lambda = 0.245$, $\sigma = 0.0328611$, $\widehat{\sigma}_{t^*} = \sigma_0 = \sigma_1 = 0.07$. Parameters γ, ϕ, and τ are chosen to equal 10, 0.35, and 6, respectively. In Figure 5A it is assumed that $\mu_{\eta^L} = -0.004$ and $\mu_{\psi^L} = -0.003$, while Figure 5B is based on a learning ratio of 10/1 for value firms in bad times.

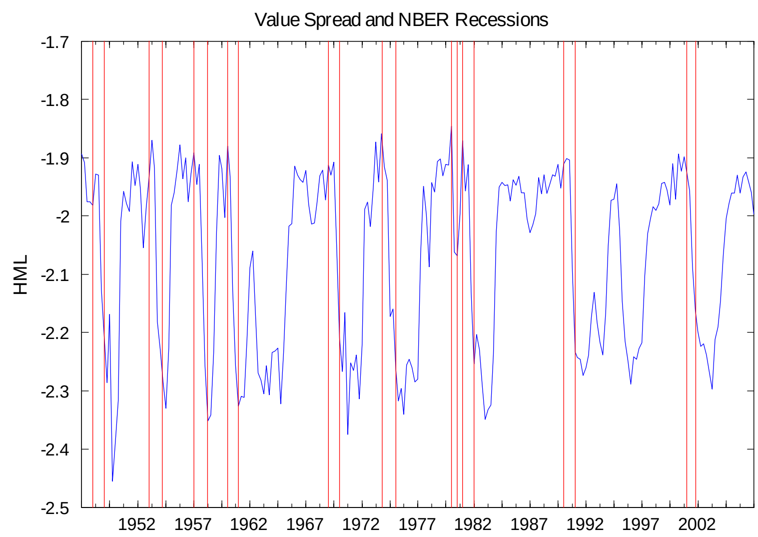

Figure 6: This figure shows the time-varying calibrated value spread for the period of 1947-2007. The value spread is the difference in the average market-to-book ratio of small and large high-M/B portfolios (Growth), and small and large low-M/B portfolios (Value). Red bars indicate the beginning and the end dates of NBER recessions. Parameter configuration is that of Table I, i.e. $\overline{\rho}^H = 0.011643$, $\overline{\rho}^L = -0.003577$, $\mu = 0.0951$, $\lambda = 0.245$, $\sigma = 0.0328611$, $\widehat{\sigma}_{t^*} = \sigma_0 = \sigma_1 = 0.07$. Parameters γ, ϕ, and τ are chosen to equal 9, 0.35, and 5.6, respectively. Time-variation in the cross-section of market-to-book ratios is due to two effects induced by learning. First, beliefs about the state of the economy as reflected by θ_t are time-varying. The latter obtains as a filtered probability, applying a Hamilton Markov-Switching Model to quarterly real GDP data (as provided by U.S. BEA). Second, the cross-section is affected by the time-varying learning ratio. The chosen learning ratio is 10/1 for value firms during recessions. Time-variation of the overall ratio is governed by θ_t.