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An alternative valuation model for contingent claims

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Abstract

This paper studies contingent claim valuation in a Lucas-type exchange economy. The derived fundamental valuation equation differs from its Cox–Ingersoll–Ross production-economy counterpart in that it is expressed in terms of the direct utility function and an exogenous output process, thus offering superior tractability. We apply our approach to derive closed-form solutions for bond, bond option, individual stock, and stock option prices, under a more general setting than allowable in the Cox–Ingersoll–Ross framework. The resulting interest rate and stock price dynamics are empirically plausible. Moreover, our stock option pricing formula with stochastic volatility and interest rates can reconcile certain puzzling empirical regularities, including the volatility smile.

Key words: Fundamental valuation equation; Stocks; Bonds; Stock options; Interest rate derivatives
JEL classification: G10; G12; G13

1. Introduction

The well-known valuation framework of Cox, Ingersoll, and Ross (henceforth, CIR) (1985a) has been the basis for recent equilibrium models of contingent claims valuation and of the term structure of interest rates. Their work has, for

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example, provided the necessary push for the development of more general option valuation models beyond Black and Scholes (1973). Like the Merton (1973a) intertemporal asset pricing model, however, their fundamental valuation equation is expressed in terms of both the indirect utility function and the endogenous wealth process. As such, two steps are required in order to price stocks, bonds, or any contingent claims: (i) solve the Hamilton–Jacobi–Bellman equation for the representative agent's optimal consumption–investment policy and the indirect utility function, and (ii) substitute the indirect utility and the derived wealth process into the fundamental valuation equation and solve for the claim's price. Unfortunately, Merton (1971) notes that a closed-form solution for the indirect utility function or the optimal policy cannot be found unless the agent has a log period utility or the investment opportunities are nonstochastic. For this reason, virtually all existing equilibrium models for contingent claims as well as for the term structure of interest rates assume a log utility function.[1]

The first purpose of this paper is to show how to circumvent the above-mentioned difficulty of the CIR (1985a) valuation model by adopting a continuous-time Lucas (1978) exchange economy.[2] A Lucas-type economy provides the flexibility to impose an exogenous consumption-portfolio plan. Once such a plan is specified, the imposed aggregate output can be substituted into Breeden's (1979) pricing relation, which is then used, together with the application of Ito's lemma, to produce an alternative valuation equation for contingent claims that depends only on the exogenous output and the direct utility of consumption.[3] As a result, security valuation can be conducted without facing the difficulty of the nonlinear Bellman equation.

To illustrate the wider applicability of our approach, we examine pricing issues for bonds, bond options, individual stocks, and stock options under the two most

[1]For a partial list, see Bailey and Stulz (1989), Chen and Scott (1992), CIR (1985b), Longstaff (1990), Longstaff and Schwartz (1992), and Scott (1996). The few exceptions are the discrete-time models of Amin and Ng (1993), Brennan (1979), Rubinstein (1976), Sun (1992), and Turnbull and Milne (1991), which typically allow the representative agent to have a power utility function. On the surface it appears that one can derive contingent claims valuation models in a more general setting using a discrete-time framework than using its continuous-time counterpart, clearly contradictory to the fact that a continuous-time framework typically offers higher, rather than lower, tractability.

[2]The relative advantage of a pure-exchange economy has also been exploited by, for instance, Sun (1992) and Wang (1996), to derive term structure models. But the issues addressed in these studies are different from those in this paper. Other characterizations of a pure-exchange economy can be found in Gennote and Marsh (1993), Goldstein and Zapatero (1996), and He and Leland (1993). All these studies assume the existence of a single aggregate firm.

[3]Note that the type of economies studied in this paper also differs from the one in Breeden (1979) in a fundamental way. As in the CIR (1985a) and Merton (1973a) models, consumption in Breeden's model is endogenous. Consequently, in order to literally use Breeden's pricing equation to value contingent claims, one would need to first solve for the optimal consumption process in closed form before going any further. Thus, the same difficulty as with the CIR valuation model applies to the Breeden model.

widely used classes of utility functions: the power and the exponential utility class. Economically, these two classes are interesting because the power utility functions form the constant relative risk aversion (CRRA) family while the exponential utility functions represent the constant absolute risk aversion (CARA) family. In the dynamic economy that we study, there are two systematic state variables that each follow a mean-reverting square-root process. The term structure model developed here contains most elements of Longstaff and Schwartz's (1992) model, and in comparison our model has at least two additional features: (i) the term structure depends on the agent's risk aversion and (ii) the term premium in our model has more plausible properties such as the ability to take any desired shape. The closed-form bond option pricing formula is also a two-factor model that depends on risk aversion. Its structure resembles those in Heston (1993) and Scott (1996) in that the probabilities that the option expires in the money are recovered by inverting their respective characteristic functions, and it is different from the formulas in CIR (1985b) and Longstaff and Schwartz (1992) in which the probabilities are obtained by integrating over a bivariate chi-square distribution function.

In deriving the stock price formula, we assume for each individual firm a continuous dividend process that is proportional to an exponential function of the systematic and firm-specific state variables, with the proportion linear in the same state variables. The class of dividend policies covered by this assumption includes those that lead to a constant dividend yield and those that produce a dividend yield equal to the short-term interest rate. The endogenously determined stock price and its dynamics have many empirically desirable properties. First, the rate of stock price appreciation has a predictable component and expected stock returns are time-varying, which is consistent with recent findings on stock return predictability. Second, the resulting volatility of stock returns consists of a systematic and an idiosyncratic risk component. This volatility structure is similar to the one used in Amin and Ng (1993), except that in their case it is an exogenously imposed feature whereas here it is a derived equilibrium property. This property is also in line with the growing empirical evidence that individual stock volatility is not only stochastic over time, but correlated with systematic or market-wide volatility. Third, the correlation between changes in stock price and in stock volatility can be negative or positive as well as time-varying, depending on the structural parameter values and the state of the economy. According to Bakshi, Cao, and Chen (1997), Bates (1995), and Longstaff (1994), such a stock return-volatility structure may be necessary for the resulting equity option pricing model to reconcile certain empirical regularities such as the 'volatility smile' (Rubinstein, 1985, 1994).

Given the empirical desirability of both the term structure of interest rates and the stock price process, the closed-form stock option pricing formula developed in this paper should surpass existing models in several dimensions. First, determining the stock option price jointly and simultaneously with the bond and stock prices guarantees internal consistency, in contrast with the partial equilibrium models of

Amin and Ng (1993), Heston (1993), Hull and White (1987), Longstaff (1994), Merton (1973b), Stein and Stein (1991), and Wiggins (1987). Second, the option pricing formula obtains whether the agent has a power or an exponential utility function, as opposed to the log utility function commonly assumed in the existing literature. Third, our option pricing model admits both stochastic interest rates and stochastic volatility, whereas existing models that extend the classic Black–Scholes (1973) formula allow either stochastic volatility and constant interest rates (e.g., Heston, 1993; Hull and White, 1987; Stein and Stein, 1991; and Wiggins, 1987) or constant volatility but stochastic interest rates (e.g., Amin and Jarrow, 1992; and Goldstein and Zapatero, 1996), but not both.[4] Fourth, option prices in our model depend on two factors (in addition to the underlying stock price), which means that the model should be able to capture more variations in option prices, both across different strikes or maturities and over time, than the existing single-factor counterparts. Finally, and perhaps most importantly, the underlying stock for our option pricing model pays a stochastic dividend yield. This is a major departure from the existing option pricing literature in which a constant dividend yield is typically assumed (e.g., Merton, 1973b) or, when a stochastic dividend yield is allowed, closed-form option prices are not available.

To demonstrate certain properties, we provide an example economy in which the structural parameter values are all chosen so that the initial term structure is in line with some known features of its real-life counterpart. First, we show that interest rates have a hump-shaped relation with, and stock prices are decreasing in, risk aversion. Second, equity call option prices are decreasing and convex in risk aversion. Third, our equity option model can help resolve the volatility smile puzzle that has been empirically documented for option markets. More precisely, when option prices determined by our model are used as inputs into the Black–Scholes formula to back out the implied volatility, the implied volatility exhibits a general U-shaped pattern across strike prices, especially for short-term options.

The exchange-economy valuation approach presented in this paper shares an important feature with the partial-equilibrium approach of Constantinides (1992). Constantinides has the pricing kernel (or the marginal utility) process directly specified, whereas we have the utility of consumption and the aggregate output process given separately, which in some sense amounts to specifying the marginal utility process directly. It should be stressed, however, that our equilibrium approach imposes more restrictions on the economy than the Constantinides approach. To see this, note that in an economy with multiple individual firms, an arbitrarily imposed consumption-equity holdings plan may not be supported by any general equilibrium, unless the stock prices of individual

[4] Amin and Ng (1993) and Heston (1993) do consider stochastic interest rates and stochastic volatility. But, they do not have a closed-form solution for option prices. Scott (1996) is an exception, and the difference between his model and ours is noted later.

firms are consistently and endogenously determined. Whereas the Constantinides approach stops at specifying the marginal utility process, our model requires not only the consistent specification of output processes at both the aggregate and the individual firm levels, but also the endogenous determination of individual stock prices in a way that supports the exogenous consumption-equity holdings plan. This is also a crucial modeling aspect that distinguishes an exchange-economy framework from its production-economy counterpart. In a CIR-type production economy, individual stock prices are partly exogenous and the model is closed by making equity holdings endogenous. In contrast, equity holdings in an exchange economy are exogenous and the model is completed by making the individual stock prices endogenous.[5]

The paper is organized as follows. Section 2 introduces a continuous-time economy and derives the alternative valuation equation for contingent claims. Section 3 describes a dynamic economy with power utility investors and then solves for the term structure of interest rates and bond option prices. In Section 4, we develop a pricing formula first for dividend-paying stocks and next for stock options with stochastic volatility and stochastic interest rates. In Section 5, we show that all results from Sections 3 and 4 hold even in economies with exponential utility. Section 6 uses an artificial economy to study properties of the stock option pricing model. Concluding remarks are offered in Section 7, and proof and derivation of results are given in the Appendix.

2. An alternative valuation model

Consider a continuous-time exchange economy of the Lucas (1978) type, in which a sole perishable consumption good is produced and markets are dynamically complete. The stochastic environment is determined by the standard $(M+N)$-dimensional vector Brownian motion $\omega = (\omega_{X_1}, \ldots, \omega_{X_M}, \omega_{Z_1}, \ldots, \omega_{Z_N})'$, where each component process ω_{X_m} or ω_{Z_n} is independent of every other

[5]As demonstrated by Constantinides (1992) (also see Sun, 1992), every non-log-utility exchange economy can be transformed into an equivalent log-utility production economy in which security prices stay the same as in the original exchange economy. This means that a given set of security prices can be supported by either a non-log-utility exchange economy or a log-utility production economy with the requisite investment opportunities. In interpreting such an equivalence result, however, one should distinguish between the *primitive* economy and its *transformed* counterpart. For example, the agent may have a power utility function in the primitive economy but a log utility in the transformed. To see the relative advantage of an exchange-economy framework from another angle, suppose the primitives are such that investors have non-log-utility and the investment opportunities are stochastic. Then, if one starts with a production economy setup, the asset pricing problem becomes, as discussed earlier, unsolvable in closed form. Therefore, a production-economy setup does not allow one to answer such questions as 'what happens to security prices when investors' risk aversion changes?', while its exchange-economy counterpart does.

component of the vector process.[6] Implicitly, the first M components, $\omega_X = (\omega_{X_1}, \ldots, \omega_{X_M})'$, together describe the systematic sources of uncertainty. These M sources are equivalently reflected by M economy-wide state variables X_m, for $m = 1, \ldots, M$, where $X = (X_1, \ldots, X_M)'$ follows a vector diffusion process:

$$dX(t) = \mu_X(t,X)\,dt + \sigma_X(t,X)\,d\omega_X(t)\,, \tag{1}$$

with $X(0) > 0$, where the drift $\mu_X(t,X)$ is an M-dimensional vector of expected instantaneous changes (per unit time) in $X(t)$ and the diffusion term $\sigma_X(t,X)$ is a full-rank $M \times M$ local covariance matrix between changes in $X(t)$ and changes in $\omega_X(t)$. Both the drift and the diffusion terms are functions of only time t and the general state of the economy as represented by $X(t)$, and they satisfy the local Lipschitz and growth conditions (e.g., footnote 4 of CIR, 1985a), implying that there is an unique solution to the stochastic differential equations. The local Lipschitz and growth conditions are also maintained for the other stochastic differential equations throughout the paper.

The other N components, $\omega_Z = (\omega_{Z_1}, \ldots, \omega_{Z_N})'$, together represent all firm-specific sources of uncertainty, for a total of N individual firms. These risk sources are more directly represented by an N-vector diffusion process $Z = (Z_1, \ldots, Z_N)'$, where

$$dZ(t) = \mu_Z(t,Z)\,dt + \sigma_Z(t,Z)\,d\omega_Z(t)\,, \tag{2}$$

with $Z(0) > 0$. The drift $\mu_Z(t,Z)$ is an N-vector of expected local changes in $Z(t)$ and $\sigma_Z(t,Z)$ is a full-rank $N \times N$ local covariance matrix between changes in $Z(t)$ and changes in $\omega_Z(t)$. It is again assumed that $\mu_Z(t,Z)$ and $\sigma_Z(t,Z)$ are smooth functions of $Z(t)$ and t. As ω_X and ω_Z are independent of each other and as the drift and diffusion terms of $dZ(t)$ and those of $dX(t)$ are unrelated to each other, X and Z are also independent of each other. This, however, does not rule out the possibility of correlation among the state variables in Z.

The N production firms (or Lucas trees) in the economy each produce the single perishable good as output or dividend. Unlike CIR's (1985a, b) production economy, each firm n's production decision is exogenous, with its dividend flow D_n being a twice-continuously differentiable function of X, Z, and time t. Thus, a firm's dividend growth can depend on the aggregate state of the economy as well as the state of individual firms. By Ito's lemma, $D = (D_1, \ldots, D_N)'$ follows a vector Ito process:

$$dD(t) = \mu_D(t,D,X,Z)\,dt + \sigma_D(t,D,X,Z)\,d\omega(t), \quad t \geq 0\,, \tag{3}$$

[6] In this paper, all random variables are defined on a given complete probability space $(\Omega, \mathscr{F}, \Pr)$. A stochastic process c is then a collection of random variables $\{c(t) \colon t \geq 0\}$ on $(\Omega, \mathscr{F}, \Pr)$. Throughout the paper we use the standard filtration $\{\mathscr{F}_t \colon t \geq 0\}$ generated by the vector Brownian motion ω. Conditional expectations, $E_t(\cdot)$, are defined according to this filtration. Every time-t variable, such as $c(t)$ and $q(t)$, is taken to be $\mathscr{F}_t$-measurable. For simplicity, we suppress the qualifier 'almost surely' when taking equality between two random variables.

with $D(0) > 0$, where $\mu_D(t,D,X,Z) = (\mu_{D_1}(t),\ldots,\mu_{D_N}(t))'$ is the vector of expected time-t instantaneous dividend changes for the N firms, and $\sigma_D(t,D,X,Z)$ is the $N \times (M+N)$ matrix of local covariances between changes in $D(t)$ and changes in $\omega(t)$. The aggregate output, $q = \sum_{n=1}^{N} D_n$, must also follow an Ito process:

$$\mathrm{d}q(t) = \mu_q(t,q,X)\,\mathrm{d}t + \sigma_q(t,q,X)\,\mathrm{d}\omega_X(t), \quad t \geqslant 0, \tag{4}$$

with $q(0) > 0$, where $\sigma_q(t,q,X)$ is an $1 \times N$ vector of local covariances between $\mathrm{d}q(t)$ and $\mathrm{d}\omega_X(t)$. In other words, the aggregate output is assumed to depend only on systematic factors X. There are $N+1$ equations from (3) and (4) that form a joint system. Given that the firms' output processes are exogenous to our model, we have essentially N degrees of freedom in specifying the processes for q and D and the remaining process has to be such that this system of $N+1$ stochastic differential equations is satisfied. For modeling convenience, we will typically specify the process q from the outset.

