Leveraged Lévy Processes as Models for Stock Prices

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Abstract

Adopting a constant elasticity of variance formulation in the context of a general Lévy process as the driving uncertainty we show that the presence of leverage effects in this form has the implication that asset price processes satisfy a scaling hypothesis. We develop forward Partial Integro-Differential Equations under a general Markovian setup, and show in two examples (both continuous and pure jump Lévy) how to use them for option pricing when stock prices follow our leveraged Lévy processes. Using calibrated models we then show an example of simulation based pricing and report on the adequacy of using leveraged Lévy models to value equity structured products.

1 Introduction

In the class of one dimensional Markov models, Local Volatility models (Dupire (1994); Derman and Kani (1994)) are models with continuous sample pahts that synthesize the surface of all traded European options, with instantaneous volatility of stock returns defined as a deterministic function of time and stock price to capture the instantaneous surface. Later Carr, Geman, Madan, and Yor (2004) extend such models into Lévy processes where a local uncertainty admits not only a Gaussian component but also includes jumps that allow for both skewness and excess kurtosis. Both Local Volatility and Local Lévy processes are nonparametric structures with local volatility accessing all its skewness from the leverage effect engineered by the dependence of volatility on the asset price. Local Lévy processes have an additional source of skewness and have been shown to enhance forward return skews and reduce at the money volatilities. We attempt to address the question whether one can formulate a simpler model with leverage which will be enough to achieve surface synthesis and reasonable forward return skews. A natural conjecture is to generalize the Constant Elasticity of Variance models (CEV, Cox (1996)Cox and Ross (1976) and Schroder (1989)) into the domain of Lévy processes. This motivates our research presented here.

The objective of this article is to enhance the applicability of Lévy processes as models for stock prices. We entertain the hypothesis that leverage considerations are relevant in

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describing the evolution of asset returns both statistically and risk neutrally and address this leverage effect directly in the context of a Lévy process as the driving uncertainty. It has been observed in Konikov and Madan (2002) that these homogeneous Lévy processes (e.g. NIG, VG, CGMY models) impose strict conditions on the term structure of the risk-neutral variance, skewness, and kurtosis. Specifically, the variance rate is constant over the term. It may be desirable to incorporate a richer behavior by introducing a leverage consideration. On constructing the "Leveraged Lévy" model, we build in the desired levels of leverage by introducing a local speed function (Carr, Madan, Geman, and Yor (2004)) that depends on the level of the asset prices and time, which affects the expected local quadratic variation in an explicit manner.

The first question we would like to address is whether we can build a tractable model under such setup. We shall make the assumption that the process of interest, $S = (S(t), t \geq 0)$, is of Lévy type rather than literally being a Lévy process. This means that the characteristics (μ, σ, ν) are allowed to be state dependent, which is of key importance for capturing essential features of financial data. A classical result, due to John Lamperti, establishes a one-to-one correspondence between a class of strictly positive Markov processes that are self-similar or semi-stable (according to John Lamperti), and the class of one-dimensional Lévy processes. This correspondence is obtained by suitably time-changing the exponential of the Lévy process. We show that our proposed leveraged Lévy processes are closely linked to processes introduced by John Lamperti (1962, 1972), and we term them the Lamperti processes associated with Lévy processes. Time homogeneous diffusions as well as processes with state dependent jump intensities and jump distributions are special cases. This expands the set of models to include most popular time homogeneous strongly Markovian semimartingales. As in the Lévy case, these processes can conveniently be parameterized by parameterizing the triplet of semimartingale characteristics which now uniquely define the process in terms of its conditional drift, its conditional volatility, its conditional intensity and distribution of its jumps.

In the example of the Leveraged Lévy model in the diffusion case (as our benchmark), Brownian motion is the only uncertainty. Since the family of Squared Bessel processes (BESQ) is the only family of continuous semi-stable Markov processes, we study our model by incorporating BESQ processes as the Lamperti processes. In the jump case, we consider only the pure jump Lévy processes, that is the ones without exposure to Brownian motions. It is argued in Carr, Geman, Madan, and Yor (2002) that the use of a jump process with infinite activity, i.e. one allowing infinitely many jumps in any time interval, effectively subsumes the need for an additional diffusion component. We therefore replace the local diffusive risk neutral dynamics in local volatility models (Dupire (1994); Derman and Kani (1994)) by a local exposure to a Lévy process. Essentially our idea is to replace Brownian motion with a Lévy process running at what we call the local speed function. In analogy with the local volatility function, this local speed function is still a deterministic function of the level of the stock price and time, $A(S, t)$ that measures the speed at which the Lévy process is running at time t when the stock price is at the level S. The Lévy process involved in this procedure is fixed through time, with only its speed that is space time dependent. In the case of Brownian motion, scaling and time changing are equivalent operations by the scaling property of Brownian motion, but for general Lévy processes these are different operations. Time changing leads to tractable results while scaling is much more complicated.

We study the case where the underlying Lévy process is taken to be CGMY process, which is introduced by Carr, Geman, Madan, and Yor (2002).

The second question is whether we can develop a Black-Scholes like Partial Differential Equation in order to achieve a option pricing method, and how to numerically implement such method and how well does the model perform in calibrating real data. The Markov property of the price allows us to express option prices as solutions of Partial Integro-Differential Equations (PIDEs) which involve, in addition to a (possibly degenerate) second-order differential operator, a non-local integral term which requires specific treatment both at the theoretical and numerical level. Such PIDEs have been used by several authors to price options in model with jumps (Andersen and Andersen (2000), Chan (1999), D'Halluin, Forsyth, and Labahn (2003), Matache, Petersdorff, and Schwab (2004)). In Section 3 , both backward and forward PIDEs are derived in a general context of Markov processes. Forward PIDE is preferred to backward PIDE since it obtains option prices on the whole surface of all strikes and maturities after one run of solving. By numerically solving the forward PIDE, the model parameters are calibrated to actual market data of option prices.

We anticipate that the use of such leveraged Lévy processes will shed better light on the exact role of leverage in financial markets both statistically and risk neutrally. Our reasoning is that since Lévy processes can internally explain both fat-tailedness and skewness without the addition of leverage, the estimation of a significant leverage effect is probably just that and not a proxy for other well known and stylized features of the return density.

The third question we study is what is the impact on future volatility when such model locks in local skewness and how our model enable an assessment of the impact of leverage on the valuation of claims otherwise analyzed in a zero leverage context. By implementing Monte Carlo simulations, we show how to use simulated data from the leveraged Lévy model to price for example Forward Start options. Forward start options are building blocks for Cliquet option, which is one of the fast-growing Equity Structured Products developed recently in the financial markets. The calculated forward implied volatility surfaces show that the model can potentially preserve volatility skew of both the shape and the level. The results here show that our model by using dynamics that lock in levels of local skewness of return evolutions, is adequate for calibrating volatility surfaces and pricing more reliably the path dependent equity structured products.

The rest of the article is organized as follows. In Section 2, we develop the leveraged Lévy model for stock price dynamics and relate such model to Lamperti processes. Section 3 presents a general derivation of the backward and forward Partial Integro Differential Equations for all Markov processes. Section 4 studies two examples of leveraged Lévy processes, in both diffusion and for pure jump CGMY settings with numerical results from calibration shown in Section 5. Section 6 shows one example of simulation based pricing of forward start options using calibrated models. Section 7 concludes.

2 Leveraged Lévy Model

The leverage effect is where negative return sequences are associated with increases in the volatility of stock returns. The leverage effect was studied in some early work by Black (1976), while it motivated the introduction of the EGARCH model of Nelson (1991) and the threshold ARCH model of Glosten, Jagannathan, and Runkle (1993). An economic theory behind such effects is discussed by Campbell and Kyle (1993).

The literature now contains many examples of log prices modeled as Lévy processes. These processes are processes of independent and identically distributed increments with constant volatilities and as a consequence are devoid of a leverage effect. To incorporate leverage into these models we follow the local Lévy development of Carr, Geman, Madan, and Yor (2004) by introducing a time change that depends on the level of the asset price. This dependence affects the expected local quadratic variation in an explicit manner and hence builds in desired levels of leverage. We start with a very basic setup to introduce the main theorem and fundamental properties of this model.

