Cash Flow Risk, Discounting Risk, and the Equity Premium Puzzle

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Stylized Features and Wish List

Large Spread of Equities Over Bonds [Mehra and Prescott (1985 JME)].

- Need a theory of equity premium that produces an equity premium of about 7%.
- Feature quite robust, as there is also a large spread of equities over bonds across other industrialized security markets (Dimson, Marsh, and Staunton (2003)).
- Ideal if the candidate models can simultaneously explain:
  1. Price path of the S&P 500,
  2. Equity risk premium, and,
  3. Treasury yield curve.
- Difficult and challenging problem.

100 Years of Equity Risk Premium (1900-2001)

Country	Relative to Bills Geomet. Mean	Relative to Bills Arith. Mean	Relative to Bills SD	Relative to Bonds Geomet. Mean	Relative to Bonds Arith. Mean
Australia	7.0	8.5	17.2	6.3	7.9
Canada	4.4	5.7	16.7	4.2	5.7
France	7.1	9.5	23.9	4.6	6.7
Germany	4.6	10.0	35.2	6.3	9.6
Italy	6.6	10.6	32.5	4.6	8.0
Japan	6.4	9.6	27.9	5.9	10.0
Sweden	5.3	7.4	21.9	4.9	7.1
UK	4.5	6.2	19.9	4.2	5.5
US	5.6	7.5	19.7	4.8	6.7
World	4.6	5.9	16.5	4.3	5.4

Crux of our Approach

Asset pricing models under the perfect-markets assumption implies

$$P_t = \int_t^\infty E_t\left[\frac{M_u}{M_t} D_u\right] du, \qquad \text{and,}$$

$$\mu_t - r_t = -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, \frac{dP_t}{P_t}\right)/dt,$$

where $E_t[\cdot]$ is the time-t conditional expectation operator with respect to the objective probability measure, and M_t is the pricing kernel.

Our goal is to explore two issues:

- If we can replace the empirically difficult-to-estimate marginal utility by a pricing-kernel function of observables and then specify both the primitive process for discounting and the exogenous cash flow stream, we will have an equilibrium asset pricing model based on observable state variables. There may be errors in consumption measurement.
- Dividend-based models have generally disapointed. Earnings-based models may be better specified and assess its potential for equity premium modeling.

Modeling Assumption 1: Dividend-Earnings Mapping

Market-portfolio has a constant dividend-payout ratio (plus noise), α, with dividend dynamics,

$$D_t\,dt = \alpha\,Y_t\,dt + d\,Z_t, \qquad 1 \geq \alpha \geq 0. \tag{1}$$

- Y_t is the aggregate earnings-per-share (EPS) flow at t and hence $Y_t\,dt$ is the total EPS over the interval from t to $t + dt$.
- $d\,Z_t$ is the increment to a martingale process with zero mean.

Modeling Assumption 2: Cash Flow Evolution

Under the objective probability measure, earnings Y_t is assumed to follow:

$$\frac{dY_t}{Y_t} = G_t\,dt + \sigma_y\,dW_t^y, \tag{2}$$

$$d\,G_t = \kappa_g\left(\mu_g^* - G_t\right)\,dt + \sigma_g\,dW_t^g, \tag{3}$$

for constants σ_y, κ_g, μ_g^* and σ_g.

- The long-run mean for both G_t and actual EPS growth $\frac{dY_t}{Y_t}$ is μ_g^*, and the speed at which G_t adjusts to μ_g^* is reflected by κ_g.
- Y_t is observable and G_t can be obtained from analyst estimates, we can learn about the equity premium based on readily identifiable and observable state variables.

Modeling Assumption 3: Discounting Process

As in Constantinides (1992), M_t follows an Ito process satisfying,

$$\frac{dM_t}{M_t} = -r_t\, dt - \sigma_m\, dW_t^m, \tag{4}$$

for a constant σ_m. Discounting rate, r_t, follows (for constants κ_r, μ_r^* and σ_r):

$$d\, r_t = \kappa_r\, (\mu_r^* - r_t)\; dt + \sigma_r\, dW_t^r. \tag{5}$$

- Suppose $M_t = C_t^{-\gamma}$. Then,

$$\frac{dM_t}{M_t} = -\gamma \frac{dC_t}{C_t} + \frac{1}{2}\gamma(1+\gamma)\left(\frac{dC_t}{C_t}\right)^2 \tag{6}$$

$$\mu_t - r_t = \gamma\, \mathrm{Cov}_t\left(\frac{dC_t}{C_t}, \frac{dP_t}{P_t}\right)/dt \tag{7}$$

$$r_t\, dt = \gamma\, E_t\left(\frac{dC_t}{C_t}\right) - \frac{1}{2}\gamma(1+\gamma)\, E_t\left(\frac{dC_t}{C_t}\right)^2. \tag{8}$$

  Thus, unlike Mehra and Prescott (1985) and Weil (1989), we independently model r_t dynamics. Modeling M_t avoids unobservability issues.