Each firm n has one equity share issued and continuously traded, the holder of which is entitled to the full dividend flow $\{D_n(t)\colon t \geqslant 0\}$. Let $S_n(t)$ be the time-t ex-dividend price of the nth stock. For now, assume that the stock price processes to be endogenously determined, $S = (S_1,\ldots,S_N)'$, together form a vector Ito process. In addition, the economy has one (real) instantaneous risk-free bond and M zero-net-supply financial claims traded at each time t, so that the market is dynamically complete. Let $F_m(t)$ be the time-t price of the mth financial claim and $R(t)$ the time-t instantaneous interest rate. To allow for such financial claims as futures contracts, let $D_m(t)$ be the time-t payout flow for the mth claim. For a claim yielding no payout prior to maturity, its $D_m(t)$ is clearly zero for each t.

As in CIR (1985a) and Lucas (1978), assume that there is a representative agent who is an expected utility maximizer with preferences given below:

$$u(c) \equiv E_0\left\{\int_0^{\infty} \mathrm{e}^{-\rho t} U(c(t))\,\mathrm{d}t\right\}, \tag{5}$$

where ρ is the time preference parameter, $c(t)$ is the amount of time-t consumption, and the period utility $U\colon \Re \to \Re$ is twice continuously differentiable such that $U_c > 0$ and $U_{cc} < 0$, with the subscripts on U denoting partial derivatives.[7] The representative agent is initially endowed with the equity share of each firm.

We follow Lucas (1978) in first treating the asset price processes S_n and F_m as given and examining what the agent's optimal consumption-portfolio policy must satisfy, and then imposing an exogenous optimal consumption-portfolio policy to derive the asset prices endogenously. Let $a_n(t)$ be the number of shares held of

[7]For our general discussion, the period utility function can be state-dependent as in CIR (1985a) and all of our characterizations in this section will not be affected. It may even be allowed to exhibit other forms of state- or path-dependence such as habit formation as in Constantinides (1990). In those latter cases, however, some of the characterizations may have to be modified.

firm n and $b_m(t)$ the number of units held of the mth financial claim at time t. The agent's problem is to solve

$$\max_{a,b,c} u(c), \tag{6}$$

subject to the budget constraint

$$\begin{aligned} \mathrm{d}W(t) = {} & \sum_{n=1}^{N} a_n(t)[\mathrm{d}S_n(t) + D_n(t)\,\mathrm{d}t] + \sum_{m=1}^{M} b_m(t)[\mathrm{d}F_m(t) + D_m(t)\,\mathrm{d}t] \\ & - c(t)\mathrm{d}t + \left[W(t) - \sum_{n=1}^{N} a_n(t)S_n(t) - \sum_{m=1}^{M} b_m(t)F_m(t)\right] R(t)\,\mathrm{d}t\,, \end{aligned} \tag{7}$$

with $W(0) = \sum_{n=1}^{N}[S_n(0) + D_n(0)] > 0$, where $W(t)$ is the time-t wealth (in terms of units of the consumption good) generated by the plan (a,b,c).

Take any claim whose value depends on the aggregate output, firms' dividends and stock price levels, the state of the economy, and the state of each firm. That is, we can write $F(t,q,D,S,X,Z)$ as the time-t price of such a claim. Let $Q(t)$ be the claim's time-t payout flow that can also be a function of q, D, S, X, and Z. Assume that F is at least twice-continuously differentiable in every argument. For now, further assume that the optimal consumption process c from (6) is an Ito process. Then, from the existing literature on consumption-based asset pricing, the first-order conditions for the problem in (6) lead to the following restrictions on the optimal policy (a,b,c):

$$R(t) = \rho + \left(-\frac{c\,U_{cc}}{U_c}\right)\mu_c(t) - \frac{1}{2}\frac{c^2 U_{ccc}}{U_c}\sigma_c^2(t)\,, \tag{8}$$

(see Eq. (21) of Breeden, 1986), where $\mu_c(t)$ and $\sigma_c(t)$ are the conditional expected value and standard deviation of instantaneous consumption growth, respectively, and

$$\left[\mu_F(t) + \frac{Q(t)}{F(t)}\right] - R(t) = \left(-\frac{c\,U_{cc}}{U_c}\right)\mathrm{Cov}_t\left(\frac{\mathrm{d}F(t)}{F(t)}, \frac{\mathrm{d}c(t)}{c(t)}\right)\,, \tag{9}$$

(see, e.g., Eq. (17) of Breeden, 1979; or Eq. (30) of CIR, 1985a), where $\mu_F(t)$ is the conditional expected rate of price change of the claim and $\mathrm{Cov}_t(\cdot,\cdot)$ is the conditional covariance operator divided by dt. That is, given the collection of processes for $R(t)$, $S_n(t)$, and $F_m(t)$, the optimal policy must satisfy the above equations, including a version of (9) with F and Q, respectively, replaced by S_n and D_n. These two equations represent the core of the consumption-based asset pricing theory.

In the Lucas-type economy, the optimal policy is given outside of the consumption-portfolio optimization problem (6). That is, with the aggregate output being perishable and exogenous, the rational agent must in equilibrium consume the entire output at each time point: $c(t) = q(t)$. This consumption plan is financed by the equilibrium portfolio policy in which $a_n(t) = 1$ for $n = 1, \ldots, N$ and $b_m(t) = 0$ for $m = 1, \ldots, M$. This imposed consumption-portfolio policy should not only be optimal for the agent but also make the goods and the security markets clear, provided that the security prices are supportive of such an exogenous plan. In other words, if we impose this plan on the optimization problem (6), we then interpret Eqs. (8)–(9) as restrictions on interest rate $R(t)$ and asset prices $S_n(t)$ and $F_m(t)$. Relying on these restrictions, we arrive at the alternative fundamental valuation equation.

Theorem 1. Let $H = (q, D', S', X', Z')'$ be the vector of the $(1 + M + 3N)$ determining variables of claim price F. Let F_H be the column vector of partial derivatives of $F(t)$ with respect to $H(t)$, and F_{HH} be the matrix of second-order derivatives of $F(t)$ with respect to $H(t)$. Then, in light of Ito's multiplication rule, the equilibrium price for any claim, $F(t)$, is a solution to the following PDE (with the time subscript dropped for convenience):

$$[F_t + Q - RF]\,\mathrm{d}t + F_H'\mu_H\,\mathrm{d}t + \tfrac{1}{2}\,\mathrm{d}H'F_{HH}\,\mathrm{d}H = \left(-\frac{q\,U_{qq}}{U_q}\right)F_H'\,\mathrm{d}H\,\frac{\mathrm{d}q}{q}\,, \tag{10}$$

subject to the relevant boundary conditions for the contingent claim as dictated by the terms of the contract, where μ_H is the (vector of) expected rates of change in $H(t)$ and

$$R(t) = \rho + \left(-\frac{q(t)U_{qq}}{U_q}\right)\frac{\mu_q(t, q, D, X)}{q(t)} - \frac{1}{2}\frac{q^2(t)U_{qqq}}{U_q}\left(\frac{\sigma_q(t, q, D, X)}{q(t)}\right)^2 . \tag{11}$$

The valuation PDE in (10) applies to any contingent claim satisfying the smoothness condition, whether interest rate or equity sensitive. The left-hand side of the equation is obtained via applying Ito's lemma to $F(t, q, D, S, X, Z)$, while the right-hand side determines the equilibrium risk compensation for the claim. As is, the valuation PDE in (10) is not yet complete for contingent claims (e.g., equity options) that depend on the vector $S(t)$, because the stock prices are to be endogenously determined as well. To make the valuation model complete, note from Grossman and Shiller (1982) and Lucas (1978) that the time-t stock price for the nth firm should be

$$S_n(t) = E_t \int_t^{\infty} \mathrm{e}^{-\rho(v-t)}\,\frac{U_q(q(v))}{U_q(q(t))}\,D_n(v)\,\mathrm{d}v\,, \tag{12}$$

which means that $S_n(t)$ can be a function of $q(t)$, $D(t)$, $X(t)$, $Z(t)$, and time t. Consequently, letting $\hat{H} = (q, D', X', Z')'$, $S_n(t)$ must solve the following version of the PDE (10):

$$[(S_n)_t + D_n - RS_n]\,\mathrm{d}t + (S_n)'_{\hat{H}}\,\mu_{\hat{H}}\,\mathrm{d}t + \tfrac{1}{2}\,\mathrm{d}\hat{H}'\,(S_n)_{\hat{H}\hat{H}}\,\mathrm{d}\hat{H} = \left(-\frac{q\,U_{qq}}{U_q}\right)(S_n)'_{\hat{H}}\,\mathrm{d}\hat{H}\,\frac{\mathrm{d}q}{q}\,, \tag{13}$$

subject to the boundary condition:

$$\lim_{T\to\infty} E_t\left\{\mathrm{e}^{-\rho(T-t)}\,\frac{U_q(q(T))}{U_q(q(t))}\,S_n(T)\right\} = 0\,, \tag{14}$$

where the notation convention from Theorem 1 is adopted, except that $(S_n)_t$, for instance, stands for the partial derivative of $S_n(t)$ with respect to t. This boundary condition is often referred to as the transversality condition, which helps ensure the existence of the integral in (12).

The endogenous nature of stock prices renders it redundant to express any contingent claim price $F(t)$ as a function of $S(t)$, when $S(t)$ is at the same time assumed to depend on $q(t)$, $D(t)$, $X(t)$, and $Z(t)$. We nonetheless choose to include $S(t)$ in $H(t)$ and hence in the PDE (10) out of the consideration that contingent claims such as equity options are sometimes direct functions of their underlying stock price.

The valuation equation in (10) is the Lucas-type exchange-economy counterpart to the fundamental valuation equation of CIR (1985a). Major differences should, however, be noted between the two valuation models. First, recall that CIR's valuation PDE is expressed in terms of both the value function and the endogenous wealth. In contrast, the PDE in (10) involves only the direct utility of consumption, $U(c)$.

More fundamentally, the difference between CIR's valuation model and ours lies in whether the optimal consumption-portfolio policy or the vector stock price process $S(t)$ is exogenously specified, while letting the other part be endogenously determined. In CIR's production-based economy, the stock price processes $S(t)$ of the firms are fixed from outside of the model and it is up to the agent to decide at what capacity he would operate the fixed production processes. That is, the agent's investment decisions have to be so as to make the exogenous stock price processes consistent with equilibrium. But to find the agent's optimal consumption–investment policy requires solving the Hamilton–Jacobi–Bellman equation which is a *nonlinear PDE*. As is known, nonlinear PDEs are difficult to solve either in closed form or by numerical methods, as there are no well-developed numerical methods for solving such PDEs. Moreover, there are no easily applied general conditions that guarantee the existence of a solution. Merton (1990, Ch. 6) provides more discussion on these points.

In contrast, the optimal consumption–investment policy is fixed from outside of the present Lucas-type exchange economy. Here, it is the stock prices $S(t)$ that have to adjust within the model, until they reach a level at which the agent finds it optimal to hold the whole supply of each firm's stock. In some sense, the need to solve the *linear PDE* in (13) is a substitute for the need to solve, in the case of CIR (1985a) and Merton (1973a), the nonlinear partial differential Bellman equation. But, this linear PDE of the parabolic type is far easier to solve, and even if closed-form solutions are not obtainable, there are well-developed numerical methods. Furthermore, there are known existence and uniqueness conditions for a solution to this type of linear PDEs. Therefore, at least for asset valuation purposes, our exchange-economy-based approach is much more tractable.

Since Breeden's (1979) consumption CAPM falls essentially in the same class of production-economy models as CIR (1985a) and Merton (1973a), it is just as difficult to apply as the other two. Furthermore, aggregate consumption is endogenously determined reflecting individual firms' production and investors' consumption decisions, eliminating the flexibility to assume a tractable aggregate consumption process from outside of the model. In other words, even if one can solve for the endogenous consumption, closed-form expressions for contingent claim prices may still be unobtainable unless the derived aggregate consumption happens to have the right structure. In contrast, because the aggregate output/consumption process in our exchange economy is exogenous, it offers researchers flexibility to choose the 'right' output process so that a closed-form solution to the PDEs (10)–(13) can be found.

3. Interest rates and interest rate derivatives in an economy with CRRA agents

In the remainder of the paper, we apply the alternative valuation model to price (i) bonds and bond options and (ii) stocks and stock options, under either power or exponential utility functions. We first work with an economy in which the representative agent has a general power utility function:

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \tag{15}$$

where $\gamma > 0$ is the coefficient of relative risk aversion and in which there are two independent systematic state variables, conveniently denoted by X and Y, each following a mean-reverting square-root process:

$$dX(t) = \kappa_x(\theta_x - X(t))\,dt + \sigma_x\sqrt{X(t)}\,d\omega_x(t), \tag{16}$$

$$dY(t) = \kappa_y(\theta_y - Y(t))\,dt + \sigma_y\sqrt{Y(t)}\,d\omega_y(t), \tag{17}$$

where all the parameters are positive constants and (ω_x, ω_y) is a standard vector Brownian motion (corresponding to the vector process ω_X in the preceding section). The aggregate output is a function of only X and Y, with its dynamics given by

$$\frac{\mathrm{d}q(t)}{q(t)} = (\mu_q + \beta_x X(t) + \beta_y Y(t))\,\mathrm{d}t + \sigma_{q,x}\sqrt{X(t)}\,\mathrm{d}\omega_x(t) + \sigma_{q,y}\sqrt{Y(t)}\,\mathrm{d}\omega_y(t). \tag{18}$$

Clearly, (q, X, Y) are jointly Markov.

The output process in (18) closely resembles the one assumed in Longstaff and Schwartz (1992): expected output growth depends on $X(t)$ and $Y(t)$, except that here the conditional volatility of output growth is also linear in both state variables. In this economy with stochastically changing production and investment opportunities and power utility function, the difficulty to solve in closed form for the indirect utility of wealth or for the optimal consumption plan is well known (Merton, 1971). This is the reason that Longstaff and Schwartz (1992), among others, choose to rely on the log utility function, which is a special case of the power utility in (15).

Solving the stochastic differential equation in (18), we have the time-τ output given by

$$q(\tau) = q(0)\exp\left\{\int_0^\tau [\mu_q + (\beta_x - \tfrac{1}{2}\sigma_{q,x}^2)X(t) + (\beta_y - \tfrac{1}{2}\sigma_{q,y}^2)Y(t)]\,\mathrm{d}t + \sigma_{q,x}\int_0^\tau \sqrt{X(t)}\,\mathrm{d}\omega_x(t) + \sigma_{q,y}\int_0^\tau \sqrt{Y(t)}\,\mathrm{d}\omega_y(t)\right\}, \tag{19}$$

with $q(0) > 0$. In this economy, the equilibrium risk premium for any contingent claim F is

$$\left[\mu_F(t) + \frac{Q(t)}{F(t)}\right] - R(t) = \gamma\sigma_{q,x}\sqrt{X(t)}\mathrm{Cov}_t\left(\frac{\mathrm{d}F}{F}, \mathrm{d}\omega_x\right) + \gamma\sigma_{q,y}\sqrt{Y(t)}\mathrm{Cov}_t\left(\frac{\mathrm{d}F}{F}, \mathrm{d}\omega_y\right), \tag{20}$$

which is obtained by substituting the utility function in (15) and the processes in (16)–(18) into Eq. (9).

