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Spanning and derivative-security valuation☆

Gurdip Bakshi*, Dilip Madan

Robert H. Smith School of Business, University of Maryland, College Park, MD 20742, USA

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Abstract

This article provides the economic foundations for valuing derivative securities. In particular, it establishes how the characteristic function (of the future uncertainty) is basis augmenting and spans the payoff universe of most, if not all, derivative assets. From the characteristic function of the state-price density, it is possible to analytically price options on any arbitrary transformation of the underlying uncertainty. By differentiating (or translating) the characteristic function, limitless pricing and/or spanning opportunities can be designed. The strength and versatility of the methodology is inherent when valuing (1) average-interest options, (2) correlation options, and (3) discretely monitored knock-out options. © 2000 Elsevier Science S.A. All rights reserved.

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* Corresponding author. Tel.: + 301-405-2261; fax: + 301-405-0359.

E-mail address: gbakshi@rhsmith.umd.edu (G. Bakshi)

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1. Introduction

The notion that options complete markets, pioneered in Ross (1976), is at the core of modern financial economics. Despite its theoretical attractiveness, however, the idea of expanding the asset space via European options has remained mostly an abstraction. With a few exceptions, it has not resulted in any valuation simplifications. Clearly, the set of applications in which the option price has been exploited, directly or indirectly, to value other contingent claims (in its basis) is potentially sparse (e.g., the pricing of elementary securities). One reason for this is that, although options span other securities, they are complex to value at the outset. For a general stochastic structure, the difficulty stems primarily from the lack of analyticity of the option payoff, a feature that has hampered closed-form option pricing characterizations. For instance, outside of the canonical log-normal asset pricing or the Bessel interest rate class, the fundamental valuation equation for the option price is mostly overwhelming. Even when options are nonredundant securities and the option price is analytical, derivative-security pricing is still not so tractable to closed-form formulations: the positioning in the continuum of options is a priori inexplicit. Confronted with such issues, the objective of this paper is to introduce a spanning entity with the ability to overcome the aforementioned valuation barriers. Specifically, we show that the future uncertainty's characteristic function indeed possesses the qualities that one should seek in a desirable spanning engine. First, its valuation is substantially more amenable (than options) to analytical constructions. Second, the underlying basis is analytical and orthonormal. Third, it jointly and simultaneously induces closed-form representation of every contingent claim (options inclusive) covered by its span. As reliance is on fundamental properties of characteristic functions, their validity is independent of how the remaining uncertainty is visualized and the sources of randomness.

To understand the intuition behind each of these statements, it is worthwhile to observe the composition of the characteristic function. From a valuation standpoint, the entity represents the price of a security that promises the holder a trigonometric payoff contingent on the remaining uncertainty. From a mathematical and economic viewpoint, it is the Fourier transform of the state-price density function (the product of the risk-neutral density and the discounting factor). As is well acknowledged from Fourier theory, the characteristic functions (for a vast class) are infinitely differentiable, from which they also inherit their smoothness and hence valuation tractability. Needless to say, for most valuation problems that economists consider pragmatic and interesting, the valuation equations for characteristic functions are remarkably simple, even though their counterparts for state-price density, or the option price, are twisted. Actually, one can count on centuries-old probability theory to arrive at the characteristic functions for a comprehensive class of stochastic processes. In one extreme, a large cohort of pure-jump price processes are recognized and mathematically represented through their characteristic functions.

Strictly speaking, the span via options and the span via characteristic functions are completely interchangeable (subject to some regularity conditions). Granted, the characteristic function is recoverable from options, and the reverse holds as well – they are competitors for describing their span of claims. But in light of the above discussion, that is only true in theory. Nevertheless, since the payoff on characteristic functions is separable into trigonometric sine and cosine, it has the added distinctive trait that pricing and/or spanning can be achieved through its differentiation or translation (as many times as one would like). The superiority of the characteristic function as a primary set of spanning securities is also apparent as the polynomial basis can be generated from differentiation, whereas the opposite direction is delicate and involves summations of infinite series. Also as one might anticipate, the positioning in the continuum of characteristic functions can be designed scientifically by drawing on inverse Fourier theory. Thus, so long as the characteristic function of the state-price density is readily computable, the valuation of any arbitrary claim can be internally accomplished.