2.1 The General Model

Let the stock price process be denoted by $S_t = (S(t), t \geq 0)$ and let $X_t = (X(t), t \geq 0)$ be the log price process with $X_t = \ln S_t$. Additionally let $Z_t = (Z(t), t \geq 0)$ be a Lévy process that we wish to see leveraged. To begin with, we focus our attention on the case of constant elasticity of variance and define the desired local speed adjustment $A(S_t)$, at spot price S, for the lévy process as $A(S_t) = {S_t}^a$. It follows that the desired model for the log price is the following time changed Lévy process,

$$X(t) = Z\left(\int_0^t S(u)^a \, du\right) = Z\left(\int_0^t e^{aX(u)} \, du\right). \tag{1}$$

Here the specific time change for the Lévy process is defined as $T_t = (T(t), t \geq 0)$ where

$$T(t) = \int_0^t e^{aX(u)} \, du.$$

We note that if $(k(x), x \in \mathbb{R})$ is the Lévy density for the Lévy process Z then the process $X(t)$ has a Lévy system with jump compensation measure $\nu_X(dx, dt)$ given by

$$\nu_X(dx, dt) = e^{aX(t)} k(x) dx dt = S(t)^a k(x) dx dt \tag{2}$$

and the compensator splits into a product of two functions. The first depends on the asset price and incorporates leverage while the second addresses the specific jump sizes.

The risk neutral dynamics for the stock price are now given by

$$\begin{aligned} dS(t) &= (r-l)S(t_-)dt + \sigma(S(t_-), t)dW(t) \\ &+ \int_{-\infty}^{\infty} S(t_-)(e^x - 1)(m(dx, dt) - \nu_x(dx, dt)) \end{aligned}$$

where $m(dx, dt)$ is the counting measure associated with the jumps in the logarithm of the stock price. r and l denoted the continuously compounded interest and dividend rate respectively.

We term the class of processes S_t for which $X_t = \ln S_t$ is defined by (1) as a (Z_t, a)-*Leveraged Lévy processes* in recognition of specific space dependence of the Lévy system observed in (2). Leveraged Lévy processes are defined with reference to the specific form of the time change employed and they may or may not be martingales. When modeling the historical or true process, martingale restrictions do not arise. From a risk neutral perspective, however, we are interested in asset price processes being discounted martingales. We may accommodate this requirement by writing a model directly for the forward price as a martingale with leverage depending on the level of the forward price. The spot price is then defined as the discounted forward price. To get the forward price to be a martingale we choose the Lévy process such that its exponential is a martingale. The incorporation of our suggested time change leaves it a martingale.

2.2 Lamperti and Leveraged Lévy

We now relate our modeled stock price process (leveraged Lévy) to a class of semi-stable Markov processes that we term Lamperti processes. This relation is made possible by the close tie between these Lamperti processes and Lévy processes studied by Lamperti (1972). We derive Theorem one as illustrating this precise relationship.

Given any one-dimensional Lévy process $Z = Z_t = (Z(t), t \geq 0)$, Lamperti associated with such a Lévy process a positive Markov process that we denote by $L_t^{(Z)} = (L_t^{(Z)}, t \geq 0)$ and call it the Lamperti process associated with Z. The process $L^{(Z)}$ is implicitly defined by the time change

$$\tau_t^{(Z)} = \int_0^t e^{Z(s)}\, ds$$

whereby $L^{(Z)}$ is defined as satisfying

$$e^{Z(t)} = L^{(Z)}(\tau_t^{(Z)}) = L^{(Z)}\left(\int_0^t e^{Z(s)}\, ds\right) \tag{3}$$

One may deduce from the independence and stationarity of Z that $L^{(Z)}$ satisfies the scaling property

$$(L_{at}^{(Z)}, 0 \leq t; P_x) = (aL_t^{(Z)}, t \geq 0; P_{x/a})$$

where P_x is the law of $L^{(Z)}$ starting at x. In Lamperti (1972) Lamperti calls such positive processes as semi-stable Markov processes and then shows that there is a one-to-one correspondence between such semi-stable processes and Lévy processes via equation (3). This result leads to the following theorem, and we thank Marc Yor for its development.

Theorem 1 *Every (Z_t, a)-Leveraged Lévy processes $\{S_t\}$ is a Lamperti process raised to a power. Specifically,*

$$S_t = \left[L_t^{(-aZ)}\right]^{(-1/a)}. \tag{4}$$

Proof. The relationship between Lamperti processes and leveraged Lévy processes follows on considering the exponential of $aX(t)$ for a leveraged Lévy process $S(t)$. From equation (1) we see that

$$\exp(aX(t)) = \exp\left(aZ(\int_0^t e^{aX(u)}\, du)\right). \tag{5}$$

Defining

$$Y(t) = \exp(aX(t)), \tag{6}$$

we deduce that

$$Y(t) = \exp\Big(aZ(\int_0^t Y(u)\,du)\Big). \tag{7}$$

By defining the inverse time change $\zeta(t)$ of $T(t)$ by $T(\zeta(t)) = t$, precisely

$$\int_0^{\zeta(t)} e^{aX(u)}\,du = t, \tag{8}$$

and evaluating both (6) and (7) at $\zeta(t)$, we obtain

$$Y(\zeta(t)) = \exp\big(aX(\zeta(t))\big), \tag{9}$$

and

$$Y(\zeta(t)) = \exp(aZ(t)), \tag{10}$$

hence,

$$\exp(aZ(t)) = \exp\big(aX(\zeta(t))\big). \tag{11}$$

The time change $\zeta(t)$ is identified by differentiating (8) to obtain

$$dt = e^{aX(\zeta(t))}d\zeta(t) = e^{aZ(t)}d\zeta(t),$$

or equivalently

$$d\zeta(t) = e^{-aZ(t)}d(t).$$

It follows that

$$\zeta(t) = \int_0^t e^{-aZ(s)}\,d(s). \tag{12}$$

From (10) we have

$$\exp(-aZ(t)) = Y^{-1}\Big(\int_0^t e^{-aZ(s)}\,ds\Big), \tag{13}$$

and we observe Y^{-1} is the Lamperti process L^{-aZ} associated with $-aZ(t)$.
But as $Y(t) = \exp(aX(t))$ we have that

$$X(t) = \frac{1}{a}\ln(Y(t)) = -\frac{1}{a}\ln(L^{-aZ}), \tag{14}$$

recalling that $S(t) = \exp(X(t))$ we see (4) holds. ■

Given the close connection between leveraged Lévy processes and the Lamperti processes, we review the connections between the infinitesimal generator of the Lévy process and that for the associated Lamperti process. Denote $\mathcal{L}$ and L the infinitesimal generators of Z_t and L_t respectively. We have the following relations (Yor, Carmona, and Petit (1994)):

$$Lf(l) = \frac{1}{l}\mathcal{L}(f\circ\exp)(\ln l); \quad \mathcal{L}f(z) = e^z L(f\circ\ln)(e^z). \tag{15}$$

The proof of the second equation similarly follows. We shall use both equations to go back and forth between Lévy processes and the associated Lamperti processes.

3 A General Derivation of Forward Partial Integro Differential Equation

In this section we show the derivation of a forward Partial Integro-Differential Equation (PIDE) in a very general setup where both diffusion and jumps are considered. The main results equation (26) and its specifications in equations (34) and (38) for diffusion and jump cases respectively, are the basis for numerical analysis in the following sections. In case where diffusion is the only uncertainty in the model, the integral part of the equation will certainly vanish and regular PDE will be obtained. A partial integro-differential equation (PIDE) of option price in underlier and time is usually referred to as the backward equation, such as the Black-Scholes PDE. A forward equation is one of option price in strike and maturity. The forward PIDE once derived is practically more attractive since it allows us to calculate the option prices on the whole surface of maturities and strikes in one execution of the solver.

We have a real valued Markov process $(X(t), t > 0)$ with generator I_X acting on functions f given by

$$I_X(f) = a(x,t)\frac{\partial}{\partial x}f + \frac{1}{2}b(x,t)\frac{\partial^2}{\partial x^2}f + \int_{-\infty}^{\infty}[f(x+\nu) - f(x) - \nu\frac{\partial}{\partial x}f]k(x,t,\nu)\,d\nu.$$

We suppose that one may take the truncation function $h(\nu)$ to be the identity function $h(\nu) = \nu$. This merely requires that we have a special semimartingale structure for the jump and the integrability condition

$$E\left[\int_0^t \int_{-\infty}^{\infty} (\nu^2 \wedge |\nu|)k(X(u), u, \nu)\,d\nu\,du\right] < \infty.$$

This condition is satisfied for a large class of models that we shall work with. In fact we shall consider locally square integrable semimartingales that satisfy the stronger condition

$$E\left[\int_0^t \int_{-\infty}^{\infty} \nu^2 k(X(u), u, \nu)\,d\nu\,du\right] < \infty.$$

Let $q(t, x, T, y)$ be the transition density for the process at level y at time $T > t$, given it is at level x at time t. We are interested in first developing the forward equation for q in the arguments T, y and shall suppress the dependence on t, x. We then apply this result to European options values and develop the forward equation for option price values.