3.1. Bond valuation and the yield curve

Before introducing the processes for individual firms, let us first examine a pure discount bond that pays one unit of consumption in τ periods and whose time-t price is $B(t, \tau)$. Following a variational argument and using the utility function in (15), we have $B(t, \tau)$ given by

$$B(t, \tau) = \mathrm{e}^{-\rho\tau} E_t\left(\frac{q(t+\tau)}{q(t)}\right)^{-\gamma}. \tag{21}$$

Substituting (19) into this equation and realizing that (ω_x, ω_y) is a standard Brownian motion, we conclude that the above conditional expectation can only be a function of $X(t)$ and $Y(t)$, so that we can write the bond price as $B(t, \tau; X, Y)$. Substituting $B(t, \tau; X, Y)$ into the fundamental valuation equation in (10) and specializing it to the present context produces the following PDE for the discount bond:

$$\frac{1}{2}\sigma_x^2 X \frac{\partial^2 B}{\partial X^2} + [\kappa_x \theta_x - (\kappa_x + \lambda_x)X] \frac{\partial B}{\partial X} + \frac{1}{2}\sigma_y^2 Y \frac{\partial^2 B}{\partial Y^2}$$
$$+[\kappa_y \theta_y - (\kappa_y + \lambda_y) Y] \frac{\partial B}{\partial Y} - \frac{\partial B}{\partial \tau} - RB = 0, \qquad (22)$$

subject to the following boundary condition: $B(t + \tau, 0; X, Y) = 1$, where $\lambda_x \equiv \gamma \sigma_x \sigma_{q,x}$, $\lambda_y \equiv \gamma \sigma_y \sigma_{q,y}$. Using a standard separation-of-variable technique, the solution for the bond price is

$$B(t, \tau; X, Y) = \exp[-(\rho + \gamma \mu_q)\tau - \alpha_x(\tau) - \alpha_y(\tau) - \varrho_x(\tau) X(t) - \varrho_y(\tau) Y(t)], \qquad (23)$$

where

$$\alpha_x(\tau) = \frac{2\kappa_x \theta_x}{\sigma_x^2} \left\{ \ln\left[1 + \frac{(1 - e^{-\vartheta_x \tau})(\kappa_x + \lambda_x - \vartheta_x)}{2\vartheta_x}\right] + \tfrac{1}{2}[\vartheta_x - \kappa_x - \lambda_x]\tau \right\},$$

$$\alpha_y(\tau) = \frac{2\kappa_y \theta_y}{\sigma_y^2} \left\{ \ln\left[1 + \frac{(1 - e^{-\vartheta_y \tau})(\kappa_y + \lambda_y - \vartheta_y)}{2\vartheta_y}\right] + \tfrac{1}{2}[\vartheta_y - \kappa_y - \lambda_y]\tau \right\},$$

$$\varrho_x(\tau) = \frac{2\gamma\{\beta_x - \frac{1}{2}(1 + \gamma)\sigma_{q,x}^2\}(1 - e^{-\vartheta_x \tau})}{2\vartheta_x + (\kappa_x + \lambda_x - \vartheta_x)(1 - e^{-\vartheta_x \tau})}, \qquad (24)$$

$$\varrho_y(\tau) = \frac{2\gamma\{\beta_y - \frac{1}{2}(1 + \gamma)\sigma_{q,y}^2\}(1 - e^{-\vartheta_y \tau})}{2\vartheta_y + (\kappa_y + \lambda_y - \vartheta_y)(1 - e^{-\vartheta_y \tau})}, \qquad (25)$$

with $\vartheta_x \equiv [(\kappa_x + \lambda_x)^2 + \gamma\sigma_x^2\{2\beta_x - (1 + \gamma)\sigma_{q,x}^2\}]^{1/2}$ and $\vartheta_y \equiv [(\kappa_y + \lambda_y)^2 + \gamma\sigma_y^2 \{2\beta_y - (1 + \gamma)\sigma_{q,y}^2\}]^{1/2}$. Intuitively, λ_x and λ_y stand for the factor risk premiums, respectively, for X and Y. The two functions, $\varrho_x(\tau)$ and $\varrho_y(\tau)$, can be interpreted as the τ-period bond's sensitivity to, respectively, X risk and Y risk. As can be checked, both $\varrho_x(\tau)$ and $\varrho_y(\tau)$ are increasing in τ, which means that longer-term bonds tend to be more sensitive to both X and Y risk. By Ito's lemma, the bond price dynamics are described by

$$\frac{dB(t, \tau)}{B(t, \tau)} = \{R(t) - \lambda_x \varrho_x(\tau) X(t) - \lambda_y \varrho_y(\tau) Y(t)\}dt - \sigma_x \varrho_x(\tau)\sqrt{X(t)}\, d\omega_x(t)$$
$$-\sigma_y \varrho_y(\tau)\sqrt{Y(t)}\, d\omega_y(t). \qquad (26)$$

The implied τ-period yield to maturity is

$$R(t,\tau) \equiv -\frac{\ln[B(t,\tau)]}{\tau}$$

$$= \rho + \gamma\mu_q + \frac{\alpha_x(\tau)+\alpha_y(\tau)}{\tau} + \frac{\varrho_x(\tau)}{\tau}X(t) + \frac{\varrho_y(\tau)}{\tau}Y(t) \qquad (27)$$

which represents a two-factor term structure of interest rates. The instantaneous interest rate is given by

$$R(t) = \rho + \gamma[\mu_q + \beta_x X(t) + \beta_y Y(t)] - \frac{\gamma(1+\gamma)}{2}[\sigma_{q,x}^2 X(t) + \sigma_{q,y}^2 Y(t)]\,. \qquad (28)$$

This two-factor term structure of interest rates specializes to the one given in Longstaff and Schwartz (1992) if we take $\gamma \to 1$ and $\sigma_{q,x} = 0$. Other single- and multi-factor term structure models, such as Bakshi and Chen (1996a), CIR (1985b), Duffie and Kan (1996), and Sun (1992), are nested within our model as well. Our model thus possesses most of the properties shared by existing two-factor term structure models in that it affords a richer set of term structure shapes, interest rates are generally not perfectly correlated, and the yield curve will generally not shift in parallel over time. However, since the utility function here is more general than assumed in most existing models and since the output volatility is driven by two state variables, rather than by one as in Longstaff and Schwartz (1992), our model has its own unique appealing features. First, the instantaneous interest rate is a function of the risk aversion level and the current state of the economy. Depending on the magnitude of the structural parameters and the state of the economy, interest rates can be both higher and lower as the coefficient γ increases because the reciprocal of γ is also the intertemporal elasticity coefficient.[8] As γ increases, the intertemporal elasticity coefficient decreases, which means the equilibrium interest rate has to increase in order to induce the agent to substitute future consumption for current consumption. This effect of an increase in γ is reflected by the second term on the right-hand side of Eq. (28). On the other hand, an increase in γ also means a higher level of risk aversion on the part of the agent, which tends to depress the interest rate. This negative effect is captured by the last term in Eq. (28). For this reason, the overall effect

[8]The fact that a dual role is played by γ is an undesirable aspect of the standard expected utility framework. Because of this, a change in γ can be due to either a change in risk-taking attitude or a change in intertemporal consumption substitutability. With this understanding in mind, we nonetheless proceed to interpret a change in γ as reflecting a change in risk-taking attitude, largely in order to streamline the discussion.

of a higher γ on $R(t)$ is mixed.[9] Compared to the log-utility economies in CIR (1985b) and Longstaff and Schwartz (1992), therefore, economies with $\gamma \neq 1$ may have higher or lower interest rates.

Second, the term premium in our model possesses more realistic properties. As noted by Constantinides (1992), the term premium in CIR-type single-factor models is either monotonically increasing or monotonically decreasing in the term to maturity, which is counterfactual. Like the term structure model of Constantinides (1992), ours allows the term premium to have many types of shape, including the common monotonically increasing, decreasing, humped, or inversely humped term premium shapes. More specifically, the term premium is

$$\begin{aligned} TP(t,\tau) &\equiv \frac{1}{\mathrm{d}t} E_t \left[\frac{\mathrm{d}B(t,\tau)}{B(t,\tau)} \right] - R(t) \\ &= -\gamma\sigma_x\sigma_{q,x}\,\varrho_x(\tau)X(t) - \gamma\sigma_y\sigma_{q,y}\,\varrho_y(\tau)Y(t)\,. \end{aligned} \tag{29}$$

To see what happens within our model, divide the discussion into four cases. In the first case, $\sigma_x\,\sigma_{q,x} > 0$ and $\sigma_y\,\sigma_{q,y} > 0$. In this case, innovations in output growth are positively correlated with those in the state variables. Thus, both X and Y risks represent 'positive' systematic risks that require a positive premium (i.e., both λ_x and λ_y must be positive). But since bond prices in this case are negatively related to both X and Y (see Eq. (26)), bonds in effect act as insurance against unfavorable movements in X and Y and hence in output q. This explains why the term premium is negative in this case, regardless of the term to maturity τ. Furthermore, as $\varrho_x(\tau)$ and $\varrho_y(\tau)$ are increasing in τ, longer-term bonds provide better insurance against unfavorable movements in q and are hence better hedging instruments, which implies that the term premium will be monotonically decreasing in τ. Now consider the second case in which $\sigma_x\,\sigma_{q,x} < 0$ but $\sigma_y\,\sigma_{q,y} > 0$. In this case, X represents 'negative' systematic risk. Since bond prices are negatively related to X, the component of risk in any bond due to X requires positive risk compensation, which is why the first term on the right-hand side of Eq. (29) is positive under this scenario. In addition, the first term is increasing in τ. On the other hand, for the same reason given above, the second term in Eq. (29) is negative and decreasing in τ. As a result of these two opposite effects, the shape of the term premium in relation to τ can be nonmonotonic, depending on the structural parameters and the state of the economy. In the third case, $\sigma_x\,\sigma_{q,x} < 0$ and $\sigma_y\,\sigma_{q,y} < 0$, which is just the opposite of the first case. The term premium will then be monotonically increasing in τ. In the last case, $\sigma_x\,\sigma_{q,x} > 0$ and $\sigma_y\,\sigma_{q,y} < 0$. This is similar to the second case, and nonmonotonic term premium shapes can arise. In addition, depending on the signs of the

[9] A similar point is made by Wang (1996) in a different context. Wang examines the impact of investor heterogeneity on the term structure of interest rates. For the case of investors with a power utility function, he also derives a closed-form term structure.

structural parameters and the state of the economy, an increase in risk aversion can mean a higher or lower term premium for any given τ.

Observe that in the case of Longstaff and Schwartz (1992), $\sigma_{q,x} = 0$, which means the first term on the right-hand side of Eq. (29) is zero. Since $\varrho_y(\tau)$ is monotone in τ, their model only permits either monotonically increasing or monotonically decreasing term premium shapes.

3.2. Bond option valuation

Consider a European call option that matures in τ periods and is written on a $\tilde{\tau}$-period pure discount bond, $B(t,\tilde{\tau})$, where $\tilde{\tau} > \tau$. Let $G(t,\tau)$ be the time-t price of the call. Using a standard argument, we have

$$G(t,\tau) = \mathrm{e}^{-\rho\tau} E_t\left[\left(\frac{q(t+\tau)}{q(t)}\right)^{-\gamma} \max\{0, B(t+\tau,\tilde{\tau}-\tau) - K\}\right], \tag{30}$$

where K is the exercise price. By the output process in (19) and the bond price dynamics in (26), the conditional expectation in (30) can only be a function of $X(t)$ and $Y(t)$, which allows us to write $G(t,\tau;X,Y)$. This means that the PDE in (22) must also apply to $G(t,\tau;X,Y)$, except that the boundary condition becomes: $G(t+\tau,0;X,Y) = \max\{0, B(t+\tau,\tilde{\tau}-\tau) - K\}$. Solving the PDE, we obtain the bond option price:

$$G(t,\tau;X,Y) = B(t,\tilde{\tau})\,P_1(t,\tau;X,Y) - K\,B(t,\tau)\,P_2(t,\tau;X,Y), \tag{31}$$

with the two probabilities, P_1 and P_2, given by

$$P_j(t,\tau;X,Y) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi\tilde{K}}\,\hat{f}_j(t,\tau,X,Y;\phi)}{\mathrm{i}\phi}\right]\mathrm{d}\phi \quad \text{for } j = 1,2, \tag{32}$$

where Re[·] stands for the real value part of the expression, i stands for imaginary numbers, $\tilde{K} \equiv \ln[K] + (\rho + \gamma\mu_q)(\tilde{\tau}-\tau) + \alpha_x(\tilde{\tau}-\tau) + \alpha_y(\tilde{\tau}-\tau)$, and the characteristic functions $\hat{f}_j$, $j = 1,2$, are provided in Eqs. (A.4) and (A.5), respectively, in the Appendix.

The bond option formula in (31) shares the same functional form with many of the known bond option formulas: the bond option price is determined by the price of a discount bond multiplied by the probability function minus the present value of the payoff from optimal exercise at expiration. Among the models, however, the probability functions can be quite different. For example, in Constantinides (1992), Jamshidian (1989), and Turnbull and Milne (1991), each probability is determined by a cumulative normal distribution function. In the single-factor model of CIR (1985b) and the two-factor generalizations in Chen and Scott (1992) and Longstaff and Schwartz (1992), the probabilities that the option expires in the money are determined by a bivariate noncentral chi-square distribution function. Compared to these existing models, our model in (31) has a few distinct features. First, the probabilities in (31) are easier to estimate than their counterpart in Chen and Scott (1992) and Longstaff and Schwartz (1992). Recall that this part of our setup is similar to theirs in that both interest rates and interest rate volatility are stochastic. In our case, however, the calculation of P_1 and P_2 involves taking the single integral over the characteristic function [see Eq. (32)], whereas in their cases computing the two probabilities involves evaluating the double integral over a bivariate noncentral chi-square distribution function. As Heston (1993) points out, integrating over the characteristic function numerically can be conducted quite efficiently because the characteristic function declines rapidly in ϕ.

Second, the bond option formula in (31) is derived under a general power utility. It applies to any economy with the assumed stochastic environment and with constant relative risk aversion. It allows one to examine how bond option prices respond to a change in risk-taking attitudes, whereas such an exercise is not possible using existing models because of the log utility assumed.

Finally, both the instantaneous interest rate and its volatility are linear in the unobservable $X(t)$ and $Y(t)$. As in Longstaff and Schwartz (1992), $R(t)$ and its volatility, or any two yields $R(t,\tau_1)$ and $R(t,\tau_2)$, can be used to substitute out $X(t)$ and $Y(t)$ in the bond and bond option formulas, so that one can conveniently implement the bond and bond option model by relying only on observable variables. For further details on this point, see, among others, Bakshi and Chen (1996b), CIR (1985b), and Longstaff and Schwartz (1992). Thus far, we have focused on pure discount bond call options. Along the lines of discussion in Longstaff (1990), Turnbull and Milne (1991), and Chen and Scott (1992, 1995), our approach can also be applied to value other interest rate derivatives (e.g., options on coupon bonds, compound options, and interest rate caps) and relate their prices to the level of risk aversion in the economy. Details are omitted here.