From a practical perspective, the observation that a generic derivative-security pricing problem is equivalent to solving just for the characteristic function is promising, and potentially vital. To pursue the above central theme and to gauge the associated simplification more rigorously, we adopted the topic of option valuation for a benchmark analysis.[1] In filling this vacuum, a key security decomposition is first established: the traditional European call can be unbundled into its primitives consisting of (i) the discount bond price, (ii) the scaled-forward price (the fair price of a commitment to deliver the underlying asset at expiration), and (iii) two Arrow–Debreu securities (or delta claims). Unique to our treatment, however, each primitive security is spanned by the payoff on characteristic functions and can hence be valued recursively through their manipulation. More significantly, options written on arbitrary (smooth) functions of the underlying uncertainty can be priced from the same rudimentary building block, i.e., the characteristic function of the state-price density. This and the original valuation task are rendered feasible without conjecturing (ad hoc) solutions to the fundamental valuation equation of each call (provided the characteristic function of the state-price density can be determined by

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[1] During the writing of this paper, Duffie et al. (1998) have extended one of our option pricing results (in an earlier version) to the affine Markov jump-diffusion class (see the discussion in Section 2.1 and in the proof of their Propositions 2 and 3). Our present work is different from theirs in ways that will become apparent from Theorem 1 and the explanation thereafter. For one, we make precise what collection of securities are spanned by characteristic functions and then exploit this insight to value all contingent claims and not just options.

solving the conditional expectation or the corresponding valuation equation). Our examination of the problem also yields the following insight: When the option claim is on the exponential of the uncertainty (as in equity or bond option models), the characteristic functions corresponding to each Arrow–Debreu security are translates of one another. When the option is on the level of uncertainty, the characteristic function for the first (second) Arrow–Debreu security is constructed from the differentiation (translation) of the primitive characteristic function. In all such option problems in which the two characteristic functions are in different parametric classes, the embedded Arrow–Debreu securities are heterogeneous in their probability compositions as a rule.[2]

To expound on the finer aspects of our approach, we consider the explicit pricing of (a) average-rate interest rate options, (b) correlation options, and (c) discretely monitored knock-out options. In each application, the characteristic function of the respective uncertainty is instrumental in spanning/closed-form pricing. In transitioning to average-rate options, we assume that the short interest rate is governed by a square-root process as in Cox et al. (1985). Our inquiry imparts quite a few basic insights concerning average-rate claims. First, the call price is the (average) scaled-forward price multiplied by a delta claim minus the product of the discount bond and the second delta claim multiplied by the adjusted strike price. To obtain the adjusted strike price, one must deduct the past average interest from the contractual exercise price. Second, the density of the remaining uncertainty (the continuous sum) has no analytical representation but its characteristic function possesses an easy-to-interpret exponential-affine structure. Finally, the characteristic function for the first (second) delta claim is obtained from differentiation (translation) of the original characteristic function. Our inspection also uncovers the finding that the second delta security is noncentral chi-squared distributed, while its twin counterpart does not share the same parent distribution.