- Approach provides a number of interest rate parameters that can be separately calibrated to the observed Treasury yield curve and outside of the equity market.

Modeling Assumption 4: Correlation Structure

All state-variable factors are priced (i.e., r_t,Y_t,G_t) and receive compensation:

- Shocks to expected earnings growth, W^g, may be correlated with both systematic shocks W^m and interest rate shocks W^r, with their respective correlation coefficients denoted by $\rho_{g,m}$ and $\rho_{g,r}$.
- Correlations of earnings shocks W^y with W^g, W^m and W^r are respectively denoted by $\rho_{g,y}$, $\rho_{m,y}$ and $\rho_{r,y}$.

Dynamics of Market Portfolio

Equity Index price P_t is Markov in G_t, r_t and Y_t. The PDE for P_t:

$$\frac{1}{2}\,\sigma_y^2\,Y^2\,\frac{\partial^2 P}{\partial Y^2} + (G - \Pi_y)\,Y\,\frac{\partial P}{\partial Y} + \rho_{g,y}\sigma_y\sigma_g\,Y\,\frac{\partial^2 P}{\partial Y\,\partial G} + \rho_{r,y}\sigma_y\sigma_r\,Y\,\frac{\partial^2 P}{\partial Y\,\partial r} +$$
$$\rho_{g,r}\sigma_g\sigma_r\,\frac{\partial^2 P}{\partial G \partial r} + \frac{1}{2}\sigma_r^2\,\frac{\partial^2 P}{\partial r^2} + \kappa_r\,(\mu_r - r)\,\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_g^2\frac{\partial^2 P}{\partial G^2}$$
$$+\kappa_g\,(\mu_g - G)\,\frac{\partial P}{\partial G} - r\,P + \alpha\,Y = 0, \tag{9}$$

subject to the transversality condition $P_t < \infty$. In the valuation PDE (9) we set,

$$\mu_g \equiv \mu_g^* - \frac{\Pi_g}{\kappa_g}, \tag{10}$$

$$\mu_r \equiv \mu_r^* - \frac{\Pi_r}{\kappa_r}, \tag{11}$$

which are, respectively, the long-run means of G_t and r_t under the risk-neutral probability measure defined by the pricing kernel M_t.

Risk Premium on Factors

It can be shown that

$$\Pi_y \equiv -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, \frac{dY_t}{Y_t}\right)/dt, \tag{12}$$

$$\Pi_g \equiv -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, dG_t\right)/dt, \tag{13}$$

$$\Pi_r \equiv -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, dr_t\right)/dt, \tag{14}$$

are the risk premium for the earnings shocks, expected earnings growth, and interest rate, respectively.

- Π_r can be estimated from the Treasury yield curve.
- Π_y and Π_g can only be inferred from $\{P_t\}_{t=1,\ldots,T}$ as Y_t and G_t risks are non-traded.

The solution to the market portfolio PDE (9) is of the form:

$$P_t = \alpha\, Y_t \int_0^\infty \overline{p}[t,u;G,r]\,du, \tag{15}$$

where $\overline{p}[t,u;G,r]$ can be interpreted as the time-t price of a claim that pays \$1 at a future date $t+u$, and:

$$\overline{p}[t,u;G,r] = \exp\left(\,\varphi[u] - \varrho[u]\,r_t + \vartheta[u]\,G_t\,\right), \tag{16}$$