4. Valuation of stocks and stock options

We now turn to valuing individual stocks and European options written on these securities. To keep the results manageable, in this section we specialize the two-systematic-factor economy of the preceding section to one with a single systematic factor X. That is, set $\beta_y = \sigma_{q,y} = 0$ in Eq. (18). As a consequence, the term structure of interest rates has a single factor and the instantaneous interest rate in (28) changes to

$$R(t) = \rho + \gamma\mu_q + \gamma\left[\beta_x - \frac{(1+\gamma)\sigma_{q,x}^2}{2}\right]X(t)\,. \tag{33}$$

This change in the systematic factor structure does not affect the generality of the results to follow and it is only to make the presentation more focused.[10]

4.1. Individual stocks

Each firm's own state variable is an autonomous diffusion process $Z_n(t)$, for $n = 1,\dots,N$, where

$$\mathrm{d}Z_n(t) = \kappa_{z_n}(\theta_{z_n} - Z_n(t))\,\mathrm{d}t + \sigma_{z_n}\sqrt{Z_n(t)}\,\mathrm{d}\omega_{z_n}(t)\,. \tag{34}$$

Here, as a slight departure from Section 1, the N firms' sources of uncertainty represented by $(\omega_{z_1},\dots,\omega_{z_N})$ can be correlated with each other (which makes the Z_n's possibly correlated with each other), but each Z_n is by itself a Markov process. Assume that each firm's dividend flow $D_n(t)$ is a function of $X(t)$ and $Z_n(t)$, implying that it depends on both systematic and firm-specific risk variables. From Eq. (12) and the utility function in (15), the time-t stock price of firm n is

$$S_n(t) = E_t \int_t^{\infty} \mathrm{e}^{-\rho(v-t)} \left(\frac{q(v)}{q(t)}\right)^{-\gamma} D_n(v,X,Z_n)\,\mathrm{d}v\,, \tag{35}$$

By Eq. (19) and the fact that X and Z_n each are autonomous processes, this conditional expectation can only be a function of $X(t)$ and $Z_n(t)$, so that we can write the stock price as $S_n(t,X,Z_n)$. By the utility function in (15) and the fundamental valuation PDE in (10), $S_n(t)$ must solve

$$\begin{aligned}&\frac{1}{2}\,\sigma_x^2 X\,\frac{\partial^2 S_n}{\partial X^2} + [\kappa_x\theta_x - (\kappa_x + \lambda_x)X]\,\frac{\partial S_n}{\partial X} + \frac{1}{2}\,\sigma_{z_n}^2\,Z_n\,\frac{\partial^2 S_n}{\partial Z_n^2}\\ &\quad + [\kappa_{z_n}\theta_{z_n} - \kappa_{z_n} Z_n]\frac{\partial S_n}{\partial Z_n} + D_n - R\,S_n = 0\,,\end{aligned} \tag{36}$$

subject to the transversality condition that

$$\lim_{T\to\infty} E_t\left\{\mathrm{e}^{-\rho(T-t)}\left(\frac{q(T)}{q(t)}\right)^{-\gamma} S_n(T,X,Z_n)\right\} = 0\,. \tag{37}$$

As existing work in a discrete-time context by Abel (1988), Bossaerts and Green (1989), Long and Plosser (1983), and Lucas (1978) and in a continuous-time context by Bakshi and Chen (1996a), Gennotte and Marsh (1993), and Goldstein

[10] To put the state variable Y back to the stock and the stock option price formulas, one only needs to add an analogous Y-related term to wherever an X-related term occurs. Each added Y-related term is identical to the corresponding X-related term, except that the variable is replaced by Y. Similarly, other independent systematic state variables can be added to the model.

and Zapatero (1996) has demonstrated,[11] finding closed-form solutions for stock prices is typically not possible for an arbitrary dividend process, especially when the utility function $U(c)$ is other than the log utility. For this reason, we need to assume a particular structure for firms' dividend policies.

Consider the case in which dividend flows for the first $N-1$ firms are of the following form:

$$D_n(t) = g_n(t)\exp[\Gamma_{n,x}X(t) + \Gamma_{n,z}Z_n(t)], \quad \forall t \geqslant 0\,, \tag{38}$$

for $n = 1,\ldots,N-1$, where

$$g_n(t) = \psi_n + \psi_{n,x}X(t) + \psi_{n,z}Z_n(t)\,, \tag{39}$$

with $\psi_n \geqslant 0$. This dividend structure is quite general and includes many known dividend policies as special cases, further discussion of which is given shortly. Since the aggregate output process $q(t)$ has already been specified, the dividend flow for the Nth firm must be $D_N(t) = q(t) - \sum_{n=1}^{N-1} D_n(t)$, to ensure internal consistency.

We start with the valuation of the first $N-1$ firms' stocks and leave the valuation of the Nth firm and the market portfolio to a later point. For convenience, take a generic firm out of the $N-1$ firms and *suppress the subscript n on each variable/parameter*, e.g., use $D(t)$, $S(t)$, $Z(t)$, Γ_x, Γ_z, $g(t)$, ψ, ψ_x, and ψ_z. Substituting the dividend function into the PDE in (36) and solving it, we arrive at the endogenous stock price:

$$S(t) = A\exp[\Gamma_x X(t) + \Gamma_z Z(t)]\,, \tag{40}$$

where

$$A = \frac{\psi}{\rho + \gamma\mu_q - \kappa_x\theta_x\Gamma_x - \kappa_z\theta_z\Gamma_z} \geqslant 0\,, \tag{41}$$

subject to the structural parameter restrictions that

$$\rho + \gamma\mu_q - \kappa_x\theta_x\Gamma_x - \kappa_z\theta_z\Gamma_z > 0 \tag{42}$$

$$\psi_x - \frac{\psi[\gamma\beta_x - \frac{1}{2}\gamma(1+\gamma)\sigma_{q,x}^2 + (\kappa_x + \lambda_x)\Gamma_x - \frac{1}{2}\sigma_x^2\Gamma_x^2]}{\rho + \gamma\mu_q - \kappa_x\theta_x\Gamma_x - \kappa_z\theta_z\Gamma_z} = 0 \tag{43}$$

$$\psi_z - \frac{\psi\Gamma_z(\kappa_z - \frac{1}{2}\sigma_z^2\Gamma_z)}{\rho + \gamma\mu_q - \kappa_x\theta_x\Gamma_x - \kappa_z\theta_z\Gamma_z} = 0\,. \tag{44}$$

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[11] All these authors have focused on deriving and studying a closed-form stock price solution for the market portfolio, but not for individual stocks. Typically, they (except Abel) specialize the utility function to the log case in order to find a closed-form solution for the market portfolio price. In the present paper, we solve for the stock prices of both individual stocks and the market portfolio, under either power or exponential utility functions.

The inequality in (42) is due to the transversality condition, while (43) and (44) guarantee that the dividend process and the resulting stock price are consistent with equilibrium. To see the necessity of restrictions (43) and (44), apply Ito's lemma to the stock price in (40):

$$\frac{dS(t)}{S(t)} = \mu_S(t)\,dt + \sigma_1\sqrt{X(t)}\,d\omega_x(t) + \sigma_2\sqrt{Z(t)}\,d\omega_z(t)\,, \tag{45}$$

where $\sigma_1 \equiv \Gamma_x\sigma_x$, $\sigma_2 \equiv \Gamma_z\sigma_z$, and

$$\mu_S(t) = \tfrac{1}{2}\Gamma_x^2\sigma_x^2\,X(t) + \kappa_x\,\Gamma_x[\theta_x - X(t)] + \tfrac{1}{2}\Gamma_z^2\sigma_z^2\,Z(t) + \kappa_z\,\Gamma_z[\theta_z - Z(t)]\,. \tag{46}$$

On the other hand, the equilibrium restriction in (20) implies that $\mu_S(t)$ must also satisfy

$$\begin{aligned}\mu_S(t) &= -\frac{D(t)}{S(t)} + R(t) + \gamma\Gamma_x\sigma_{q,x}\sigma_x X(t)\\ &= -\frac{\psi}{A} - \frac{\psi_x}{A}X(t) - \frac{\psi_z}{A}Z(t) + R(t) + \Gamma_x\lambda_x X(t).\end{aligned} \tag{47}$$

In equilibrium, the right-hand side of (46) must equal the right-hand side of (47). This imposes two restrictions on the structural parameter values, as given in (43) and (44). When the choice of the structural parameter values violates these restrictions, the stock price solution will no longer be consistent with the dividend policy.

Dividend yield for such firms is linear in the state variables, namely, $D(t)/S(t) = (1/A)\{\psi + \psi_x X(t) + \psi_z Z(t)\}$. This class of dividend policies includes many known cases. The following are some examples:

1. Set $\psi_x = \psi_z = 0$ and choose Γ_x and Γ_z according to (43) and (44), so that $\psi \neq 0$. This corresponds to a constant proportional dividend policy with a constant dividend yield;
2. Set $\psi = \psi_x = \psi_z = 0$, which means the firm never pays any dividend, now or at any time in the future. Then, $A = 0$ and the stock price is zero at each time t;
3. Set $\Gamma_x = \Gamma_z = 0$, in which case the stock price is a constant (i.e., A) and the dividend yield is the same as the instantaneous interest rate $R(t)$;
4. Set $\Gamma_x = 0$, but $\Gamma_z \neq 0$. In this case, the firm's stock price is stochastic but has no exposure to systematic risk. It is exposed only to the firm-specific risk Z. Consequently, the expected instantaneous rate of total gains from holding the stock must be the same as the interest rate $R(t)$. The dividend yield is equal to $g(t)/A = R(t) + \Gamma_z[\kappa_z - \kappa_z\theta_z - \frac{1}{2}\sigma_z^2\Gamma_z]\,Z(t)$. That is, the firm has its dividend yield indexed to the short-term interest rate (plus noise).

Among the comparative statics, $S(t)$ is decreasing in relative risk aversion γ, time preference parameter ρ, and time-invariant expected output growth μ_q, but increasing in Γ_x and Γ_z. Note that there is a long-run steady-state distribution for both $X(t)$ and $Z(t)$, with their respective long-run means given by θ_x and θ_z. As the expected appreciation rate of stock price, $\mu_S(t)$, is a linear combination of the two independent gamma variates $X(t)$ and $Z(t)$, it must also have a steady-state distribution, with its long-run mean equal to $\frac{1}{2}\Gamma_x^2\sigma_x^2\,\theta_x + \frac{1}{2}\Gamma_z^2\sigma_z^2\,\theta_z$.

Other distinct features of the equilibrium stock price can also be noted. First, its instantaneous return volatility (i.e., variance) is, by Eq. (45),

$$V(t) \equiv \sigma_1^2 X(t) + \sigma_2^2 Z(t)\,, \tag{48}$$

which is linear in both the systematic and the idiosyncratic state variables. Here, the systematic volatility component is determined by the coefficient σ_1: the higher the coefficient, the more systematic risk the stock has and the more highly correlated its volatility is with the market-wide volatility factor. For instance, when $\sigma_1 = 0$, the stock has no systematic risk and only idiosyncratic risk; when $\sigma_2 = 0$, the stock has only systematic risk. This derived feature for stock price volatility is plausible given the considerable and growing empirical evidence that the volatility of an individual stock is not only stochastic over time but also highly correlated with the overall market volatility. For instance, Bates (1995) and Wiggins (1987) demonstrate that the volatilities of stocks are positively correlated with each other and highly correlated with market volatility. In a recent paper on option valuation, Amin and Ng (1993) also have the return volatility of an individual stock comprising a systematic and an unsystematic component, although in their case such a volatility structure is an exogenously assumed feature, rather than an endogenous feature as in our model.

Over time, the stock volatility follows an Ito process:

$$\begin{aligned} \mathrm{d}V(t) = {} & \{\sigma_1^2\kappa_x[\theta_x - X(t)] + \sigma_2^2\kappa_z[\theta_z - Z(t)]\}\,\mathrm{d}t + \sigma_1^2\sigma_x\sqrt{X(t)}\,\mathrm{d}\omega_x(t) \\ & +\sigma_2^2\sigma_z\sqrt{Z(t)}\,\mathrm{d}\omega_z(t)\,, \end{aligned} \tag{49}$$

which is obtained by applying Ito's lemma to (48). Like the expected appreciation rate $\mu_S(t)$, $V(t)$ has a steady-state distribution, with mean $\sigma_1^2\theta_x + \sigma_2^2\theta_z$. This volatility process is different from those used in Heston (1993), Hull and White (1987), Stein and Stein (1991), and Wiggins (1987), in which the stochastic volatility is typically driven by a single systematic risk source. Another difference between these existing models and ours lies in the correlation structure between stochastic volatility and interest rates. For example, the instantaneous interest rate and the underlying asset volatility are assumed to be perfectly correlated in Heston (1993), whereas in our model these two entities are generally not perfectly correlated, as can be seen by comparing Eq. (48) with Eq. (33). In practice, it is unlikely for a stock's volatility to be perfectly correlated with interest rates.

Finally, the conditional covariance (per unit time) between volatility change and stock return is stochastic over time:

$$\mathrm{Cov}_t\left(\frac{\mathrm{d}S(t)}{S(t)}, \mathrm{d}V(t)\right) = \Gamma_x^3 \sigma_x^4 X(t) + \Gamma_z^3 \sigma_z^4 Z(t), \tag{50}$$

which can clearly be positive or negative, depending on the sign and the magnitude of Γ_x and Γ_z. Bakshi, Cao, and Chen (1997) and Longstaff (1994) argue that time-varying correlations between stock return and its volatility are important for avoiding the skewness-related biases (e.g., 'volatility smile') found in existing option pricing models. In addition, the volatility process in (49) may also help explain the empirical findings of excess conditional kurtosis in stock returns. Larger (absolute) values for Γ_x and Γ_z can, for instance, make the stock volatility process more volatile, which can in turn bring the distributional properties of *modeled* stock returns in line with those implied by *actual* option prices. Overall, our endogenous stock price and its return volatility have a rich enough structure that we expect the resulting stock option pricing model, which is derived next, to possess many empirically appealing properties as well.