Our innovations can also be applied to options written on more than one asset and especially outside of the log-normal environment. Generating the

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[2] Characteristic functions have been used for claims pricing by a number of authors, starting with Heston (1993). However, our reasons for reexamining characteristic functions are somewhat different from his. First, Heston's goal is to solve a uniquely parameterized stochastic volatility option model and to analytically determine the characteristic function for each Arrow–Debreu security. Our main object of interest, in contrast, is the characteristic function of the state-price density. Second, our study elaborates how a generic payoff can be spanned by either a continuum of characteristic functions or by a continuum of calls. Third, by integrating the spanning and pricing properties of the characteristic function, we put on a firm footing how it is that a large class of payoff functions can be built and valued (with or without univariate/multivariate exercise regions). By drawing on this distinction and then breaking up a call into its pricing components, extant valuation steps (i.e., the complex practice of solving each Arrow–Debreu security separately and then guessing their solution) can be circumvented altogether. Other similarities/differences between Heston and existing work will be reviewed later.

analytical solution to the joint characteristic function of the two assets lies at the crux of valuation (it will span all claims contingent on the joint uncertainty). In a distinctive example of our own, we stipulate a payoff structure in which (i) the call is exercised only when the (gross) return on each asset exceeds a prespecified threshold (i.e., calls on correlation), and (ii) each asset innovation is cross-correlated and possesses a common volatility factor. Naturally, the call (put) option is in the money when the returns are positively correlated in a rising (declining) market. As articulated in Zhang (1998), correlation derivatives are desirable for coping with cross-market or cross-currency (commodity) risks. In the context of equity markets, they even allow investors to position on a stock/sector relative to a market index. These contracts are precisely what Ross (1976) and Nachman (1988) have labeled complex options and joint simple options, respectively. In any case, as the composite payoff is a product of two calls, its solution structure requires four delta securities in analytical form. Each security can be interpreted as the expectation of a unity payoff conditional on both calls expiring in the money. As this is an option on the exponential of the joint uncertainty, each characteristic function is translated from the joint characteristic function (and is thus in the same parametric class). In recovering each Arrow–Debreu security price, we adapt a result from Shephard (1991) and extend the one-dimensional Fourier inversion formulation (i.e., Kendall and Stuart, 1977) to a multidimensional setting. But this development cannot be adopted verbatim and depends on the valuation problem at hand, and on the exercise region of the calls. This style of reasoning is evident in the valuation of discretely monitored knock-out options.

This paper is organized as follows. How characteristic functions facilitate in the spanning and pricing of contingent securities is made exact in Section 2. Section 3 is devoted to the pricing of average-rate interest rate options. Section 4 refines the methodology to cover claims written on more than a single asset. A pricing formula for discretely monitored knock-out options (N-dimensional generalization) is proposed in Section 5. Concluding remarks are offered in Section 6. The proof of each result can be found in the appendix.

2. Spanning and pricing via characteristic functions

To go directly to the center of the derivative-security pricing problem and to their spanning underpinnings, consider a generic European-style call option contract with expiration date $t+\tau$, strike price K, and claim payoff as follows:

$$\max(0, X[s(t+\tau), r(t+\tau), y(t+\tau)] - K), \tag{1}$$

where, for completeness of analysis, the payoff on the call is contingent on the price of a traded asset $s(t)$, the spot interest rate $r(t)$, and the vector of state variables $y(t)$. To suppress unnecessary notation, write $X[s(t+\tau), r(t+\tau),$

$y(t+\tau)]$ as $X(t+\tau)$ and define the exercise region of the call as $\mathscr{X} \equiv \{X(t+\tau) > K\}$. Let $X(t) > 0$ with probability one for all t and $\Omega \equiv \{X(t+\tau) > 0\}$. Provided certain regularity conditions are satisfied, the time t price of the option contract, denoted $C(t,\ \tau;\ K)$, is

$$C(t,\ \tau;\ K) = E_t^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right)\max(0, X(t+\tau)-K)\right\} \tag{2}$$

$$= \int_{\mathscr{X}} \exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right)[X(t+\tau)-K]q(v)\,\mathrm{d}v, \tag{3}$$