where

$$\begin{aligned}
\varphi[u] \quad \equiv \quad & -\Pi_y\,u + \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\left(u + \frac{1-e^{-2\kappa_r u}}{2\kappa_r} - \frac{2(1-e^{-\kappa_r u})}{\kappa_r}\right) \\
& - \frac{\kappa_r\mu_r + \sigma_y\sigma_r\rho_{r,y}}{\kappa_r}\left(u - \frac{1-e^{-\kappa_r u}}{\kappa_r}\right) \\
& + \frac{1}{2}\frac{\sigma_g^2}{\kappa_g^2}\left(u + \frac{1-e^{-2\kappa_g u}}{2\kappa_g} - \frac{2}{\kappa_g}(1-e^{-\kappa_g u})\right) \\
& + \frac{\kappa_g\mu_g + \sigma_y\sigma_g\rho_{g,y}}{\kappa_g}\left(u - \frac{1-e^{-\kappa_g u}}{\kappa_g}\right) \\
& - \frac{\sigma_r\sigma_g\rho_{g,r}}{\kappa_r\kappa_g}\left(u - \frac{1}{\kappa_r}(1-e^{-\kappa_r u}) - \frac{1}{\kappa_g}(1-e^{-\kappa_g u}) + \frac{1-e^{-(\kappa_r+\kappa_g)u}}{\kappa_r+\kappa_g}\right)
\end{aligned} \tag{17}$$

$$\varrho[u] \quad \equiv \quad \frac{1 - e^{-\kappa_r u}}{\kappa_r}, \qquad \vartheta[u] \equiv \frac{1 - e^{-\kappa_g u}}{\kappa_g}, \tag{18}$$

subject to the transversality condition that

$$\mu_r - \mu_g > \frac{\sigma_r^2}{2\,\kappa_r^2} - \frac{\sigma_r \sigma_y \rho_{r,y}}{\kappa_r} - \frac{\sigma_g \sigma_r \rho_{g,r}}{\kappa_g \kappa_r} - \Pi_y + \frac{\sigma_g^2}{2\kappa_g^2} + \frac{\sigma_g \sigma_y \rho_{g,y}}{\kappa_g}. \tag{19}$$

- Model price for the market-portfolio is the summed value of a continuum of claims that each pay at a future time an amount respectively determined by the earnings process.
- Presence of an integral in (15) should not hamper the applicability of the model as the integral can be computed numerically.
- Gordon model: Both G_t and r_t are constant over time: $G_t = g$ and $r_t = r$. Both M_t and Y_t now follow geometric Brownian motion. In this case, $P_t = \frac{\alpha\, Y_t}{r + \Pi_y - g}$ provided $r + \Pi_y - g > 0$.
- In our economic setting valuation is more complex as both discounting and cash flow forecasts have to be simultaneously assessed at the same time, and in the entire time-continuum.

Dynamics of Equity Premium

Based on (1) and the pricing solution (15), the equity premium is,

$$\begin{aligned}
\mu_t - r_t &= -\mathrm{Cov}_t\left(\frac{dM_t}{M_t}, \frac{dP_t}{P_t}\right)/dt, \\
&= \Pi_y \frac{Y_t}{P_t}\frac{\partial P_t}{\partial Y_t} + \Pi_g \frac{1}{P_t}\frac{\partial P_t}{\partial G_t} + \Pi_r \frac{1}{P_t}\frac{\partial P_t}{\partial r_t} \\
&= \Pi_y + \Pi_g \left(\frac{\int_0^\infty \overline{p}[t,u;G,r] \times \vartheta[u]\, du}{\int_0^\infty \overline{p}[t,u;G,r]\, du}\right) - \Pi_r \left(\frac{\int_0^\infty \overline{p}[t,u;G,r] \times \varrho[u]\, du}{\int_0^\infty \overline{p}[t,u;G,r]\, du}\right)
\end{aligned}$$

- Equation (20) shows that the equity premium is a weighted sum of the risk premiums for shocks respectively due to earnings, expected earnings growth, and interest rate.
- Equilibrium equity premium is a function of r_t, the expected EPS growth, the firm's required risk premiums, and the cash flow and interest rate parameters.$\mu_t - r_t$ is mean-reverting with the state of r_t and G_t.

An Intuitive Explanation

Based on (9), we may write the stock price as,

$$P_t = \alpha \int_t^\infty E_t^Q \left(e^{-\int_t^u r_s\, ds} Y_u \right) du, \tag{20}$$

where the processes for (Y_t, G_t, r_t) under the risk-neutral Q-measure are:

$$\frac{dY_t}{Y_t} = (G_t - \Pi_y)\, dt + \sigma_y\, d\widetilde{W}_t^y, \tag{21}$$

$$d\,G_t = \kappa_g \left([\mu_g^* - \Pi_g/\kappa_g] - G_t\right) dt + \sigma_g\, d\widetilde{W}_t^g, \tag{22}$$

$$d\,r_t = \kappa_r \left([\mu_r^* - \Pi_r/\kappa_r] - r_t\right) dt + \sigma_r\, d\widetilde{W}_t^r. \tag{23}$$

1. Economically, risk-averse investors seek to discount future cash flows more heavily under the equivalent martingale measure. We should expect $\Pi_r < 0$.