4.2. Options on dividend-paying stocks

Consider a European call option written on a stock that pays a continuous stochastic dividend according to (38), with strike price K and τ periods to expiration from time t. The dollar payoff to the holder at expiration is $\max\{0, S(t+\tau) - K\}$. Its time-t price, $C(t,\tau)$, is given by

$$C(t,\tau) = E_t\left[\mathrm{e}^{-\rho\tau}\left(\frac{q(t+\tau)}{q(t)}\right)^{-\gamma} \max\{0, S(t+\tau) - K\}\right]. \tag{51}$$

Given the stock price dynamics in (45) and the restriction that $\beta_y = 0$ and $\sigma_{q,y} = 0$, the conditional expectation in (51) can only be a function of $S(t)$, $X(t)$, and $Z(t)$, which allows us to write $C(t,\tau; S, X, Z)$. The fundamental valuation PDE in Theorem 1 can thus be specialized to

$$\frac{1}{2}(\sigma_1^2 X + \sigma_2^2 Z) S^2 \frac{\partial^2 C}{\partial S^2} + \left[R - \frac{D}{S}\right] S \frac{\partial C}{\partial S} + \sigma_1 \sigma_x X S \frac{\partial^2 C}{\partial S \partial X}$$
$$+ \sigma_2 \sigma_z Z S \frac{\partial^2 C}{\partial S \partial Z} + \frac{1}{2}\sigma_x^2 X \frac{\partial^2 C}{\partial X^2} + [\kappa_x \theta_x - (\kappa_x + \lambda_x) X] \frac{\partial C}{\partial X}$$
$$+ \frac{1}{2}\sigma_z^2 Z \frac{\partial^2 C}{\partial Z^2} + [\kappa_z \theta_z - \kappa_z Z] \frac{\partial C}{\partial Z} - \frac{\partial C}{\partial \tau} - RC = 0, \tag{52}$$

where the time argument is suppressed, subject to the boundary condition that $C(t+\tau,0;S,X,Z)=\max\{0,S(t+\tau)-K\}$. The solution to this PDE is

$$C(t,\tau;S,X,Z)=S(t)\exp[-\eta(t,\tau;X,Z)]\Pi_1(t,\tau;S,X,Z) \\ -KB(t,\tau)\Pi_2(t,\tau;S,X,Z), \qquad (53)$$

where $B(t,\tau)$ is the τ-period discount bond price, and the two risk-neutralized probabilities, Π_1 and Π_2, are determined by

$$\Pi_j(t,\tau;S,X,Z)=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi\ln[K]}f_j(t,\tau,S,X,Z;\phi)}{\mathrm{i}\phi}\right]\mathrm{d}\phi, \qquad (54)$$

for $j=1,2$. The characteristic functions, f_1 and f_2, are, respectively, presented in Eqs. (75) and (76) of the appendix, and the dividend adjustment function $\eta(t,\tau;X,Z)$ is

$$\begin{aligned}\eta(t,\tau;X,Z)&\\ =\frac{\kappa_x\theta_x}{\sigma_x^2}\Bigg(&(\delta_x-\kappa_x-\lambda_x+\Gamma_x\sigma_x^2)\tau\\ &+2\ln\left[1-\frac{(\delta_x-\kappa_x-\lambda_x+\Gamma_x\sigma_x^2)(1-\mathrm{e}^{-\delta_x\tau})}{2\delta_x}\right]\Bigg)\\ +\frac{\kappa_z\theta_z}{\sigma_z^2}\Bigg(&(\delta_z-\kappa_z+\Gamma_z\sigma_z^2)\tau\\ &+2\ln\left[1-\frac{(\delta_z-\kappa_z+\Gamma_z\sigma_z^2)(1-\mathrm{e}^{-\delta_z\tau})}{2\delta_z}\right]\Bigg)\\ &+\frac{2\psi_x(1-e^{-\delta_z\tau})A^{-1}}{2\delta_x-(\delta_x-\kappa_x-\lambda_x+\Gamma_x\sigma_x^2)(1-\mathrm{e}^{-\delta_x\tau})}X(t)\\ &+\frac{2\psi_z(1-e^{-\delta_z\tau})A^{-1}}{2\delta_z\delta_z-\kappa_z+\Gamma_z\sigma_z^2)(1-\mathrm{e}^{-\delta_z\tau})}Z(t)+\frac{\psi}{A}\tau, \end{aligned} \qquad (55)$$

where $\delta_x=[(\kappa_x+\lambda_x-\Gamma_x\sigma_x^2)^2+2\sigma_x^2\psi_x A^{-1}]^{1/2}$ and $\delta_z=[(\kappa_z-\Gamma_z\sigma_z^2)^2+2\sigma_z^2\psi_z A^{-1}]^{1/2}$. When $\psi_x=\psi_z=0$, for example, the dividend yield is a constant, $D(t)/S(t)=\psi/A$, and the dividend adjustment term in (53) becomes: $\exp[-\eta(t,\tau;X,Z)]=\exp[-\frac{\psi}{A}\tau]$, which is also how dividend adjustment is done in the constant-dividend-yield option pricing formula of Merton (1973b). The term in (53), $S(t)\exp[-\eta(t,\tau;X,Z)]$, stands for the forward price of a τ-period forward contract written on the stock, multiplied by the price of a τ-period discount bond.

The individual stock option formula in (53) has several distinct features:

- The underlying asset pays a continuous stochastic dividend yield, and the class of admissible dividend policies for the equity option pricing formula is quite large;
- Both interest rates and underlying asset volatility are stochastic, with the later consisting of a systematic and an idiosyncratic volatility component;
- All prices and rates (e.g., interest rates, stock price, and option price) are jointly determined in a general equilibrium;
- The pricing model is derived with the representative agent having a power utility function (or, exponential utility, as shown later), rather than the usual log utility; and
- It is effectively a two-factor option model, with the probabilities that the option will expire in the money depending on both market-wide and firm-specific state variables.

Thus, our model goes beyond existing option models and combines together most of their desirable elements. For example, the option pricing model of Merton (1973b) allows the underlying asset to pay a continuous dividend that is in constant proportion to the stock price, but it does not allow interest rates or stock volatility to change over time. The option models in Heston (1993), Hull and White (1987), Stein and Stein (1991), and Wiggins (1987) admit stochastic volatility, but they are derived assuming that the underlying asset does not pay any dividends and that interest rates stay constant over time. A third class of option pricing models, such as Amin and Jarrow (1992) and Goldstein and Zapatero (1996), is built under the assumption that interest rates are stochastic whereas the underlying asset has constant volatility. Finally, Scott (1996) has a closed-form pricing formula for stock options with stochastic volatility and stochastic interest rates. However, like other existing ones, his model is of partial equilibrium in nature, with interest rates and stock prices specified outside of the model, and is concerned with the pricing of an option written on a stock index with only systematic risk and without dividend payments.

Our valuation model for individual stock options differs from Amin and Ng's (1993) partial equilibrium model in a crucial way. Even though in their discrete-time framework they also consider a general power utility function and a similar volatility structure for individual stocks, their option valuation formula is given in terms of an expected Black–Scholes formula with the conditional expectation taken with respect to the variance and interest rate processes. Hence, their pricing formula is not given in closed form, and the option price can only be obtained via cumbersome numerical methods. Besides, the underlying stock in their model does not pay dividends and the stock price is exogenously specified.

The aforementioned differences between existing option models and ours have many implications for understanding and correctly pricing equity options. For example, consider two extreme types of stock option, type-A options written on stocks or stock indexes with only systematic risk (i.e., $\Gamma_z = 0$ for these underlying assets) and type-B options written on stocks with only idiosyncratic risk (i.e., $\Gamma_x = 0$). For type-A options, the correct pricing formula is a version of Eq. (53) with only the systematic state variable $X(t)$ as the driving factor, whereas for type-B options the pricing formula has only the unsystematic state variable $Z(t)$ as the driving factor. Consequently, the two pricing formulas have completely different factor structures. If one fits the same option pricing formula to both types of option, large pricing errors can result. This may explain why, for instance, Whaley (1982) finds that the Black and Scholes model leads to option pricing biases that differ across stocks of distinct risk characteristics. Since the pricing models in Heston (1993), Hull and White (1987), Stein and Stein (1991), and Scott (1996) are more suitable for type-A options and since individual stocks are bound to have exposures to both systematic and idiosyncratic risk sources, one can expect these existing option models with stochastic volatility to still generate pricing biases that differ across stocks of distinct sizes or risk characteristics.

The availability of closed-form option prices makes it possible to derive comparative statics and hedge ratios analytically. For the option pricing formula in (53), we can note a few useful comparative statics. First, the call price is decreasing in the strike price: $\partial C(t,\tau)/\partial K = -B(t,\tau)\Pi_2 < 0$. Next, given the dependence of $C(t)$ on $S(t)$, $X(t)$, and $Z(t)$, there are now three different hedge ratios or delta measures. With respect to the stock price, the option delta is

$$\Delta_S \equiv \frac{\partial C(t,\tau)}{\partial S} = \mathrm{e}^{-\eta(t,\tau)}\Pi_1 \geqslant 0\,, \tag{56}$$

which is state-dependent and time-varying. (Recall that in the Black–Scholes case there is only one call option delta and that delta depends only on the underlying stock price.) With respect to $X(t)$ and $Z(t)$, the option deltas are, respectively,

$$\Delta_x \equiv \frac{\partial C(t,\tau)}{\partial X} = S(t)\mathrm{e}^{-\eta(t,\tau)}\left\{\frac{\partial \Pi_1}{\partial X} + \Gamma_x\Pi_1 - \Pi_1\frac{\partial \eta}{\partial X}\right\}$$

$$-KB(t,\tau)\left\{\frac{\partial \Pi_2}{\partial X} - \varrho_x(\tau)\Pi_2\right\}, \tag{57}$$

$$\Delta_z \equiv \frac{\partial C(t,\tau)}{\partial Z} = S(t)\,\mathrm{e}^{-\eta(t,\tau)}\left\{\frac{\partial \Pi_1}{\partial Z} + \Gamma_z\,\Pi_1 - \Pi_1\,\frac{\partial \eta}{\partial Z}\right\} - KB(t,\tau)\,\frac{\partial \Pi_2}{\partial Z}\,, \tag{58}$$

where, for $j = 1,2$ and for $h = X, Z$,

$$\frac{\partial \Pi_j}{\partial h} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[(\mathrm{i}\phi)^{-1}\mathrm{e}^{-\mathrm{i}\,\phi\ln[K]}\,\frac{\partial f_j}{\partial h}\right]\mathrm{d}\phi\,,$$

and $\partial\eta/\partial h$ can be easily determined from (55). Here, a change in $X(t)$ will lead to a change in four endogenous variables: the spot stock price, the dividend yield, the interest rate, and the stock volatility. As these four variables affect the option value in different directions, the impact of the change on $C(t, \tau)$ is ambiguous, depending on the parameter values and the state of the economy. Similarly, a change in $Z(t)$ will have an ambiguous impact on $C(t, \tau)$, even though $Z(t)$ does not affect the interest rate level. In Section 5, we resort to a calibration exercise to show how stock options respond to a change in risk aversion.

Using the hedge ratios Δ_S, Δ_x, and Δ_z, one can follow the standard practice to design a delta-neutral hedging strategy for calls, puts, and other option-like derivatives. Relying on the same logic as above, the different gamma measures can be derived for a call or a put. The details are omitted here to save space.

Finally, note that the option pricing formula in (53) is expressed as a function of $S(t)$, $X(t)$, and $Z(t)$. Among the three time-t variables, $X(t)$ and $Z(t)$ are not observable, which can hamper the practical implementability of the pricing formula. To solve this problem, observe from Eqs. (33) and (48) that both $R(t)$ and $V(t)$ are linear in $X(t)$ and $Z(t)$, which means that one can substitute out the unobservable $X(t)$ and $Z(t)$ in formula (53) by the observable short-term rate $R(t)$ and the recoverable stock volatility $V(t)$. This substitution technique, therefore, permits convenient implementation of our equity option pricing formula. See Bakshi, Cao, and Chen (1997) for such implementation details.

4.3. The Market portfolio and options on the market portfolio

The discussion in the preceding subsections applies to the first $N-1$ firms' stocks and options written on them. Now, we turn to valuing the market portfolio. Once we have resolved the valuation issue for the market portfolio, the valuation of the Nth firm's stock is trivial because its price must be the difference between the market portfolio value and the sum of the other $N-1$ firms' stock prices. Let $\bar{S}(t)$ be the time-t price of the market portfolio. We still set $\beta_y = \sigma_{q,y} = 0$ and maintain the single systematic state variable assumption. Then, as the market portfolio is a claim to the aggregate output stream of the economy, we have by Eq. (35)

$$\bar{S}(t) = q(t) E_t \int_t^{\infty} e^{-\rho(v-t)} \left(\frac{q(v)}{q(t)} \right)^{-\gamma+1} dv \,. \tag{59}$$

This price $\bar{S}(t)$ must also satisfy the fundamental valuation PDE from Theorem 1:

$$\frac{1}{2}\sigma_{q,x}^2 X q^2 \frac{\partial^2 \bar{S}}{\partial q^2} + (\mu_q + \{\beta_x - \gamma\sigma_{q,x}^2\}X)q\frac{\partial \bar{S}}{\partial q} + \frac{1}{2}\sigma_x^2 X \frac{\partial^2 \bar{S}}{\partial X^2}$$
$$+ [\kappa_x\theta_x - (\kappa_x + \lambda_x)X]\frac{\partial \bar{S}}{\partial X} + \sigma_x\sigma_{q,x} q X \frac{\partial^2 \bar{S}}{\partial q \partial X} + q - R\bar{S} = 0 \,, \tag{60}$$

subject to the transversality condition. Solving this PDE gives

$$\bar{S}(t) = \frac{q(t)}{h(X(t))}, \tag{61}$$

where

$$h(X(t)) = \left[\int_0^\infty \exp[-\{\rho + (\gamma - 1)\mu_q\}v - s(t,v;X)]\,\mathrm{d}v\right]^{-1}, \tag{62}$$

$$\begin{aligned} s(t,v;X) = {} & \frac{\kappa_x\theta_x}{\sigma_x^2}\left[2\ln\left(1 - \frac{[\varepsilon_x - \kappa_x - \frac{1-\gamma}{\gamma}\lambda_x](1-\mathrm{e}^{-\varepsilon_x v})}{2\varepsilon_x}\right)\right. \\ & \left. + \left[\varepsilon_x - \kappa_x - \frac{1-\gamma}{\gamma}\lambda_x\right]v\right] \\ & + \frac{(\gamma - 1)[\beta_x - \frac{1}{2}\gamma\sigma_{q,x}^2](1-\mathrm{e}^{-\varepsilon_x v})}{2\varepsilon_x - [\varepsilon_x - \kappa_x - \frac{1-\gamma}{\gamma}\lambda_x](1-\mathrm{e}^{-\varepsilon_x v})}X(t), \end{aligned} \tag{63}$$

letting $\varepsilon_x \equiv \sqrt{(\kappa_x + [1-\gamma/\gamma]\lambda_x)^2 + 2\sigma_x^2(\gamma - 1)\{\beta_x - \frac{1}{2}\gamma\sigma_{q,x}^2\}}$. The function $h(X(t))$ is the equilibrium dividend yield for the market portfolio. In general, it is hard to analytically solve the integral for $h(X(t))$. There are, however, two special cases in which the integral can be solved.

Case 1. Let $X(t)$ be a constant for each t (i.e., setting $\kappa_x = \sigma_x = 0$). Then, the output/dividend process q is a geometric Brownian motion, and so is the price process $\bar{S}$. The dividend yield becomes a constant, with $h(X(t)) = \rho + (\gamma - 1)\mu_q + (\gamma - 1)\{\beta_x - \frac{1}{2}\gamma\sigma_{q,x}^2\}$. The market portfolio price is thus increasing in μ_q and β_x if $\gamma < 1$, and decreasing in these parameters if $\gamma > 1$. The price $\bar{S}(t)$ is decreasing in the risk level $\sigma_{q,x}$ of the aggregate output growth if $\gamma < 1$, and increasing in the risk level if $\gamma > 1$. The solution for this case is consistent with that of Abel (1988), who studies a similar discrete-time economy.

Case 2: Let $\gamma \to 1$, which produces the often-used log utility function. Consequently, $h(X(t)) = \rho$, that is, the dividend yield for the market portfolio equals the subjective time discounting factor. Applying Ito's lemma to $\bar{S}(t)$ produces

$$\frac{\mathrm{d}\bar{S}(t)}{\bar{S}(t)} = [\mu_q + \beta_x X(t)] + \sigma_{q,x}\sqrt{X(t)}\mathrm{d}\omega_x(t). \tag{64}$$

In this case, the time t price of a European call option written on the market portfolio, with strike price K and with τ periods to maturity from time t, is

$$\bar{C}(t,\tau;S,X) = \bar{S}(t)\mathrm{e}^{-\rho\tau}\bar{\Pi}_1(t,\tau;\bar{S},X) - KB(t,\tau)\bar{\Pi}_2(t,\tau;\bar{S},X) \tag{65}$$

where $\bar{\Pi}_1$ and $\bar{\Pi}_1$ are the probabilities that can be recovered from the characteristic functions given in (A.11) and (A.12) of the Appendix. This option formula is the stochastic opportunity counterpart to the one in Rubinstein (1976). Furthermore, as the interest rates and the market portfolio volatility are still stochastic in this case, the option formula in (65) is more general than the market portfolio option formula in Goldstein and Zapatero (1996), who have stochastic interest rates but constant market volatility.