We shall also make use in our derivation of the double tail of the Lévy system defined by

$$\tilde{\tilde{k}}(x,t,\nu) = \begin{cases} \int_{-\infty}^{\nu}\int_{-\infty}^{u} k(x,t,w)\,dw\,du & \nu < 0, \\ \int_{\nu}^{\infty}\int_{u}^{\infty} k(x,t,w)\,dw\,du & \nu > 0. \end{cases}$$

The double tail integrates the tail of the Lévy measure in both directions twice and hence we refer it to as the double tail. It is important as it measures quadratic variation, which may be observed by applying integration by parts two times to get

$$\int_{-\infty}^{\infty} \tilde{\tilde{k}}(x,t,\nu)\,d\nu = \frac{1}{2}\int_{-\infty}^{\infty} \nu^2 k(x,t,\nu)\,d\nu < \infty \tag{16}$$

for locally square integrable semimartingales. In particular the double tail is itself well defined at all points of space x and times t More properties about double tail of Lévy processes can be found in Carr, Madan, Geman, and Yor (2004).

For a test function $f(y)$ we begin by defining

$$V_f(T) \quad = \quad E[f(X(T))|X(t) = a] = \int_{-\infty}^{\infty} q(y,T)f(y)\,dy.$$

Via differentiation with respect to T we get

$$\frac{\partial}{\partial T}V_f(T) = \int_{-\infty}^{\infty} q_T(y,T)f(y)\,dy. \tag{17}$$

In terms of the generator we have that

$$\begin{aligned} V_f(T) &= E[f(X(T))|X(t) = a] \\ &= f(a) + E\left[\int_t^T I_X(f)(X(u),u)\,du|X(t) = a\right] \\ &= f(a) + \int_t^T du \int_{-\infty}^{\infty} dw\ q(w,u)I_X(f)(w,u) \\ &= f(a) + \int_t^T du \int_{-\infty}^{\infty} dw\ q(w,u)\left[a(w,u)\frac{\partial}{\partial w}f + \frac{1}{2}b(w,u)\frac{\partial^2}{\partial w^2}f \right.\\ &\quad \left. + \int_{-\infty}^{\infty}[f(w+\nu) - f(w) - \nu\frac{\partial}{\partial w}f]k(w,u,\nu)\,d\nu\right]. \end{aligned}$$

Taking the partials with respect to T we obtain

$$\begin{aligned} \frac{\partial}{\partial T}V_f(T) = \int_{-\infty}^{\infty} dw\ q(w,T)&\left[a(w,T)\frac{\partial}{\partial w}f + \frac{1}{2}b(w,T)\frac{\partial^2}{\partial w^2}f \right.\\ &\left. + \int_{-\infty}^{\infty}[f(w+\nu) - f(w) - \nu\frac{\partial}{\partial w}f]k(w,T,\nu)\,d\nu\right]. \end{aligned} \tag{18}$$

We now employ integration by parts one and two times to rewrite the first two components as

$$\int_{-\infty}^{\infty} dw\ q(w,T)a(w,T)\frac{\partial}{\partial w}f(w) = -\int_{-\infty}^{\infty} dy\ f(y)\frac{\partial}{\partial y}[q(y,T)a(y,T)], \tag{19}$$

$$\int_{-\infty}^{\infty} dw\ q(w,T)\frac{1}{2}b(w,T)\frac{\partial^2}{\partial w^2}f(w) = \int_{-\infty}^{\infty} dy\ f(y)\frac{1}{2}\frac{\partial^2}{\partial y^2}[q(y,T)b(y,T)], \tag{20}$$

For the analysis of the jump integral we proceed as follows

$$\begin{aligned}
&\int_{-\infty}^{\infty} dw\ q(w,T) \int_{-\infty}^{\infty} [f(w+\nu) - f(w) - \nu \frac{\partial}{\partial w} f] k(w,T,\nu)\, d\nu \\
=& \int_{-\infty}^{\infty} dw\ q(w,T) \int_{-\infty}^{\infty} \tilde{\tilde{k}}(w,T,\nu) \frac{\partial^2}{\partial \nu^2} f(w+\nu)\, d\nu \\
=& \int_{-\infty}^{\infty} dy\ \frac{\partial^2}{\partial y^2}[f(y)] \int_{-\infty}^{\infty} d\nu\ q(y-\nu,T) \tilde{\tilde{k}}(y-\nu,T,\nu) \\
=& \int_{-\infty}^{\infty} dy\ f(y) \frac{\partial^2}{\partial y^2}\Big[\int_{-\infty}^{\infty} d\nu\ q(y-\nu,T) \tilde{\tilde{k}}(y-\nu,T,\nu)\Big]. \qquad (21)
\end{aligned}$$

Substituting equations (19, 20, 21) back into equation (18) we obtain

$$\begin{aligned}
\frac{\partial}{\partial T} V_f(T) = \int_{-\infty}^{\infty} dy\ f(y) \Big[&-\frac{\partial}{\partial y}[q(y,T)a(y,T)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[q(y,T)b(y,T)] \\
&+ \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} d\nu\ q(y-\nu,T) \tilde{\tilde{k}}(y-\nu,T,\nu) \Big]. \qquad (22)
\end{aligned}$$

Comparing equations (17) and (22) for all test functions f we deduce

$$\begin{aligned}
q_T(y,T) = &-\frac{\partial}{\partial y}[q(y,T)a(y,T)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[q(y,T)b(y,T)] \\
&+ \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} d\nu\ q(y-\nu,T) \tilde{\tilde{k}}(y-\nu,T,\nu). \qquad (23)
\end{aligned}$$

Applying this result we derive a forward equation for the prices of European options in the strike and maturity arguments. For our Markov process we take the logarithm of the stock price, $X(t) = \ln(S(t))$. For the risk neutral process we have the generator for X given by

$$\begin{aligned}
I_X(f)(x,u) &= \big(r - l - \frac{\sigma^2}{2} + \omega(x,u)\big)\frac{\partial}{\partial x} f + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} f \\
&\quad + \int_{-\infty}^{\infty} [f(x+\nu) - f(x) - \nu \frac{\partial}{\partial x} f] k(x,u,\nu)\, d\nu,
\end{aligned}$$

where r is the interest rate and l is the dividend yield. We allow for a general space time dependent Lévy system that permits processes of infinite variation while the diffusion component is relatively simplistic and uninteresting and will in most applications be in fact assumed to be null. The risk neutral drift is $r - l$. The exponential compensation of the jump component in the log price process is

$$\omega(x,u) = -\int_{-\infty}^{\infty} (e^{\nu} - 1 - \nu) k(x,u,\nu)\, d\nu.$$

In this particular case, we may write

$$\begin{aligned}
q_T(y,T) = &-\frac{\partial}{\partial y}\big[\big(r - l - \frac{\sigma^2}{2} + \omega(y,T)\big) q(y,T)\big] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2} q(y,T) \\
&+ \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} d\nu\ q(y-\nu,T) \tilde{\tilde{k}}(y-\nu,T,\nu). \qquad (24)
\end{aligned}$$

We now consider the price of a European call option which is given by

$$C(K,T) = e^{-rT}\int_{\ln K}^{\infty} q(y,T)(e^y - K)\,dy.$$

It follows that

$$C_T = -rC + e^{-rT}\int_{\ln K}^{\infty} q_T(y,T)(e^y - K)\,dy, \tag{25}$$

and

$$\begin{aligned} C_K &= -e^{-rT}\int_{\ln K}^{\infty} q(y,T)\,dy, \\ e^{-rT}\int_{\ln K}^{\infty} e^y q(y,T)\,dy &= C - KC_K, \\ KC_{KK}(K,T) &= e^{-rT} q(\ln K, T). \end{aligned}$$

Substituting from equation (24) for q_T we obtain the forward PIDE[1]

$$\begin{aligned} C_T = & -lC - (r-l)KC_K + \frac{\sigma^2}{2}K^2 C_{KK} \\ & + K^2\int_{-\infty}^{\infty} e^{-\nu} C_{KK}(Ke^{-\nu},T)\tilde{\tilde{k}}(\ln K - \nu, T, \nu)\,d\nu \\ & + \int_K^{\infty} dU \int_{\ln\frac{U}{K}}^{\infty} d\nu\; Ue^{-\nu} C_{KK}(Ue^{-\nu},T)\tilde{\tilde{k}}(\ln U - \nu, T, \nu) \\ & - \int_0^{K} dU \int_{-\infty}^{\ln\frac{U}{K}} d\nu\; Ue^{-\nu} C_{KK}(Ue^{-\nu},T)\tilde{\tilde{k}}(\ln U - \nu, T, \nu), \end{aligned} \tag{26}$$

where the final three integrals may be seen as the costs of jumps to the strike, plus the costs of downcrossing and the costs of upcrossing. The *at*, *down* and *upcrossing* costs are all measured by the likelihood times the level of the post jump double tail.