2. Investors tend to be less optimistic about future cash flows under the equivalent martingale measure than under the physical probability measure. Intuitively, we should have $\Pi_y > 0$ and $\Pi_g > 0$.

Benchmark Comparative Statics for Equity Premium

In benchmark comparative statics calculations $r_t = 5.68\%$ and $G_t = 7.48\%$ which are market observed values as of July 1998 and correspond to S&P 500 index level of 1174.

1. Equity premium is increasing in both G_t and μ_g^*, but decreasing in both r_t and μ_r^*.
2. Equity premium is much more sensitive to μ_g^* (μ_r^*) than to G_t (r_t).
3. Model equity premium increases with EPS growth volatility σ_y, the volatility of expected EPS growth σ_g, and the volatility of the interest rate σ_r. Risks as measured by these parameters raise the required compensation to shareholders.

Data and Proxies

- S&P 500 index is the proxy for the market-portfolio.
- Three data inputs: expected EPS growth G_t, interest rate r_t, current EPS Y_t, plus model parameters needed.
- I/B/E/S did not start collecting analyst EPS estimates until January 1982. Thus, our focus is on the sample period from January 1982 to July 1998.
- Current-year S&P 500 EPS (i.e., FY1) is taken to be the proxy for Y_t.
- Analyst-expected EPS growth from the current (FY1) to the next fiscal-year (FY2) is the measure for G_t.
- We use the 3-month Treasury yield or those implied by the Kalman-filter as candidates for r_t in estimation and calibration.
- Panel of Treasury yields. We choose Treasury securities with constant maturity of 6 months, 2 years, 5 years, and 10 years.

Table 1

Table 1

	Average	**Std.**	**Max.**	**Min.**
Price-to-Earnings Ratio	15.10	4.13	26.47	7.28
Expected Earnings Growth	10.13%	5.31%	26.13%	0.09%
Interest Rate	6.98%	2.13%	14.68%	5.68%
Monthly Equity Premium	0.0073 8.7%	0.040	0.162	-0.200

Interest Rate Risk Premium

- Explain the Treasury yield curve as close as possible with model. Hence, our approach circumvents the risk-free rate puzzle outlined in Weil (1989).
- Sign and magnitude of the interest rate risk premium is addressed using the Kalman filtering approach and a panel of Treasury bond yields (6 months, 2 years, 5 years, and 10 years).
- Estimated interest-rate risk premium, Π_r, has a point estimate of -0.00201 (i.e., -20 basis points).
- Estimated Π_r can drive a substantial wedge between the risk-neutral and the physical interest rate processes. Calculation shows that Π_r raises the risk-neutral interest-rate drift by 86.9 basis points.

Table 2: Kalman Filtering Estimation

Parameter	κ_r	σ_r	μ_r^*	Π_r	Log-Lik
r_t	0.2313	0.0128	0.0728	-0.0020	1804.93
process	(0.0135)	(0.0008)	(0.0022)	(0.0005)	

	6-months	**2-years**	**5-years**	**10-years**
Median Absol. Pricing Errors (bp)	37	25	35	50
Squared-root Mean Squared Errors (bp)	48	33	44	59

MLE of G_t Process

- Unavailability of contingent claims written directly on the G_t process precludes a joint estimation of the expected EPS growth processes.
- Propose a two-step procedure to estimate Π_g. First, we exploit the transition density function to estimate the structural parameters, $\Theta_g \equiv \{\kappa_g, \mu_g^*, \sigma_g\}$, of the G_t process in (3). Second, taking Θ_g as given, we estimate Π_g, along with other unknown parameters, based on the time-series of S&P 500 index.
- MLE based on the transition density (Nowman (1997) and Bergstrom (1984)). MLE parameter estimates are (the standard errors in parenthesis):

$$\kappa_g = 1.4401 \quad (0.4411) \tag{24}$$

$$\mu_g^* = 0.1024 \quad (0.0153) \tag{25}$$

$$\sigma_g = 0.0894 \quad (0.0047) \tag{26}$$

with an average log-likelihood value of 2.29575.

Compensation for Cash Flow Risk and the Equity Premium

5 unknown parameters:

$$\Theta \equiv \{\Pi_g, \Pi_y, \alpha, \sigma_y, \rho\}, \tag{27}$$

are still required to determine the price of the market portfolio, P_t, in (15).