For other cases, the corresponding pricing formula for options written on the market portfolio is typically not obtainable, since the exact dynamics for $\bar{S}(t)$ are not known (unless the integral in $h(X(t))$ can be explicitly solved).

5. Contingent claims valuation with exponential utility

Using the alternative valuation approach, we can demonstrate that the results established in the two preceding sections also apply to economies with exponential utility functions. Continuing with the earlier notation, assume that the representative agent has an exponential period utility:

$$U(c) = -\mathrm{e}^{-\gamma c}, \tag{66}$$

where γ is now interpreted as the coefficient of absolute risk aversion. To arrive at the previous pricing formulas, we only need to change the output process in (18) to

$$\begin{aligned} \mathrm{d}q(t) &= (\mu_q + \beta_x X(t) + \beta_y Y(t))\,\mathrm{d}t + \sigma_{q,x}\sqrt{X(t)}\,\mathrm{d}\omega_x(t) \\ &\quad + \sigma_{q,y}\sqrt{Y(t)}\,\mathrm{d}\omega_y(t), \end{aligned} \tag{67}$$

and keep the processes for state variables X and Y as given in (16) and (17), respectively. Here, change in output, rather than growth in output, is driven by a linear combination of the state variables. With this output process and the exponential utility function, the equilibrium risk premium for any contingent claim F is the same as given in (20). Take the τ-period discount bond as an example. In this economy,

$$\begin{aligned} B(t,\tau) &= \mathrm{e}^{-\rho\tau} E_t[\mathrm{e}^{-\gamma[q(t+\tau)-q(t)]}] \\ &= E_t \exp\left\{ -(\rho + \mu_q\gamma)\tau - \gamma \int_t^{t+\tau} [\beta_x X(v) + \beta_y Y(v)]\,\mathrm{d}v \right. \\ &\quad \left. - \gamma\sigma_{q,x} \int_t^{t+\tau} \sqrt{X(v)}\,\mathrm{d}\omega_x(v) - \gamma\sigma_{q,y} \int_t^{t+\tau} \sqrt{Y(v)}\,\mathrm{d}\omega_y(v) \right\}. \end{aligned}$$

Thus, $B(t,\tau)$ is still only a function of $X(t)$ and $Y(t)$. As the risk premium for the bond and the processes for X and Y remain unchanged from before, the PDE in (22) applies to this bond as well. As a result, the τ-period bond price in this exponential utility economy is the same as in (23). Similarly, the bond option price remains the same as determined in (31).

Let the firm-specific state variables Z_n and the dividend policy be as given in (34) and (38), respectively. Following the same reasoning as above, we conclude that (i) the stock price solution in (40) and its dynamics in (45) still hold and (ii) the PDE in (52) and hence the closed-form pricing formula in (53) apply to a European call option written on such a stock in this economy.

In summary, the term structure, the stock prices, and the bond and stock option pricing formulas remain unaltered, whether the representative agent has a power utility or an exponential utility function. This statement is, however, made only in terms of the functional forms for the respective pricing formulas. The two types of economy can be quite different from one another. For one thing, the output in each type of economy follows a distinct process.

6. Properties of option prices

The purpose of this section is twofold. First, using an artificial economy, we seek to understand how interest rates, term premiums, stock prices, and stock option prices respond to a change in risk aversion. Given that existing models are often free of the risk aversion parameter, this calibration exercise should serve a special role. Second, we study whether our general option pricing model can reconcile certain differences between existing option models and empirical regularities.

The values chosen for the structural parameters are reported in the notes to Figs. 1 and 3. These parameter values are chosen such that the resulting term structure of interest rates and stock price volatility are empirically plausible. For example, the term structure of interest rates corresponding to $\gamma = 2.0$ is upward sloping, with $R(t) = 8.10\%$, $R(t, 0.50) = 8.29\%$, $R(t, 1) = 8.47\%$, $R(t, 2) = 8.88\%$, and $R(t, \infty) = 12.30\%$, and the steady-state standard deviation of the spot interest rate is 6.28%. The stock return standard deviation is 36.88% and the dividend yield is 3.23%, which are not unreasonable. Furthermore, stock volatility is negatively correlated with stock returns, and interest rates are positively correlated with stock volatility.

Let us first look at the impact of risk aversion on interest rates. Note that when $\gamma = 1$ (i.e., the often-used log utility case), the term structure changes to $R(t) = 4.73\%$, $R(t, 0.50) = 4.85\%$, $R(t, 1) = 4.95\%$, $R(t, 2) = 5.13\%$, and $R(t, \infty) = 7.16\%$, which is still upward sloping but quite different from its counterpart at $\gamma = 2$. Fig. 1 plots the response of the term structure to a change in risk aversion. The interest rate for any maturity has a hump-shaped relationship with γ: it

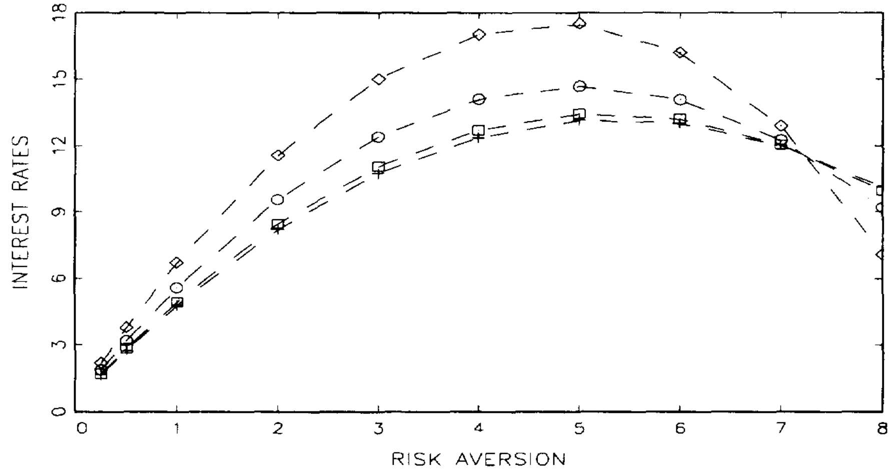

Fig. 1. Effect of risk aversion on interest rates.

Note: The (+)-curve, (□)-curve, (○)-curve, and ◇-curve, respectively, plot the yield-to-maturity on a τ-year discount bond when τ varies from $\tau = 0.25$, $\tau = 1$, $\tau = 2$, to $\tau = 30$. The calculations are based on the following structural parameter values: $\rho = 0.005$, $\mu_q = 0.02$, $\beta_x = 0.04$, $\beta_y = 0.06$, $\sigma_{q,x} = 0.12$, $\sigma_{q,y} = -0.12$, $\kappa_x = 0.15$, $\theta_x = 0.45$, $\sigma_x = 0.12$, $\kappa_y = 0.20$, $\theta_y = 0.75$, $\sigma_y = 0.10$, and time-t (initial) values of $X(t) = 0.25$ and $Y(t) = 0.35$.

first increases and then decreases, with $\gamma = 5$ (approximately) being the turning point. This is exactly as expected. Recall that the elasticity of intertemporal substitution is the inverse of the risk aversion parameter γ. As γ increases, initially the (positive) effect of intertemporal substitution dominates and then the (negative) risk aversion effect on interest rates dominates. While this response pattern applies to every maturity, long-term interest rates are more sensitive (with a steeper slope), as long-term bonds are riskier and hence more responsive to changes in risk-taking attitudes. Fig. 2 plots the term premium in relation to γ. The term premium on short-term bonds is increasing in γ, while for long-term bonds, such as for the 30-year bond, it displays a hump-shaped response. This finding should not be surprising given the two opposite effects of γ on interest rates.

Fig. 3 plots the sensitivity of the stock price to γ. As expected, based on Eq. (40), the stock price is decreasing in γ. For the European stock calls reported in Fig. 4, we fix the term to expiration at $\tau = 0.25$ (a quarter of a year) and let the strike price change: $K = \$90$, $K = \$100$, and $K = \$110$. For each call option, the reported prices are normalized (i.e., divided) by its price corresponding to $\gamma = 2$, which is done for ease of comparison. Observe the following features from Fig. 4: (i) the call prices, regardless of strike price, are decreasing and convex in γ and (ii) the sensitivity of the call price differs substantially across strike prices,

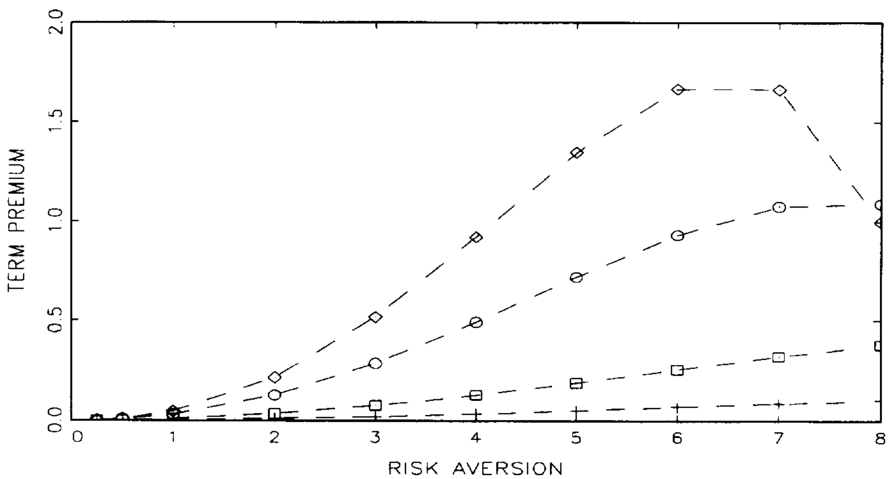

Fig. 2. Effect of risk aversion on the term premium.

Note: The (+)-curve, (□)-curve, (⊙)-curve, and (◇)-curve, respectively, plot the term premium on a τ-year discount bond when τ varies from $\tau = 0.25$, $\tau = 1$, $\tau = 2$, to $\tau = 30$. All the calculations are based on the structural parameter values displayed in the note to Fig. 1.

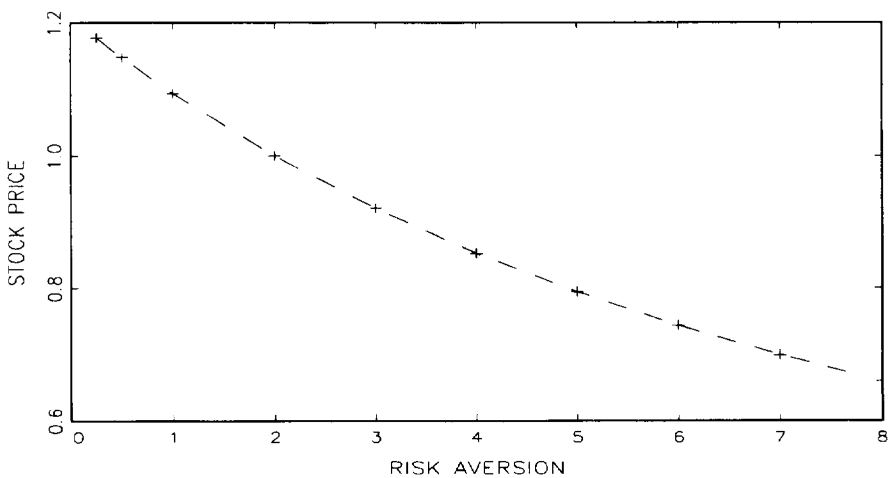

Fig. 3. Effect of risk aversion on the stock price.

Note: The reported values in the graph are normalized by the price of the stock corresponding to $\gamma = 2$. The calculations are based on the following structural parameter values: $\rho = 0.005$, $\mu_q = 0.02$, $\beta_x = 0.04$, $\beta_y = 0$, $\sigma_{q,x} = 0.12$, $\sigma_{q,y} = 0$, $\kappa_x = 0.15$, $\theta_x = 0.45$, $\sigma_x = 0.12$, $\kappa_z = 0.10$, $\theta_z = 0.55$, $\sigma_z = 0.40$, $\psi = 67.09$, $\Gamma_x = -1.75$, $\Gamma_z = -1.25$, and time-t (initial) values of $X(t) = 0.25$ and $Z(t) = 0.35$.

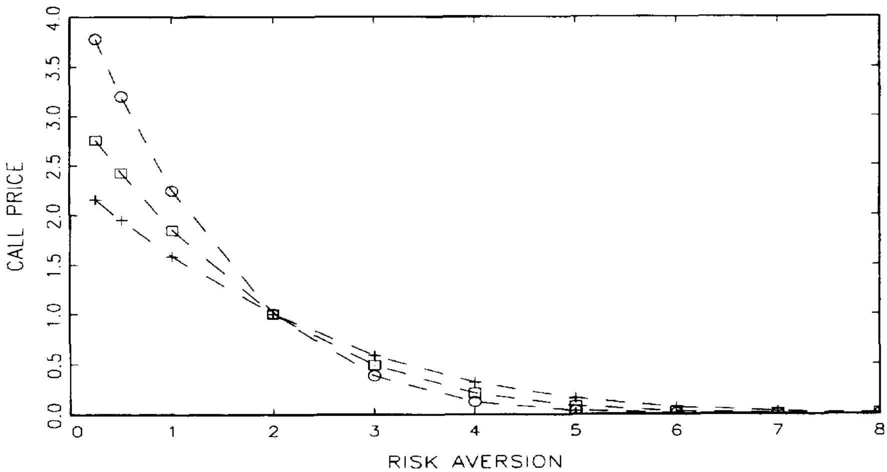

Fig. 4. Effect of risk aversion on stock options.

Note: The (+)-curve, (□)-curve, and (⊙)-curve, respectively, plot the call option price when the strike price varies from $K = 90$, $K = 100$, to $K = 110$. The time-to-expiration for the options is $\tau = 0.25$ years and the spot stock price is \$100. The reported values in the graph are normalized by the price of the call option corresponding to $\gamma = 2$. All the calculations are based on the parameter values displayed in the note to Fig. 3.

with in-the-money calls being the least sensitive and the out-of-the-money calls the most sensitive. Observe from Eq. (53) that γ can affect call option values through its impact on the stock price, the dividend yield, and the bond price: a higher γ means a lower $S(t)$, a higher $\eta(t,\tau)$, and an initially lower and then higher $B(t,\tau)$. The negative impact of an increase in γ on stock prices in general dominates.