4 Two Specific Leveraged Lévy Models

We wish to maintain the growth rate of the stock and allow for an additional parameter in the volatility structure relative to what we were considering in the general formulation of Section 2.1. The growth rate for the stock is set to be μ, and we suggest that the local relative volatility be of the form

$$\sigma\left(\frac{S_t}{S_0}e^{-\mu t}\right)^{\alpha},$$

so that we have with $\alpha = 0$, a constant volatility, while for $\alpha \neq 0$ we do not engineer any trend into the volatility, given the stock price is assumed to be trending at rate μ.

[1] The detailed steps are available upon request.

Noticing the leverage parameter a in the definition of the local speed function (in Section 2.1), $A(S_t) = S_t^a$ has been set to $a = 2\alpha$ to reflect the sense of "variance", and in fact now

$$A(S_t, t) = \sigma^2 (\frac{S_t}{S_0} e^{-\mu t})^{2\alpha}.$$

Moreover, we would like $S_t e^{-\mu t}$ to be a positive martingale and hence take as a generic model the stochastic exponential of a martingale. Let us still call this exponential martingale Lévy process $Z(t)$ for a moment, and also incorporate the change in volatility using time changes, the specific formulation now becomes

$$S_t e^{-\mu t} = S_0 \exp\left(Z\left(\int_0^t A(S_u, u) du \right)\right) \tag{27}$$

After making trivial changes to the proof of Theorem 1, we obtain the general model for the stock price in this set-up

$$S_t = S_0 e^{\mu t} \{L^{(-2\alpha Z)}(\sigma^2 t)\}^{-\frac{1}{2\alpha}} = S_0 e^{\mu t} \{\sigma^2 L^{(-2\alpha Z)}(t)\}^{-\frac{1}{2\alpha}}. \tag{28}$$

It is a Lamperti process raised to a power. The Lamperti process $L^{(-2\alpha Z)}$ is associated with Lévy process $-2\alpha Z$. The second equality is achieved from the scaling property of the Lamperti process.

4.1 The Diffusion Case

We now construct as a benchmark a leveraged diffusion model. Void of jumps, the Brownian component is the only uncertainty that drives the model. Hence, $Z(t) = W(t) - \frac{1}{2} t$ and (27) is

$$S_t e^{-\mu t} = S_0 \exp\left(W\left(\int_0^t \sigma^2 (\frac{S_u}{S_0} e^{-\mu u})^{2\alpha} \, du \right) - \frac{1}{2} \int_0^t \sigma^2 (\frac{S_u}{S_0} e^{-\mu u})^{2\alpha} \, du \right) \tag{29}$$

A Bessel Squared Process of dimension δ (BESQ$^\delta$) $Z_t^{(\delta)}$ has SDE

$$dZ_t^{(\delta)} = \delta dt + 2\sqrt{Z_t^{(\delta)}} dW_t$$

Since BESQ is the only family of continuous Lamperti processes[2] we replace the Lamperti process in (28) with $Z_t^{(\delta)}$, where by Martingale property $\delta = \frac{1}{\alpha} + 2$. We have for stock price model:

$$S_t = S_0 e^{\mu t} \{\sigma^2 Z_t^{(\delta)}\}^{\frac{2-\delta}{2}}. \tag{30}$$

where $Z_t^{(\delta)}$ is BESQ process of dimension δ. Knowing the SDE for $Z_t^{(\delta)}$ and (30), the SDE of S_t is easily derived by applying Ito's Lemma

$$dS_t = \mu S_t dt + \sigma(2 - \delta) \left(\frac{S_t}{S_0 e^{\mu t}} \right)^{\frac{1}{\delta - 2}} S_t dW_t. \tag{31}$$

[2] The Lamperti Representation as in (3) relating BESQ process and Brownian motion is $\exp 2(W(t) + \nu t) = BESQ^{(\delta)}(\int_0^t \exp 2(W(s) + \nu s) \, ds)$, where $\delta = 2\nu + 2$ (see e.g. Williams (1974)).

In the risk-neutral setting $\mu = r - l$, and our generalized SDE is

$$dS_t = (r-l)S_t dt + \sigma(2-\delta)\left(\frac{S_t}{S_0 e^{(r-l)t}}\right)^{\frac{1}{\delta-2}} S_t dW_t. \tag{32}$$

We use the result in Section 3 and develop the forward equations in our diffusion case as a benchmark. Consider the log price process $X_t = \ln(S_t)$, and then apply Ito's lemma to (32) to obtain

$$dX_t = \left[r - l - \frac{1}{2}\sigma^2(2-\delta)^2\left(\frac{e^{X_t}}{S_0 e^{(r-l)t}}\right)^{\frac{2}{\delta-2}}\right]dt + \sigma(2-\delta)\left(\frac{e^{X_t}}{S_0 e^{(r-l)t}}\right)^{\frac{1}{\delta-2}} dW_t. \tag{33}$$

We know that in this case, we have a diffusion model without the jump component. The generator of X_t is

$$I_X(f)(x,u) = a(x,u)\frac{\partial f}{\partial x} + \frac{1}{2}b(x,u)\frac{\partial^2 f}{\partial x^2},$$

where

$$a(x,u) = r - l - \frac{1}{2}\sigma^2(2-\delta)^2\left(\frac{e^{X_t}}{S_0 e^{(r-l)t}}\right)^{\frac{2}{\delta-2}},$$

$$b(x,u) = \sigma^2(2-\delta)^2\left(\frac{e^{x}}{S_0 e^{(r-l)u}}\right)^{\frac{2}{\delta-2}}.$$

Hence, by the result in (24), we have

$$\begin{aligned} q_T(y,T) &= -\frac{\partial}{\partial y}[a(y,T)q(y,T)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[b(y,T)q(y,T)] \\ &= -(r-l)\frac{\partial}{\partial y}q(y,T) + \frac{\sigma^2(2-\delta)^2}{2(S_0 e^{(r-l)T})^{\frac{2}{\delta-2}}}\frac{\partial}{\partial y}[e^{\frac{2y}{\delta-2}}q(y,T)] \\ &\quad + \frac{\sigma^2(2-\delta)^2}{2(S_0 e^{(r-l)T})^{\frac{2}{\delta-2}}}\frac{\partial^2}{\partial y^2}[e^{\frac{2y}{\delta-2}}q(y,T)]. \end{aligned}$$

Substituting $q_T(y,T)$ into (25), we have

$$C_T = -lC - (r-l)KC_K + \frac{\sigma^2(2-\delta)^2 K^{\frac{2\delta-2}{\delta-2}}}{2(S_0 e^{(r-l)T})^{\frac{2}{\delta-2}}} C_{KK}(K,T)$$

We switch to the log strike space by defining[3] $k = \ln K$.
Define

$$c(k,T) = C(e^k,T) = C(K,T),$$

[3]This transformation will benefit the discretization of the system. Because strike prices collected from real market are often sparsely distributed in a large range, which brings difficulty to the discretization of the system when we always require the spacing of K be small enough to achieve convergence. By transforming from strike space to log strike space, we can achieve the small spacing without making the system huge.

so that

$$KC_K(K,T) = c_k(k,T), \qquad C_{KK}(K,T) = e^{-2k}[c_{kk}(k,T) - c_k(k,T)].$$

then the forward PDE becomes

$$c_T(k,T) = -lc - (r-l)c_k(k,T) + \frac{\sigma^2(2-\delta)^2 e^{\frac{2k}{\delta-2}}}{2(S_0 e^{(r-l)T})^{\frac{2}{\delta-2}}}[c_{kk}(k,T) - c_k(k,T)] \tag{34}$$

4.2 The Pure Jump CGMY Case

In this section we consider pure jump Lévy processes as our building blocks for constructing the Leveraged Lévy model. It is argued in Carr, Geman, Madan, and Yor (2002) that the use of a jump process with infinite activity, i.e. one allowing infinitely many jumps in any time interval, effectively subsumes the need for an additional diffusion component. We replace Brownian motion with a Lévy process running at what we call the local speed function and this local speed function is still a deterministic function of the level of the stock price and time. The Lévy process involved in this chapter is the CGMY process which was introduced in the 2002 paper by Carr, Geman, Madan, and Yor (2002).