Following the lead in fixed-income and option pricing, Θ is estimated using the time-series of market prices. Define the model price-to-earnings ratio as:

$$\mathrm{pe}_t \equiv \frac{P_t}{Y_t} = \alpha \int_0^\infty \overline{p}[t, u; G, r] \, du, \tag{28}$$

and let $\widetilde{\mathrm{pe}}_t$ be the month-t observed price-to-earnings ratio. Our estimation procedure tries to find a Θ to solve,

$$\mathrm{RMSE} \equiv \min_{\Theta} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \left(\alpha \int_0^\infty \overline{p}[t, u; G, r] \, du - \widetilde{\mathrm{pe}}_t \right)^2}, \tag{29}$$

subject to the transversality condition in (19). See Table 3.

Equity Premium and Table 3

- Market-implied expected-EPS-growth risk premium, $\Pi_g = 0.145\%$, is surprisingly small relative to the market-implied earnings risk premium, $\Pi_y = 6.531\%$.
- With $\Pi_r = -0.002$, the sample $-\Pi_r \left(\frac{\int_0^\infty \overline{p}[t,u;G,r] \times \varrho[u]\, du}{\int_0^\infty \overline{p}[t,u;G,r]\, du} \right)$ is 77.16 bp. Accounting for discounting risk can help alleviate the equity premium puzzle.
- Based on (20), the overall equity premium can, thus, be calculated as:

$$\begin{aligned} \mu_t - r_t &= \Pi_y + \Pi_g \left(\frac{\int_0^\infty \overline{p}[t,u;G,r] \times \vartheta[u]\, du}{\int_0^\infty \overline{p}[t,u;G,r]\, du} \right) - \Pi_r \left(\frac{\int_0^\infty \overline{p}[t,u;G,r] \times \varrho[u]\, du}{\int_0^\infty \overline{p}[t,u;G,r]\, du} \right), \\ &= 6.53\% + 0.01\% + 0.7716\%, \\ &= 7.31\%. \end{aligned}$$

The ability of the model to generate an equity premium of 7.31% sharply contrasts many studies calibrated to the per-capita consumption data.

- One particular economic yardstick is whether the estimated risk premiums and model parameters provide a "good enough" approximation of the market's implicit valuation process. The average mean pricing error is -7.22% with a standard deviation of 23.98%, and the absolute pricing error of the S&P 500's 18.30%.

- Proper parameterization of both the discounting structure and the cash flow process is key to improving performance by an asset pricing model and to achieving a reasonable equity premium.

Table 3: Estimation and Equity Risk Premium

Estimation of the risk premiums is based on S&P 500 index observations from January 1982 to July 1998 (199 observations). We minimize the distance between the model price-to-earnings ratio and the market price-to-earnings ratio denoted by $\widetilde{\mathrm{pe}}_t$:

$$\mathrm{RMSE} \equiv \min_{\Theta} \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\alpha\int_0^{\infty}\overline{p}[t,u;G,r]\,du - \widetilde{\mathrm{pe}}_t\right)^2},$$

subject to the transversality condition
$\mu_r - \mu_g > \frac{\sigma_r^2}{2\,\kappa_r^2} + \frac{\sigma_g^2}{2\kappa_g^2} + \frac{\sigma_g\sigma_y\rho_{g,y}}{\kappa_g} - \frac{\sigma_r\sigma_y\rho_{r,y}}{\kappa_r} - \frac{\sigma_g\sigma_r\rho_{g,r}}{\kappa_g\kappa_r} - \Pi_y.$

Panel A: Estimation Based on 3-Month Treasury Rate

Π_g	Π_y	α	σ_y	ρ	RMSE	Mean(ϵ_t) {Std(ϵ_t)}	Mean($\lvert\epsilon_t\rvert$) {Std($\lvert\epsilon_t\rvert$)}	Mean($\mu_t - r_t$)
0.00145	0.0653	0.410	0.181	-0.109	3.229	-7.22% {23.98%}	18.30% {17.63}	7.312%

Concluding Remarks

1. Our theoretical approach is based on the observation that aggregate per-capita consumption is hard to measure empirically.
2. If we can replace the empirically difficult-to-estimate marginal utility by a pricing-kernel function of observables and then specify both the primitive process for discounting and the exogenous cash flow stream, we will have an equilibrium asset pricing model based on observable state variables.
3. S&P 500 index-based estimation results show that the framework is quantitatively useful in explaining the observed market equity premium.