We now demonstrate that our option pricing model can generate a *volatility smile*. According to Rubinstein (1985) and Bakshi, Cao, and Chen (1997), when actual option prices are substituted into the Black–Scholes model to back out the implied volatility, the implied volatility tends to vary across strike prices in a U-shaped manner. This volatility smile is particularly striking for short-term options and has been a persistent feature of option markets (e.g., Rubinstein, 1994). While it is beyond the scope of the present paper to subject our model to actual option data, we study the performance of our model in the following fashion. We first obtain (theoretical) option prices using our equity option pricing model, and then substitute those prices into the (dividend-adjusted) Black–Scholes formula to numerically back out the implied volatility. The test hypothesis is that if our model exhibits empirically plausible features, the model-generated option prices should also lead to an implied volatility smile. For each term to expiration, Fig. 5 plots the implied volatility against the strike price, whereas

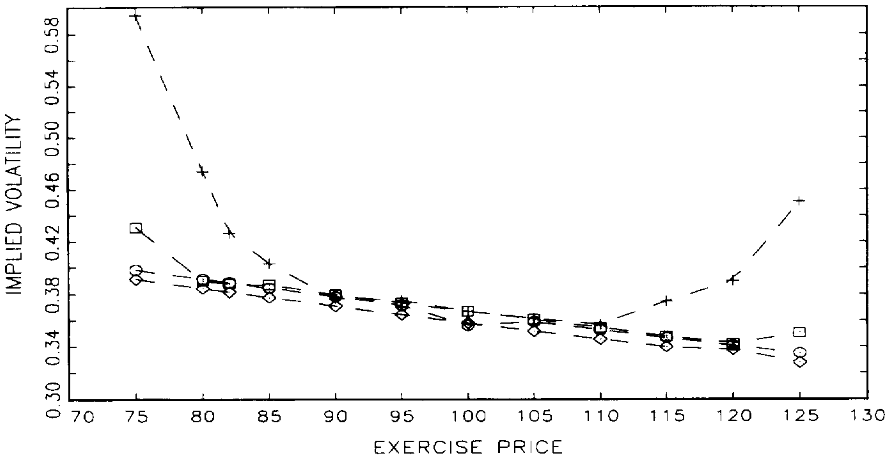

Fig. 5. Volatility smile in stock options.

Note: The (+)-curve, (□)-curve, (⊙)-curve, and ◇-curve, respectively, plot the implied stock volatility when the term-to-expiration for the options varies from $\tau = \frac{10}{365}$, $\tau = \frac{15}{365}$, $\tau = \frac{30}{365}$, to $\tau = \frac{90}{365}$. The spot stock price is \$100 and $\gamma = 2$. The implied stock volatility is obtained by inverting the (dividend-adjusted) Black–Scholes formula where the call option price is determined via Eq. (53). The structural parameter values used in the calculation are as displayed in the note to Fig. 3.

Fig. 6 displays the term structure of implied volatility. In all the calculations, $\gamma = 2.0$. Several observations can be made from these figures. First, the volatility smile is much stronger for call options with 30 days or less to expiration. For longer-term options, however, the implied volatility is declining in the strike price. Both of these features are consistent with the evidence presented in Rubinstein (1994). Second, the implied volatility for in-the-money options is substantially higher, compared to those for both at-the-money and out-of-the-money options. Take as an example options with ten days to expiration (i.e., $\tau = 10/365$ years). The implied volatility is 59.42% for $K = \$75$, 36.68% for $K = \$100$, and 45% for $K = \$125$. Recall that the time t standard deviation of the stock is 36.88%. Finally, the term structure of implied volatility for both in-the-money and out-of-the-money options also displays a volatility smile, with 30 days being the turning point of the U-curve, although for at-the-money calls, the implied volatility is virtually linear in the term to expiration.

Note that these observed features apply to option prices taken from one time point and obtained from one set of structural parameter values. As can be seen from our derivations, for instance, different time t values for X, Y, and Z can generate different volatility patterns across strike prices and across terms to expiration. In other words, the volatility patterns coming out of our model can be time-varying and state-dependent, which is again consistent with the time-varying

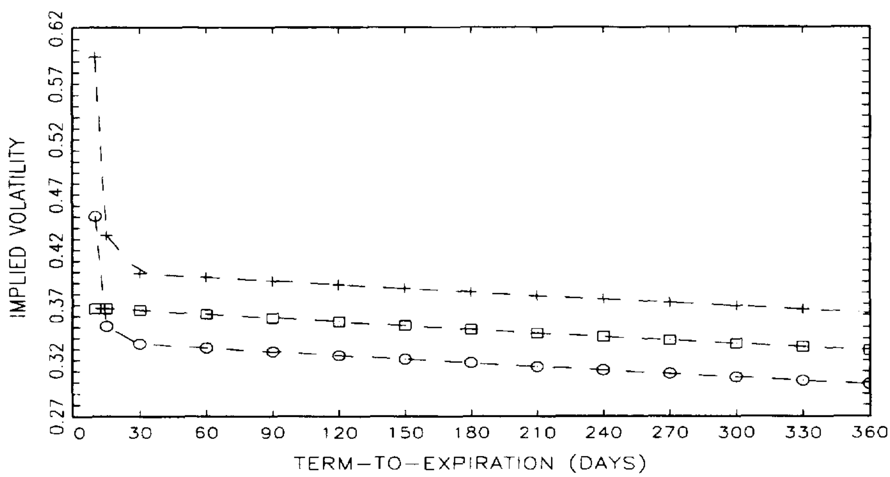

Fig. 6. Term structure of implied volatiliy.

Note: The (+)-curve, (□)-curve, and (⊙)-curve, respectively, plot the term structure of implied stock volatility when the exercise price varies from $K = \$75$, $K = \$100$ to $K = \$125$. The spot stock price is \$100 and $\gamma = 2$. The implied stock volatility is obtained by inverting the (dividend-adjusted) Black–Scholes formula where the call option price is determined via Eq. (53). The structural parameter values used in the calculation are as displayed in the note to Fig. 3.

nature of the strike price biases and volatility smiles emphasized in Rubinstein (1985).

7. Concluding remarks

The work here can be extended in different directions. For example, the alternative valuation equation can be used to study contingent claims by assuming other classes of stochastic processes. The choice of the exact stochastic structure for the economy often depends on empirical plausibility and technical tractability. For instance, systematic jumps can be introduced along the lines of Amin and Ng (1993), Bates (1996), and Scott (1996). Given the existence of jumps in real-life asset prices and economic variables, this line of research should be of significant importance. In another direction, this approach can be applied to value other types of derivative claims, such as equity forward and futures contracts, options on equity futures, and options on interest rate futures.

The bond and stock option pricing models developed here can also be tested using real-life data. Such empirical exercises can address issues related to the well-known biases associated with the Black–Scholes model as well as how option prices differ across assets of distinct systematic and idiosyncratic risk characteristics. This task, together with model implementation issues, is pursued in Bakshi, Cao, and Chen (1997). The bond and stock valuation formulas of this paper are applied and implemented in Bakshi and Chen (1996b).

Finally, non-expected utilities can also be introduced into this framework. Typically, solving the Hamilton–Jacobi–Bellman equation for the indirect utility or value function is even more difficult when nonstandard utility theories are applied (e.g., see Constantinides, 1990; Duffie and Epstein, 1992; Epstein and Zin, 1991; and Sundaresan, 1989). For these types of application, it then becomes more important to develop contingent claims valuation methods without involving the value function. This paper offers a useful direction for finding a way to conduct contingent claims valuation under nonstandard or nonexpected utility.

Appendix. Proof of results

Derivation of the bond option formula in (31)

Conjecture that the solution to the PDE (22) is of the form as given in (31). Then the functions P_j for $j = 1, 2$ must, respectively, satisfy the PDEs:

$$\frac{1}{2}\sigma_x^2 X \frac{\partial^2 P_1}{\partial X^2} + \left[\kappa_x\theta_x - (\kappa_x + \lambda_x)X + \frac{\sigma_x^2}{B(t,\tilde{\tau})}\frac{\partial B(t,\tilde{\tau})}{\partial X}X\right]\frac{\partial P_1}{\partial X}$$
$$+\frac{1}{2}\sigma_y^2 Y \frac{\partial^2 P_1}{\partial Y^2} + \left[\kappa_y\theta_y - (\kappa_y + \lambda_y)Y + \frac{\sigma_y^2}{B(t,\tilde{\tau})}\frac{\partial B(t,\tilde{\tau})}{\partial Y}Y\right]\frac{\partial P_1}{\partial Y}$$
$$-\frac{\partial P_1}{\partial \tau} = 0 \tag{A.1}$$

and

$$\frac{1}{2}\sigma_x^2 X \frac{\partial^2 P_2}{\partial X^2} + \left[\kappa_x\theta_x - (\kappa_x + \lambda_x)X + \frac{\sigma_x^2}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial X}X\right]\frac{\partial P_2}{\partial X}$$
$$+\frac{1}{2}\sigma_y^2 Y \frac{\partial^2 P_2}{\partial Y^2} + \left[\kappa_y\theta_y - (\kappa_y + \lambda_y)Y + \frac{\sigma_y^2}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial Y}Y\right]\frac{\partial P_2}{\partial Y}$$
$$-\frac{\partial P_2}{\partial \tau} = 0\,, \tag{A.2}$$

where all time subscripts are suppressed and the PDEs must be solved subject to the terminal condition:

$$P_j(t+\tau, 0; X, Y) = 1_{-\varrho_x(\tilde{\tau}-\tau)X(t-\tau)-\varrho_y(\tilde{\tau}-\tau)Y(t-\tau)\geqslant \tilde{K}}$$

for $j = 1, 2$, with $\tilde{K} \equiv \ln[K] + (\rho + \gamma\mu_q)(\tilde{\tau} - \tau) + \alpha_x(\tilde{\tau} - \tau) + \alpha_x(\tilde{\tau} - \tau)$. In a similar agreement, Heston (1993), the PDEs for P_1 and P_2 in (A.1) and (A.2), respectively, are precisely the Fokker–Planck forward equations for probability functions (see Karlin and Taylor, 1975). Hence, P_1 and P_2 must both be probability functions taking values in $[0, 1]$. The characteristic functions corresponding to these two probabilities, $\hat{f}_j(t, \tau, X, Y; \phi)$ for $j = 1, 2$, must also satisfy the respective PDEs in (A.1) and (A.2), subject to the terminal condition:

$$\hat{f}_j(t+\tau, 0, X, Y; \phi) = \exp[-i\phi \varrho_x(\bar{\tau} - \tau) X(t+\tau) - i\phi \varrho_y(\bar{\tau} - \tau) Y(t+\tau)]\,. \tag{A.3}$$

For details on this point, see Johnson and Kotz (1970). Solving the respective PDEs, we have the characteristic functions for the bond option formula given by

$$\begin{aligned}
&\hat{f}_1(t, \tau, X, Y; \phi) \\
&= \exp\Bigg\{ -\frac{\kappa_x \theta_x}{\sigma_x^2} \\
&\left[2 \ln\left(1 - \frac{[\xi_x - \kappa_x - \lambda_x - (1+i\phi)\sigma_x^2 \varrho_x(\bar{\tau} - \tau)](1 - e^{-\xi_x \tau})}{2\xi_x}\right)\right] \\
&- \frac{\kappa_y \theta_y}{\sigma_y^2}\left[2 \ln\left(1 - \frac{[\xi_y - \kappa_y - \lambda_y - (1+i\phi)\sigma_y^2 \varrho_y(\bar{\tau} - \tau)](1 - e^{-\xi_y \tau})}{2\xi_y}\right)\right] \\
&- (\rho + \gamma\mu_q)\bar{\tau} - \alpha_x(\bar{\tau} - \tau) - \alpha_y(\bar{\tau} - \tau) - (1+i\phi)\varrho_x(\bar{\tau} - \tau)\kappa_x \theta_x \tau \\
&- (1+i\phi)\varrho_y(\bar{\tau} - \tau)\kappa_y \theta_y \tau - (1+i\phi)\varrho_x(\bar{\tau} - \tau) X(t) \\
&- (1+i\phi)\varrho_y(\bar{\tau} - \tau) Y(t) - \ln[B(t, \bar{\tau})] \\
&+ \{[[\tfrac{1}{2}\sigma_x^2(1+i\phi)^2 \varrho_x^2(\bar{\tau} - \tau) + (1+i\phi)(\kappa_x + \lambda_x)\varrho_x(\bar{\tau} - \tau) \\
&- \gamma(\beta_x - \tfrac{1}{2}(1+\gamma)\sigma_{q,x}^2)](1 - e^{-\xi_x \tau})]/[\xi_x - \tfrac{1}{2}[\xi_x - \kappa_x - \lambda_x \\
&- (1+i\phi)\sigma_x^2 \varrho_x(\bar{\tau} - \tau)](1 - e^{-\xi_x \tau})\} X(t) + \{[[\tfrac{1}{2}\sigma_y^2(1+i\phi)^2 \varrho_y^2(\bar{\tau} - \tau) \\
&+ (1+i\phi)(\kappa_y + \lambda_y)\varrho_y(\bar{\tau} - \tau) - \gamma(\beta_y - \tfrac{1}{2}(1+\gamma)\sigma_{q,y}^2)](1 - e^{-\xi_y \tau}])/ \\
&[\xi_y - \tfrac{1}{2}[\xi_y - \kappa_y - \lambda_y - (1+i\phi)\sigma_y^2 \varrho_y(\bar{\tau} - \tau)](1 - e^{-\xi_y \tau})]\} Y(t) \\
&- \frac{\kappa_x \theta_x}{\sigma_x^2}[\xi_x - \kappa_x - \lambda_x - (1+i\phi)\sigma_x^2 \varrho_x(\bar{\tau} - \tau)]\tau \\
&- \frac{\kappa_y \theta_y}{\sigma_y^2}[\xi_y - \kappa_y - \lambda_y - (1+i\phi)\sigma_y^2 \varrho_y(\bar{\tau} - \tau)]\tau \Bigg\},
\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}
&\hat{f}_2(t,\tau,X,Y;\phi)\\
&= \exp\Bigg\{-\frac{\kappa_x\theta_x}{\sigma_x^2}\left[2\ln\left(1-\frac{[\xi_x^*-\kappa_x-\lambda_x-\mathrm{i}\phi\sigma_x^2\varrho_x(\tilde{\tau}-\tau)](1-\mathrm{e}^{-\xi_x^*\tau})}{2\xi_x^*}\right)\right]\\
&\quad-\frac{\kappa_y\theta_y}{\sigma_y^2}\left[2\ln\left(1-\frac{[\xi_y^*-\kappa_y-\lambda_y-\mathrm{i}\phi\sigma_y^2\varrho_y(\tilde{\tau}-\tau)](1-\mathrm{e}^{-\xi_y^*\tau})}{2\xi_y^*}\right)\right]\\
&\quad-(\rho+\gamma\mu_q)\tau-\mathrm{i}\phi\varrho_x(\tilde{\tau}-\tau)\kappa_x\theta_x\tau-\mathrm{i}\phi\varrho_y(\tilde{\tau}-\tau)\kappa_y\theta_y\tau-\mathrm{i}\phi\varrho_x(\tilde{\tau}-\tau)X(t)\\
&\quad-\mathrm{i}\phi\varrho_y(\tilde{\tau}-\tau)Y(t)-\ln[B(t,\tau)]-\frac{\kappa_x\theta_x}{\sigma_x^2}[\xi_x^*-\kappa_x-\lambda_x-\mathrm{i}\phi\sigma_x^2\varrho_x(\tilde{\tau}-\tau)]\tau\\
&\quad+\{[[\tfrac{1}{2}\sigma_x^2(\mathrm{i}\phi)^2\varrho_x^2(\tilde{\tau}-\tau)+\mathrm{i}\phi(\kappa_x+\lambda_x)\varrho_x(\tilde{\tau}-\tau)-\gamma(\beta_x-\tfrac{1}{2}(1+\gamma)\sigma_{q,x}^2)]\\
&\quad\times(1-\mathrm{e}^{-\xi_x^*\tau})]/[\xi_x^*-\tfrac{1}{2}[\xi_x^*-\kappa_x-\lambda_x-\mathrm{i}\phi\sigma_x^2\varrho_x(\tilde{\tau}-\tau)](1-\mathrm{e}^{-\xi_x^*\tau})]\}X(t)\\
&\quad+\{[\tfrac{1}{2}\sigma_y^2(\mathrm{i}\phi)^2\varrho_y^2(\tilde{\tau}-\tau)+\mathrm{i}\phi(\kappa_y+\lambda_y)\varrho_y(\tilde{\tau}-\tau)\\
&\quad-\gamma(\beta_y-\tfrac{1}{2}(1+\gamma)\sigma_{q,y}^2)](1-\mathrm{e}^{-\xi_y^*\tau})]\}/\\
&\quad[\xi_y^*-\tfrac{1}{2}[\xi_y^*-\kappa_y-\lambda_y-\mathrm{i}\phi\sigma_y^2\varrho_y(\tilde{\tau}-\tau)](1-\mathrm{e}^{-\xi_y^*\tau})]\}Y(t)\\
&\quad-\frac{\kappa_y\theta_y}{\sigma_y^2}[\xi_y-\kappa_y-\lambda_y-\mathrm{i}\phi\sigma_y^2\varrho_y(\tilde{\tau}-\tau)]\tau\Bigg\},
\end{aligned}\tag{A.5}$$