The Lévy triplet of CGMY process $Z_{CGMY}(t)$ is $(\gamma, 0, k_{CGMY}(x)dx)$, where

$$k_{CGMY}(x) = \frac{Ce^{Ax-B|x|}}{|x|^{1+Y}}$$

is the Lévy density and $A = \frac{G-M}{2}, B = \frac{G+M}{2}$. $C > 0$, $G \geq 0$, $M \geq 0$, and $Y < 2$. The condition $Y < 2$ is induced by the requirement that Lévy densities integrate x^2 in the neighborhood of 0. The characteristic exponent is

$$\eta(u) = iu\gamma + \int_{R-\{0\}} (e^{iux} - 1 - iuh(x))k_{CGMY}(x)\,dx.$$

The value of γ is dependent on the choice of the truncation function $h(x)$ and particularly

$$\gamma = \int_{R-\{0\}} h(x)k_{CGMY}(x)\,dx.$$

We know that $\exp(Z_{CGMY}(t) - t\eta(-i))$ is a Martingale, in fact we define

$$\omega = -\eta(-i) = -\int_{R-\{0\}} (e^x - 1)k_{CGMY}(x)\,dx,$$

hence $\exp(Z_{CGMY}(t) + \omega t)$ is a Martingale, and (27) is now

$$S_t e^{-\mu t} = S_0 \exp\left(Z_{CGMY}\left(\int_0^t \sigma^2 (\frac{S_u}{S_0} e^{-\mu u})^{2\alpha}\,du \right) + \omega \int_0^t \sigma^2 (\frac{S_u}{S_0} e^{-\mu u})^{2\alpha}\,du \right) \tag{35}$$

However, in this pure jump case we no longer know what the Lamperti process is specifically, our knowledge of the model is solely based on the knowledge of the underlying Lévy and what is given in the model construction. Now let us begin to derive the generator of S from the generator of Z_{CGMY}.

According to (28), our specific Lévy process which is related to the Lamperti process becomes $\xi^{(\beta)}(t) = \frac{1}{\beta}(Z_{CGMY}(t)+\omega t)$ (define $\beta = -\frac{1}{2\alpha}$). It is easy to see that the Lévy triplet for $\xi^{(\beta)}(t)$ is $(\lambda, 0, k_\xi(x)dx)$, where a and $k_\xi(x)$ are now related to CGMY process by

$$\lambda = \frac{\gamma+\omega}{\beta} = -\frac{1}{\beta}\int_{R-\{0\}} (e^x - 1 - h(x))k_{CGMY}(x)\, dx,$$

and

$$k_\xi(x)dx = \beta k_{CGMY}(\beta x)dx. \tag{36}$$

From result (44) in Appendix A, the generator of S_t (in Risk Neutral condition) is

$$\begin{aligned} I_S(f)(x,u) &= \left[(r-l) + \left(\frac{x}{S_0 e^{(r-l)u}\sigma^{2\beta}}\right)^{-1/\beta}\lambda\beta\right] x f'(x) \\ &+ \left(\frac{x}{S_0 e^{(r-l)u}\sigma^{2\beta}}\right)^{-1/\beta}\int_{R-\{0\}}\left[f(xe^{\beta\eta}) - f(x) - \beta x f'(x)h(\eta)\right]k_\xi(\eta)\, d\eta. \end{aligned}$$

We may take the truncation function to be the identity function, $h(\eta) = \eta$, and by (36) and a simple variable change, the generator of S_t can be written in terms of $k_{CGMY}(x)$

$$\begin{aligned} I_S(f)(x,u) &= \left[(r-l) + \left(\frac{x}{S_0 e^{(r-l)u}\sigma^{2\beta}}\right)^{-1/\beta}\lambda\beta\right] x f'(x) \\ &+ \left(\frac{x}{S_0 e^{(r-l)u}\sigma^{2\beta}}\right)^{-1/\beta}\int_{R-\{0\}}\left[f(xe^{\eta}) - f(x) - x f'(x)\eta\right]k_{CGMY}(\eta)\, d\eta. \end{aligned}$$

Finally, let us derive the forward PIDE in this case follow the same outline as in Section 3. For our Markov process we take the logarithm of the stock price $X(t) = \ln S(t)$, hence the generator for $X(t)$

$$I_X(g)(x,u) = a(x,u)\frac{\partial}{\partial x}g + \int_{R-\{0\}}\left[g(x+\eta) - g(x) - \eta\frac{\partial}{\partial x}g\right]k(x,u,\eta)\, d\eta,$$

where

$$a(x,u) = (r-l) + \lambda\beta\Big(\frac{e^x}{S_0 e^{(r-l)u}\sigma^{2\beta}}\Big)^{-\frac{1}{\beta}},$$

$$k(x,u,\nu) = \left(\frac{e^x}{S_0 e^{(r-l)u}\sigma^{2\beta}}\right)^{-\frac{1}{\beta}} k_{CGMY}(\nu). \tag{37}$$

Here, we have $b(x,u) = 0$, which implies no diffusion part. Hence by the result in (24), we have

$$\begin{aligned} q_T(y,T) &= -\frac{\partial}{\partial y}[(r-l+w(y,T))q(y,T)] \\ &+ \frac{\partial^2}{\partial y^2}\int_{R-\{0\}}\tilde{\tilde{k}}(y-\nu,T,\nu)q(y-\nu,T)\, d\nu. \end{aligned}$$

where

$$w(x,u) = -\int_{R-\{0\}} (e^{\nu} - 1 - \nu) k(x,u,\nu)\, d\nu.$$

Substituting $q_T(y,T)$ into (25), we obtain an equation analogous to what we have in (26) except that the diffusion part is no longer present

$$\begin{aligned}
& C_T = -lC - (r-l)KC_K \\
+ \quad & KCD_1 e^{\frac{r-l}{\beta}T} \int_K^{\infty} C_{KK}(U,T) U^{-\frac{1}{\beta}} f_1(\ln \frac{K}{U})\, dU \\
& + KCD_1 e^{\frac{r-l}{\beta}T} \int_0^{K} C_{KK}(U,T) U^{-\frac{1}{\beta}} f_2(\ln \frac{K}{U})\, dU \\
+ \quad & CD_1 e^{\frac{r-l}{\beta}T} \int_K^{\infty} dU \int_0^{K} C_{KK}(W,T) W^{-\frac{1}{\beta}} f_2(\ln \frac{U}{W}) dW \\
- \quad & CD_1 e^{\frac{r-l}{\beta}T} \int_0^{K} dU \int_K^{\infty} C_{KK}(W,T) W^{-\frac{1}{\beta}} f_1(\ln \frac{U}{W}) dW.
\end{aligned}$$

where $D_1 = (S_0 \sigma^{2\beta})^{\frac{1}{\beta}}$,and

$$\begin{aligned}
f_1(\nu) &= \int_{-\infty}^{\nu} \int_{-\infty}^{u} e^{Gw} (-w)^{-(1+Y)}\, dw\, du, \\
f_2(\nu) &= \int_{\nu}^{\infty} \int_{u}^{\infty} e^{-Mw} (w)^{-(1+Y)}\, dw\, du.
\end{aligned}$$

We switch to the log strike space by defining $k = \ln K$, then the forward PIDE becomes

$$\begin{aligned}
& c_T = -lc(k,T) - (r-l) c_k(k,T) \\
+ \quad & KCD_1 e^{\frac{r-l}{\beta}T} \left[\int_K^{\infty} [c_{kk}(\ln U, T) - c_k(\ln U, T)] U^{-\frac{1}{\beta}-2} f_1(\ln \frac{K}{U})\, dU \right. \\
+ \quad & \left. \int_0^{K} [c_{kk}(\ln U, T) - c_k(\ln U, T)] U^{-\frac{1}{\beta}-2} f_2(\ln \frac{K}{U})\, dU \right] \\
+ \quad & CD_1 e^{\frac{r-l}{\beta}T} \int_K^{\infty} dU \int_0^{K} [c_{kk}(\ln W, T) - c_k(\ln W, T)] W^{-\frac{1}{\beta}-2} f_2(\ln \frac{U}{W}) dW \\
- \quad & CD_1 e^{\frac{r-l}{\beta}T} \int_0^{K} dU \int_K^{\infty} [c_{kk}(\ln W, T) - c_k(\ln W, T)] W^{-\frac{1}{\beta}-2} f_1(\ln \frac{U}{W}) dW.
\end{aligned} \tag{38}$$

5 Risk Neutral Estimation

The parameters employed in our study are obtained by calibrating our leveraged Lévy model prices to market data. The forward PIDE (or PDE) formulation allows us to receive option prices for all maturities and all strikes after one execution of the PDE solver[4]

[4]Details on the numerical analysis such as discretization, evaluation of double tail functions f_1 f_2 using FFT, and a new scheme for fast computation as well as anayltical integration results, are omitted intentionally

The prices used in the calibration are those of all exchange traded strikes lying within 20% of the forward price on either side. The data is drawn from CRSP daily option data on $S\&P500$ for December 31, 2003. The criterion for selection of the parameters is the minimization over the parameter space of the root mean square error on an equally weighted basis between market prices and model prices, specifically:

$$\min_{\delta,\sigma} \quad z = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(marketprice_i - modelprice_i(\delta,\sigma))^2}. \tag{39}$$

or

$$\min_{\beta,\sigma,C,G,M,Y} \quad z = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(marketprice_i - modelprice_i(\beta,\sigma,C,G,M,Y))^2}. \tag{40}$$

The $S\&P500$ spot price on December 31, 2003 was 1110.5952. Market prices used in parameter estimation are for out-of-money options on account of their relative liquidity. More exactly, for strikes below the forward price we use put prices and for strikes above the forward price we use call prices. There are total 32 calls and 66 puts of 5 maturities. A summary of maturities we choose and the corresponding interest and dividend rates can be found in Table 1.

Table 1: Summary of T, r and q for $S\&P500$ on December 31, 2003.

S_0	T	r	q
1110.595285	0.139350188	0.010861433	0.01615076
1110.595285	0.215852921	0.011020496	0.016844507
1110.595285	0.464372957	0.011821679	0.01599182
1110.595285	0.713006837	0.012776774	0.015944081
1110.595285	0.96175456	0.014074942	0.01647996

The model prices are calculated on a fine mesh of both maturities and strikes and the ones that correspond to the strikes and maturities of the market prices are calculated using cubic spline interpolation[5] For the pure jump CGMY case, we calibrate all other parameters by fixing Y to be 0.5, 1.0 and 1.5. Table 2 contains the calibrated parameters.

In all cases, the calibrated values for α are all negative which indicates that modeling with the consideration of leverage is necessary. In the context of diffusions, given their documented inability to address certain aspects of the density (e.g. long-tailedness and skewness), it is possible that estimated risk neutral leverage for example is just a reflection

due to limited space. Interested readers are referred to my dissertation Xiao (2005) Chapter 6 and are welcome to contact authors for notes and discussion.

[5] On solving the PDE numerically, we also implement convergence test as ΔT and Δk approach zero to demonstrate numerical stability. The results are not shown here for space consideration.

Table 2: Estimated Parameters From Calibration

Models	Error	Parameters						Implied α
Diffusion	$z(\$)$	σ	δ					$\alpha = \frac{1}{\delta - 2}$
	1.42	0.48	1.67					-3.06
CGMY	$z(\$)$	C	G	M	Y	σ	β	$\alpha = -\frac{1}{2\beta}$
	0.62	3.75	9.76	23.48	0.5	0.48	0.24	-2.08
	0.60	3.74	5.22	20.68	1.0	0.19	0.26	-1.95
	0.60	1.79	1.87	51.35	1.5	0.12	0.29	-1.75

of well documented skewness and has little to do with volatilities actually moving with a market drop. However, we argue that since Lévy processes can internally explain both long-tailedness and skewness without the addition of leverage, the estimation of a significant leverage effect in the pure jump CGMY case is likely just that and not a proxy for other well known and stylized features of the return density.

Comparing the values of z, the minimized root mean squared error, which indicates how well the actual data is fitted by prices calculated from our calibrated model, we see that the jump model clearly has a win over the diffusion model. We show the fitted data plot for only the jump case ($Y = 0.5$) for puts in Figures 1.

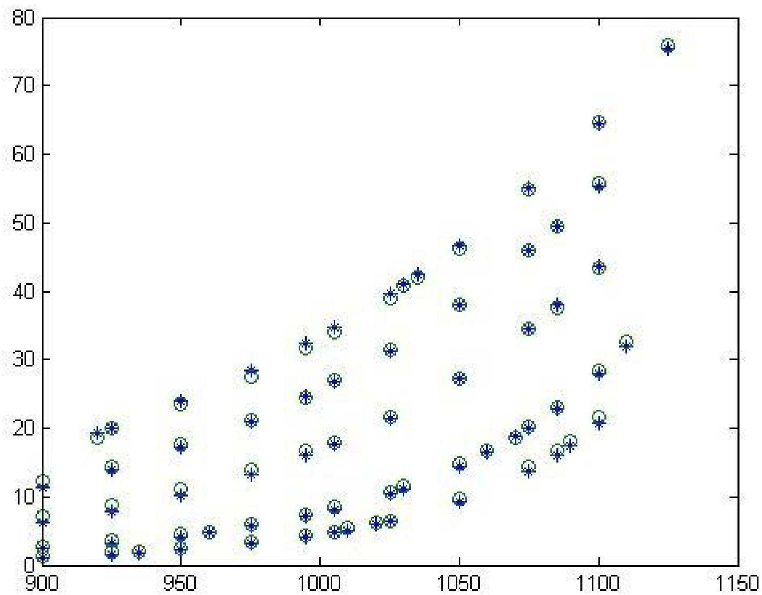

Figure 1: Data Fitting For Puts of All Strikes and Maturities on 12-31-2003, Y fixed at 0.5, $\star$ - Model prices, o - Market prices

6 An Example of Simulation Based Pricing

In this section we show an example of using our leveraged Lévy process to price forward start options and hope to show some adequacy of using such process to value equity structured products in practice. The equity structured products have cash flows defined by functions of the stock price from the contract initiation time to either maturity or early termination. The growth of derivative markets globally, coupled with more informed investor understanding of the risk and return characteristics of structured investment opportunities, has led to an enormous growth in the number and variety of equity structured products being offered by banks, mortgage banks, and building societies. See Madan and Eberlein (2006) for a series of graphs indicating the fast development of this industry globally.

Pricing forward start options is the key to pricing cliquets, which are one class of these fast growing equity structured products. A forward start option is a particular case of multi-stage options, which are derivatives allowing decisions to be made via conditions evaluated at intermediate time points during the life of the contingent claim (Etheridge (2002)). An example of such a forward start option can be found in corporate incentive stock option arrangements, where employees look forward to receiving options, which are at-the-money on the day of grant. The pricing, marking and risk management of these products are usually done by the simultaneous calibration, across strike and maturity, of a stochastic process model to the prices of vanilla options at market close. We mimic such procedure utilizing our model calibration, and use the result from calibration to generate stock price paths. The option price is then determined by computing a discounted expected cash flow of the specified payoff.

Since the leveraged pure jump model outperforms the diffusion model, we use our leveraged pure jump CGMY model and its calibrated parameters to simulate stock price paths. Recall that the stock price process is defined as

$$S_t e^{-\mu t} = S_0 \exp\left(Z_{CGMY}\left(\int_0^t A(S_u, u)\, du \right) + \omega \int_0^t A(S_u, u)\, du \right),$$

where

$$A(S_t, t) = \sigma^2 (\frac{S_t}{S_0} e^{-\mu t})^{2\alpha}$$

and $Z_{CGMY}(t) + \omega t$ is the compensated CGMY process, an exponential Martingale,

$$\omega = -\int_{R-\{0\}} (e^x - 1) k_{CGMY}(x)\, dx.$$

The risk neutral dynamics for the stock price are given by

$$dS_t = (r - l)S_t dt + S_t \int_{-\infty}^{+\infty} (e^x - 1)[m(dx, dt) - A(S_t, t)k(x)dxdt]$$

After discretization, where we assumed $\int_{t-1}^{t} A(S_u, u)du \approx A(S_{t-1}, t-1) * h$

$$S_t - S_{t-1} = (r - l)S_{t-1}h + S_{t-1}\big[Z(A(S_{t-1}, t-1)h) + \omega * (A(S_{t_1}, t-1)h)\big] \tag{41}$$

We then simulate 10000 paths of stock prices[6] using model parameters $C = 3.75$, $G = 9.76$, $M = 23.48$, $Y = 0.5$, $\alpha = -2.08$, $\sigma = 0.48$, calibrated to 2003.12.31 data with initial level being 1110.5952. Each path is for 6 years, 252 days per year. One issue in the simulation is that we observe in (41) the simulation of CGMY random variables is path dependent due to the time change $A(S_{t-1}, t-1)$. Also in each time step we must check the martingale condition which requires the average of $Z(A(S_{t-1}, t-1)h) + \omega * (A(S_{t-1}, t-1)h)$ be 0.