where

$$\begin{aligned}
\xi_x&=[\{\kappa_x+\lambda_x+\sigma_x^2(1+\mathrm{i}\phi)\varrho_x(\tilde{\tau}-\tau)\}^2-2\sigma_x^2\{(1+\mathrm{i}\phi)(\kappa_x+\lambda_x)\varrho_x(\tilde{\tau}-\tau)\\
&\qquad+\tfrac{1}{2}\sigma_x^2(1+\mathrm{i}\phi)^2\varrho_x^2(\tilde{\tau}-\tau)-\gamma(\beta_x-\tfrac{1}{2}(1+\gamma)\sigma_{q,x}^2)\}]^{1/2}\\
\xi_y&=[\{\kappa_y+\lambda_y+\sigma_y^2(1+\mathrm{i}\phi)\varrho_y(\tilde{\tau}-\tau)\}^2-2\sigma_y^2\{(1+\mathrm{i}\phi)(\kappa_y+\lambda_y)\varrho_y(\tilde{\tau}-\tau)\\
&\qquad+\tfrac{1}{2}\sigma_y^2(1+\mathrm{i}\phi)^2\varrho_y^2(\tilde{\tau}-\tau)-\gamma(\beta_y-\tfrac{1}{2}(1+\gamma)\sigma_{q,y}^2)\}]^{1/2}\\
\xi_x^*&=[\{\kappa_x+\lambda_x+\sigma_x^2\mathrm{i}\phi\varrho_x(\tilde{\tau}-\tau)\}^2-2\sigma_x^2\{\mathrm{i}\phi(\kappa_x+\lambda_x)\varrho_x(\tilde{\tau}-\tau)\\
&\qquad+\tfrac{1}{2}\sigma_x^2(\mathrm{i}\phi)^2\varrho_x^2(\tilde{\tau}-\tau)-\gamma(\beta_x-\tfrac{1}{2}(1+\gamma)\sigma_{q,x}^2)\}]^{1/2}
\end{aligned}$$

$$\xi_y^* = [\{\kappa_y + \lambda_y + \sigma_y^2 \mathrm{i}\phi \varrho_y(\tilde{\tau} - \tau)\}^2 - 2\sigma_y^2\{\mathrm{i}\phi(\kappa_y + \lambda_y)\varrho_y(\tilde{\tau} - \tau)$$

$$+ \tfrac{1}{2}\sigma_y^2(\mathrm{i}\phi)^2\varrho_y^2(\tilde{\tau} - \tau) - \gamma(\beta_y - \tfrac{1}{2}(1+\gamma)\sigma_{q,y}^2)\}]^{1/2}. \quad \square$$

Derivation of the stock option formula in (53)

The PDE in (52) can be written as

$$\frac{1}{2}(\sigma_1^2 X + \sigma_2^2 Z)\frac{\partial^2 C}{\partial L^2} + \left[R - \frac{1}{2}(\sigma_1^2 X + \sigma_2^2 Z) - \frac{\psi}{A} - \frac{\psi_x}{A}X - \frac{\psi_z}{A}Z\right]\frac{\partial C}{\partial L}$$

$$+\sigma_1\sigma_x X\frac{\partial^2 C}{\partial L \partial X} + \sigma_2\sigma_z Z\frac{\partial^2 C}{\partial L \partial Z} + \frac{1}{2}\sigma_x^2 X\frac{\partial^2 C}{\partial X^2} + [\kappa_x\theta_x - (\kappa_x + \lambda_x)X]\frac{\partial C}{\partial X}$$

$$+\frac{1}{2}\sigma_z^2 Z\frac{\partial^2 C}{\partial Z^2} + [\kappa_z\theta_z - \kappa_z Z]\frac{\partial C}{\partial Z} - \frac{\partial C}{\partial \tau} - RC = 0, \qquad \text{(A.6)}$$

where, for convenience, $L(t) \equiv \ln[S(t)]$. Conjecture that the solution is of the form as given in (53) and substitute it into (A.6) to yield the two PDEs for the two functions, Π_1 and Π_2, respectively. As in the derivation of the bond option formula, the PDEs for Π_1 and Π_2, respectively, satisfy the Fokker–Planck forward equation for probability functions. This guarantees that Π_1 and Π_2 are two valid probability functions. The corresponding characteristic functions for Π_1 and Π_2 also satisfy the same respective PDEs, subject to the boundary condition

$$f_j(t+\tau, 0, S, X, Z; \phi) = \exp(\mathrm{i}\phi \ln[S(t+\tau)]). \qquad \text{(A.7)}$$

Solving the PDEs for the two characteristic functions results in the final solution:

$$f_1(t, \tau, S, X, Z; \phi)$$

$$= \exp\left\{ -\frac{\kappa_x\theta_x}{\sigma_x^2}\left[2\ln\left(1 - \frac{[v_x - \kappa_x - \lambda_x + (1+\mathrm{i}\phi)\sigma_x\sigma_1](1 - \mathrm{e}^{-v_x\tau})}{2v_x}\right)\right]\right.$$

$$-\frac{\kappa_z\theta_z}{\sigma_z^2}\left[2\ln\left(1 - \frac{[v_z - \kappa_z + (1+\mathrm{i}\phi)\sigma_z\sigma_2](1 - \mathrm{e}^{-v_z\tau})}{2v_z}\right)\right.$$

$$\left. +[v_z - \kappa_z + (1+\mathrm{i}\phi)\sigma_z\sigma_2]\tau\right] - \frac{\kappa_x\theta_x}{\sigma_x^2}[v_x - \kappa_x - \lambda_x + (1+\mathrm{i}\phi)\sigma_x\sigma_1]\tau$$

$$+\mathrm{i}\phi(\rho + \gamma\mu_q)\tau - \frac{(1+\mathrm{i}\phi)\psi}{A}\tau + \eta(t, \tau; X, Z) + \mathrm{i}\phi\ln[S(t)]$$

$$+\frac{\{\mathrm{i}\phi\gamma[\beta_x - \frac{1}{2}(1+\gamma)\sigma_{q,x}^2] - (1+\mathrm{i}\phi)[\frac{\psi_x}{A} - \frac{1}{2}\mathrm{i}\phi\sigma_1^2]\}(1 - \mathrm{e}^{-v_x\tau})}{v_x - \frac{1}{2}[v_x - \kappa_x - \lambda_x + (1+\mathrm{i}\phi)\sigma_x\sigma_1](1 - \mathrm{e}^{-v_x\tau})}X(t)$$

$$+\frac{\{i\phi\sigma_2^2 - 2\frac{\psi_z}{A}\}(1+i\phi)(1-e^{-v_z\tau})}{2v_z - [v_z - \kappa_z + (1+i\phi)\sigma_z\sigma_2](1-e^{-v_z\tau})} Z(t)\Bigg\}, \tag{A.8}$$

and

$$\begin{aligned}
&f_2(t,\tau,S,X,Z)\\
&= \exp\Bigg\{-\frac{\kappa_x\theta_x}{\sigma_x^2}\left[2\ln\left(1-\frac{[v_x^* - \kappa_x - \lambda_x + i\phi\sigma_x\sigma_1](1-e^{-v_x^*\tau})}{2v_x^*}\right)\right]\\
&-\frac{\kappa_z\theta_z}{\sigma_z^2}\Bigg[2\ln\left(1-\frac{[v_z^* - \kappa_z + i\phi\sigma_z\sigma_2](1-e^{-v_z^*\tau})}{2v_z^*}\right)\\
&+[v_z^* - \kappa_z + i\phi\sigma_z\sigma_2]\tau\Bigg] - \frac{\kappa_x\theta_x}{\sigma_x^2}[v_x^* - \kappa_x - \lambda_x + i\phi\sigma_x\sigma_1]\tau\\
&+(i\phi - 1)(\rho + \gamma\mu_q)\tau - \frac{i\phi\psi}{A}\tau - \ln[B(t,\tau)] + i\phi\ln[S(t)]\\
&+\{[2\{(i\phi-1)\gamma[\beta_x - \tfrac{1}{2}(1+\gamma)\sigma_{q,x}^2] - i\phi[\tfrac{\psi_x}{A} - \tfrac{1}{2}(i\phi-1)\sigma_1^2]\}\\
&\times(1-e^{-v_x^*\tau})]/[2v_x^* - [v_x^* - \kappa_x - \lambda_x + i\phi\sigma_x\sigma_1](1-e^{-v_x^*\tau})]\}X(t)\\
&+i\phi\frac{\{(i\phi-1)\sigma_2^2 - 2\frac{\psi_z}{A}\}(1-e^{-v_z^*\tau})}{2v_z^* - [v_z^* - \kappa_z + i\phi\sigma_z\sigma_2](1-e^{-v_z^*\tau})} Z(t)\Bigg\},
\end{aligned} \tag{A.9}$$

where

$$\begin{aligned}
v_x &= \Bigg\{(\kappa_x + \lambda_x - (1+i\phi)\sigma_x\sigma_1)^2 - 2\sigma_x^2\Bigg(i\phi\gamma[\beta_x - \tfrac{1}{2}(1+\gamma)\sigma_{q,x}^2]\\
&\quad -(1+i\phi)\left[\frac{\psi_x}{A} - \frac{1}{2}i\phi\sigma_1^2\right]\Bigg)\Bigg\}^{1/2}\\
v_z &= \left\{[\kappa_z - (1+i\phi)\sigma_z\sigma_2]^2 - \sigma_z^2(1+i\phi)\left(i\phi\sigma_2^2 - 2\frac{\psi_z}{A}\right)\right\}^{1/2}\\
v_x^* &= \Bigg\{[\kappa_x + \lambda_x - i\phi\sigma_x\sigma_1]^2 - 2\sigma_x^2\Bigg((i\phi-1)\gamma[\beta_x - \tfrac{1}{2}(1+\gamma)\sigma_{q,x}^2]\\
&\quad -i\phi\left[\frac{\psi_x}{A} - \frac{1}{2}(i\phi-1)\sigma_1^2\right]\Bigg)\Bigg\}^{1/2}\\
v_z^* &= \left\{[\kappa_z - i\phi\sigma_z\sigma_2]^2 - \sigma_z^2 i\phi\left((i\phi-1)\sigma_2^2 - 2\frac{\psi_z}{A}\right)\right\}^{1/2}. \quad \square
\end{aligned}$$

Derivation of the option formula in (65)

The valuation PDE for a call option written on the market portfolio is

$$\frac{1}{2}\sigma_{q,x}^2\bar{S}^2 X\frac{\partial^2\bar{C}}{\partial\bar{S}^2} + (R-\rho)\bar{S}\frac{\partial\bar{C}}{\partial\bar{S}} + \sigma_{q,x}\sigma_x\bar{S}X\frac{\partial^2\bar{C}}{\partial\bar{S}\partial X}$$

$$+\frac{1}{2}\sigma_x^2 X\frac{\partial^2\bar{C}}{\partial X^2} + [\kappa_x\theta_x - (\kappa_x+\lambda_x)X]\frac{\partial\bar{C}}{\partial X} - \frac{\partial\bar{C}}{\partial\tau} - R\bar{C} = 0\,, \qquad \text{(A.10)}$$

subject to the boundary condition $\bar{C}(t+\tau,0) = \max(0,\bar{S}(t+\tau)-K)$. Using the same sequence of steps as in the derivation of the bond option and stock option formulas, first suppose that the solution is of the form as in (65) and then solve the resulting PDEs for the characteristic functions. For completeness, we give the solution for the two characteristic functions below:

$$\bar{f}_1(t,\tau,\bar{S},X;\phi) = \exp\left\{-\frac{\kappa_x\theta_x}{\sigma_x^2}\left[2\ln\left(1-\frac{[\zeta_x-\kappa_x+\mathrm{i}\phi\bar{\lambda}_x](1-\mathrm{e}^{-\zeta_x\tau})}{2\zeta_x}\right)\right.\right.$$

$$\left.+[\zeta_x-\kappa_x+\mathrm{i}\phi\bar{\lambda}_x]\tau\right] + \mathrm{i}\phi\mu_q\tau + \mathrm{i}\phi\ln[\bar{S}(t)]$$

$$\left.+\frac{\mathrm{i}\phi(2\beta_x+(\mathrm{i}\phi-1)\sigma_{q,x}^2)(1-\mathrm{e}^{-\zeta_x\tau})}{2\zeta_x-[\zeta_x-\kappa_x+\mathrm{i}\phi\bar{\lambda}_x](1-\mathrm{e}^{-\zeta_x\tau})}X(t)\right\}, \qquad \text{(A.11)}$$

$$\bar{f}_2(t,\tau,\bar{S},X;\phi)$$

$$= \exp\left\{-\frac{2\kappa_x\theta_x}{\sigma_x^2}\ln\left(1-\frac{[\zeta_x^*-\kappa_x+(\mathrm{i}\phi-1)\bar{\lambda}_x](1-\mathrm{e}^{-\zeta_x^*\tau})}{2\zeta_x^*}\right)\right.$$

$$-\frac{\kappa_x\theta_x}{\sigma_x^2}[\zeta_x^*-\kappa_x+(\mathrm{i}\phi-1)\bar{\lambda}_x]\tau$$

$$+(\mathrm{i}\phi-1)\mu_q\tau - \rho\tau + \mathrm{i}\phi\ln[\bar{S}(t)] - \ln[B(t,\tau)]$$

$$\left.+\frac{(\mathrm{i}\phi-1)(2\beta_x+(\mathrm{i}\phi-2)\sigma_{q,x}^2)(1-\mathrm{e}^{-\zeta_x^*\tau})}{2\zeta_x^*-[\zeta_x^*-\kappa_x+(\mathrm{i}\phi-1)\bar{\lambda}_x](1-\mathrm{e}^{-\zeta_x^*\tau})}X(t)\right\}, \qquad \text{(A.12)}$$

where $\bar{\lambda}_x \equiv \sigma_x \sigma_{q,x}$ and

$$\zeta_x = \{(\kappa_x - i\phi\bar{\lambda}_x)^2 - i\phi\sigma_x^2(2\beta_x + (i\phi - 1)\sigma_{q,x}^2)\}^{1/2},$$
$$\zeta_x^* = \{(\kappa_x - (i\phi - 1)\bar{\lambda}_x)^2 - (i\phi - 1)\sigma_x^2(2\beta_x + (i\phi - 2)\sigma_{q,x}^2)\}^{1/2}. \quad \square$$

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