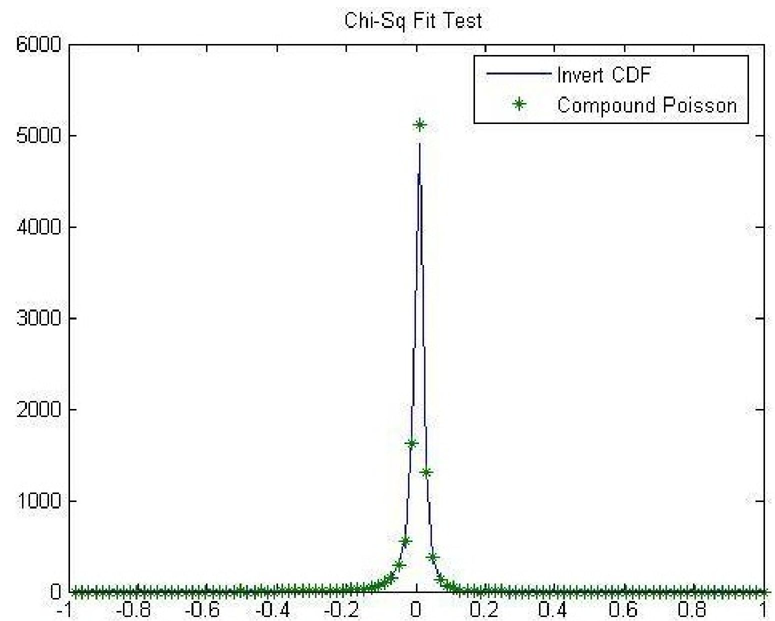

Figure 2: Chi-Squared Fit Test. H0: Compound Poisson Method generate CGMY random variables X(t), C=3.75, G=9.76, M=23.48, Y=0.5, t=0.01

On these simulated paths of stock prices, we then calculate forward start option prices. A forward starting call option with a 100 dollar notional on the gross return pays at $t + \tau$ for the strike k the cash flow

$$100\Big(\frac{S(t+\tau)}{S(t)} - k\Big)^+$$

This option has a time t price $W_t(k, \tau)$ in the model obtained given by

$$W_t(k,\tau) = E_t\Big[e^{-r\tau}\Big(\frac{S(t+\tau)}{S(t)} - k\Big)^+\Big]$$

We construct three surfaces for $t = 1, 2, 5$ years, and each surface using 21 strikes k ranging from 80 to 120 in 2 dollar intervals, and four maturities τ, ranging from 3 months to one

[6] We first simulate the CGMY random variable $Z(t)$ at time t. This can be done in many different ways such as direct compound Poisson simulation, time-changed Brownian motion (Madan and Yor (2005)) and etc. We use a direct compound Poisson simulation and show in Figure 6 a Chi-squared fit test for 10000 simulations using calibrated parameters $C = 3.75$, $G = 9.76$, $M = 23.48$, $Y = 0.5$ for $t = 0.01$. The bench mark of the test is from simulated random variables by inverse transform of the cumulative distribution function, which is obtained from Fourier transform of the characteristic function. The test accepts the null hypothesis H0, at a significance level of 0.05.

year in steps of 3 months. This gives us a total of 84 options for each of the three forward dates.

On investigating the properties of the forward return distributions embedded in our leveraged CGMY process, we construct forward Black-Merton-Scholes implied volatility surfaces $\sigma_t(k, \tau)$ from the forward gross return call prices $W_t(k, \tau)$. These are the forward, time $t > 0$, gross return implied volatility surfaces associated with our leveraged Lévy model calibrated to market implied volatilities at time 0. In Figure 3 we graph the forward implied volatility curves for the leveraged CGMY model with parameters $C = 3.75$, $G = 9.76$, $M = 23.48$, $Y = 0.5$, $\alpha = -2.08$, $\sigma = 0.48$. The curves are for fixed maturities of 3 months, 6 months, 9 months and one year. For each maturity we show the curves for the spot, one year forward, two years forward and five years forward.

We see from these graphs that the forward return distributions are closer to each other than they are to the spot distribution. Forward return distributions have lower ATM volatilities and sharper skews than those of the spot curve. These properties are also documented in Madan and Eberlein (2006) for Sato processes. However, both Skewness and level are preserved to certain extent which shows model should be useful in calibrating option surfaces using dynamics that lock in levels of local skewness of return evolutions.

7 Conclusions

In our study, we build in the leverage effect by introducing a time change dependent on the level of asset and hence affect the expected local volatility in an explicit manner. This is a fairly direct approach in the context of Lévy processes. The continuous case in our study coincides with the development of the constant elasticity of variance models. We however, conduct our investigation in the continuous case through our incorporation of BESQ process as the semi-stable Markov process. In the pure jump case with underlying time changed Lévy process being specified as CGMY process, we hope to engage the leverage effect as well as the ability of explaining long-tailedness and skewness as already being provided by using such pure jump Lévy process with infinite activity. We expect the subsequent developments will allow for both stochastic volatility and leverage by incorporating stochastic volatility into Lamperti processes.

The development of forward Partial Integro-Differential Equations is under a general setup and shows great advantage over the backward ones. In both the continuous case and the pure jump case, we show how to calibrate our model parameters by solving such forward PIDEs and compare model prices to the market data. Although the numerical approach used in the pure jump case is discussed in the context of CGMY process, it is evident that the approach can be extended to a general frame work indifferent of the choice of Lévy process and shall be similarly carried out where other Lévy processes are specified in our model.

Our method of calibration in the context of such leverage Lévy model and its results are then followed by simulation of these models with leverage consideration. It enables an assessment of the impact of leverage on the valuation of claims otherwise analyzed in a zero leverage context. Forward Start Options are the building blocks for Cliquet options as one of the fast growing Equity Structured Products. In this simulation based pricing example we show how to use our calibrated model parameters to simulate time changed CGMY process

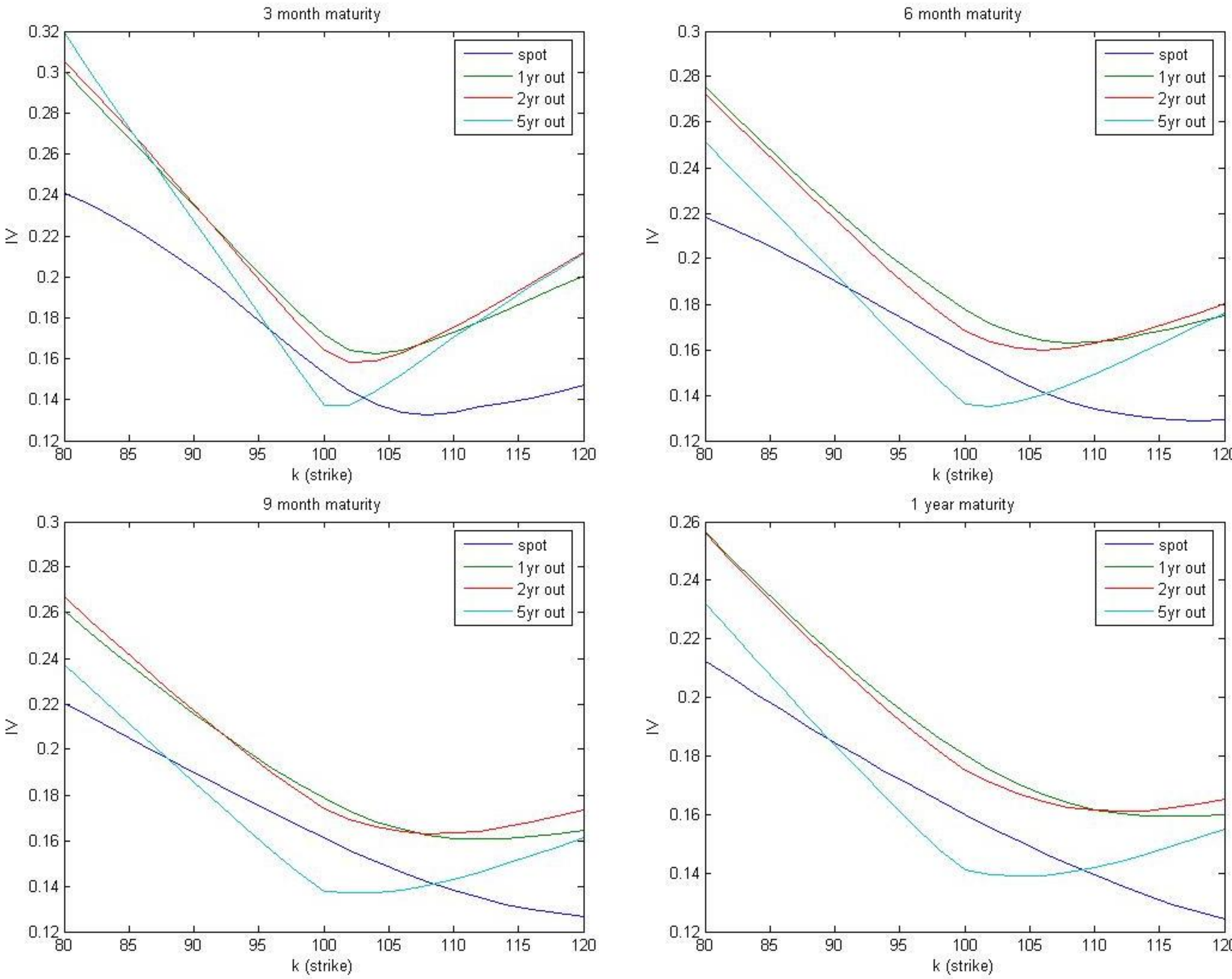

Figure 3: Forward implied volatilities of maturities 3, 6, 9 and 12 months from simulated data.

and hence the stock price dynamics. We construct forward option price surface as well as the forward implied volatility surfaces on our simulated data. The results on forward implied volatility surfaces show that forward return distributions are closer to each other than they are to the spot distribution. Forward return distributions have lower ATM volatilities and sharper skews than those of the spot curve. These properties are also documented in Eberlein and Madan (2006) for Sato processes. We also observe that by locking in the spot skewness into the model, both Skewness and level going forward are preserved to certain extent. The results here show that our model by using dynamics that lock in levels of local skewness of return evolutions, is adequate for calibrating volatility surfaces and pricing more reliably the path dependent equity structured products.

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A From Generator of ξ to Generator of S

Our model for the stock price is basically presenting S_t as a Lamperti process to some power:

$$S_t = S_0 e^{\mu t}\sigma^{2\beta}[L(t)]^{\beta},$$

where $L(t)$ is the Lamperti process associated with a Lévy process, which we now denote as $\xi(t)$. We wish to derive the generator of S from the generator of ξ. Assume the Lévy triplet for $\xi(t)$ is $(\lambda, 0, k_\xi(x)dx)$, where λ is the drift coefficient associated with a certain truncation function. According to a theorem due to Ito and Neveu (through applying results about Fourier and Fourier inversion transformation, and applying the Lévy-Khinchine formula, see Applebaum (2004) Page 139 for detail), the generator of $\xi(t)$ is

$$I_\xi f(\xi) = af'(\xi) + \int_{R-\{0\}} [f(\xi+\eta) - f(\xi) - f'(\xi)h(\eta)]k_\xi(\eta)\, d\eta.$$

From the result in Yor, Carmona, and Petit (1994) we know that I_L, the generator of the associated Lamperti $L(t)$ is related to I_ξ in the following way

$$I_L f(x) = \frac{1}{x} I_\xi(f \circ \exp)(\ln x).$$

Hence, we see that

$$I_L f(x) = \frac{1}{x}\big[axf'(x) + \int_{R-\{0\}} [f(xe^{\eta}) - f(x) - xf'(x)h(\eta)]k_\xi(\eta)\, d\eta\big].$$

We can now calculate the generator of S_t via the following computation:

$$\begin{aligned} & E[f(S_t) - f(S_0)] \\ = \; & E[f(S_0 e^{\mu t}\sigma^{2\beta}L_t^{\beta}) - f(S_0)] \\ = \; & E\left[\int_0^t f'(S_{u-})\mu S_{u-}\, du + \int_0^t I_L(f(S_0 e^{\mu u}\sigma^{2\beta}L_u^{\beta}))\, du\right]. \end{aligned} \tag{42}$$

We now work on the second term

$$\begin{aligned} & I_L(f(S_0 e^{\mu u}\sigma^{2\beta}L_u^{\beta})) \\ = \; & \frac{1}{L}\left[aL\frac{\partial}{\partial L} f(S_0 e^{\mu u}\sigma^{2\beta}L_u^{\beta}) + \int_{R-\{0\}} \big[f(S_0 e^{\mu u}\sigma^{2\beta}(L_u e^{\eta})^{\beta}) \right. \\ & \left. - f(S_0 e^{\mu u}\sigma^{2\beta}L_u^{\beta}) - L\frac{\partial}{\partial L} f(S_0 e^{\mu u}\sigma^{2\beta}L_u^{\beta})h(\eta)\big]\, k_\xi(\eta)\, d\eta\right], \end{aligned} \tag{43}$$

the term

$$\frac{\partial}{\partial L} f(S_0 e^{\mu u}\sigma^{2\beta}L_u^{\beta}) = f'(S_u)S_0 e^{\mu u}\sigma^{2\beta}\beta L_u^{\beta-1} = f'(S_u)\frac{S_u\beta}{L},$$

or equivalently,

$$L\frac{\partial}{\partial L}f(S_0e^{\mu u}\sigma^{2\beta}L_u^\beta) = \beta S_u f'(S_u).$$

We also have from $S_u = S_0e^{\mu u}\sigma^{2\beta}L_u^\beta$, that $L_u = \left(\frac{S_u}{S_0e^{\mu u}\sigma^{2\beta}}\right)^{1/\beta}$. We may now write (43) in terms of S

$$I_L(f(S_0e^{\mu u}\sigma^{2\beta}L_u^\beta)) = \left(\frac{S_u}{S_0e^{\mu u}\sigma^{2\beta}}\right)^{-1/\beta}\Bigg[a\beta S_u f'(S_u) + \int_{R-\{0\}}\big[f(S_{u-}e^{\beta\eta}) - f(S_{u-}) - \beta S_{u-}f'(S_{u-})h(\eta)\big]k_\xi(\eta)\,d\eta\Bigg].$$

Substituting back in (42), we get

$$\begin{aligned} &E[f(S_t) - f(S_0)] \\ = \ &E\Bigg[\int_0^t \mu S_{u-}f'(S_{u-})\,du + \int_0^t du\left(\frac{S_u}{S_0e^{\mu u}\sigma^{2\beta}}\right)^{-1/\beta}\Bigg[a\beta S_u f'(S_u) + \\ &\int_{R-\{0\}}\big[f(S_{u-}e^{\beta\eta}) - f(S_{u-}) - \beta S_{u-}f'(S_{u-})h(\eta)\big]k_\xi(\eta)\,d\eta\Bigg]\Bigg]. \end{aligned}$$

We know it is also true that

$$E[f(S_t) - f(S_0)] = E\left[\int_0^t I_S(f)(S_u)\,du\right],$$

so it follows that

$$\begin{aligned} I_S(f)(x) \ = \ &\mu x f'(x) + \left(\frac{x}{S_0e^{\mu u}\sigma^{2\beta}}\right)^{-1/\beta}\Bigg[a\beta x f'(x) + \\ &\int_{R-\{0\}}\big[f(xe^{\beta\eta}) - f(x) - \beta x f'(x)h(\eta)\big]k_\xi(\eta)\,d\eta\Bigg]. \end{aligned}$$

By rearranging the terms, we write the generator of S_t as

$$\begin{aligned} I_S(f)(x,u) = &\left[\mu + \left(\frac{x}{S_0e^{\mu u}\sigma^{2\beta}}\right)^{-1/\beta}a\beta\right]x f'(x) \\ + &\left(\frac{x}{S_0e^{\mu u}\sigma^{2\beta}}\right)^{-1/\beta}\int_{R-\{0\}}\big[f(xe^{\beta\eta}) - f(x) - \beta x f'(x)h(\eta)\big]k_\xi(\eta)\,d\eta. \end{aligned}$$

With l denoting the dividend yield, and changing μ to $r-l$ in I_S, we have the risk neutral form

$$\begin{aligned} I_S(f)(x,u) = &\left[(r-l) + \left(\frac{x}{S_0e^{(r-l)u}\sigma^{2\beta}}\right)^{-1/\beta}a\beta\right]x f'(x) \\ + &\left(\frac{x}{S_0e^{(r-l)u}\sigma^{2\beta}}\right)^{\frac{-1}{\beta}}\int_{R-\{0\}}\big[f(xe^{\beta\eta}) - f(x) - \beta x f'(x)h(\eta)\big]k_\xi(\eta)\,d\eta. \qquad (44) \end{aligned}$$