where $E_t^Q\{\,.\,\}$ represents the time t conditional expectation under the equivalent martingale measure (which is presumed to exist) and $q(v)$ is the risk-neutral (joint) density function of the remaining/future uncertainty: $v \equiv (\int_t^{t+\tau} r(u)\,\mathrm{d}u, X(t+\tau))$. From Breeden and Litzenberger (1978), the state-price density is simply $q(v)\exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)$. Although the basic valuation problem outlined in (2)–(3) is well known, an often-posed question is how the conditional expectation and hence the derivative-security price can be determined analytically. Clearly, when the density function, $q(v)$ (or the state-price density), is known and tractable, the valuation problem warrants no further simplification. Unfortunately, for most realistic option pricing and derivative-security valuation applications, the exercise region of the call/put is contingent on a general (vector) Markov (or non-Markov) process for which the state-price density is either unknown or cannot be expressed in terms of special functions of mathematics. As will be validated shortly, the characteristic function of the state-price density is remarkably uncomplicated (in a relative sense) for option problems of practical interest, even though the state-price density function is not. For the most part, and as we show, all that is required for option and derivative-security valuation is the closed-form formulation of the characteristic function.

As our simplifications are about exploiting the fundamental properties of characteristic functions and their span, the principal approach will be applicable regardless of the source of primitive uncertainty, whether in discrete-time, continuous-time, pure-jump, or mixture environments. Let $x(t) \equiv (s(t), r(t), y(t)')'$. Since option and claim valuation problems are conventionally cast in a diffusion setting, assume for now that $x(t)$ is a vector Markov Ito process as follows:

$$\mathrm{d}x(t) = \mu[x(t), t]\,\mathrm{d}t + \sigma[x(t), t]\,\mathrm{d}\omega(t), \tag{4}$$

where $\omega(t)$ represents a (vector) standard Brownian motion. Under the appropriate set of regularity conditions and dynamics (2), the solution to the valuation partial differential equation of the call,

$$\tfrac{1}{2}\mathrm{tr}[\sigma\sigma' C_{xx}] + \mu C_x - C_\tau - rC = 0, \tag{5}$$

subject to $C(t+\tau, 0; K) = \max(0, X(t+\tau) - K)$, is from the Feynman–Kac theorem, the conditional expectation (2). Leave the exact dynamics for $\mu[x(t), t]$ and $\sigma[x(t), t]$ unspecified for the moment.

Traditionally, researchers have directly solved such contingent claims valuation equations (e.g., Cox and Ross, 1976; Cox et al., 1985; Merton, 1973). But is that necessary or ideal for a general claims problem? Can we somehow algebraically span the underlying payoff and then price the claim? What are the distinct advantages to adopting one approach over the other? To render these statements more precise and to seek answers to the above questions, define the characteristic function of the state-price density as follows (see Lukacs, 1960):

$$f(t, \tau; \phi) \equiv E_t^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right) \times \mathrm{e}^{\mathrm{i}\phi X(t+\tau)}\right\} \tag{6}$$

$$= \int_\Omega \mathrm{e}^{\mathrm{i}\phi X(t+\tau)} \exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right) q(v)\,\mathrm{d}v, \tag{7}$$

which is implicitly the time t price of a hypothetical claim that pays $\mathrm{e}^{\mathrm{i}\phi X(t+\tau)}$ (where $i = \sqrt{-1}$ and ϕ is some parameter of the contract) at date $t+\tau$. Since $\mathrm{e}^{\mathrm{i}\phi X(t+\tau)} = \cos(\phi X(t+\tau)) + \mathrm{i}\sin(\phi X(t+\tau))$ by Euler's identity, the payoff on characteristic functions is mathematically composed of trigonometric sine and cosine.

Technically, the characteristic function formulated in (7) is well defined even without the inclusion of a time-value factor. Indeed, every admissible characteristic function in the classical theory is unity at $\phi = 0$. Certainly, what we have described in (6) is the intrinsic value of a trigonometric payoff. It is therefore not improper to call $f(t, \tau; \phi)$ a discounted (or spot) characteristic function. Subject to this caveat and to avoid introducing fresh terminology, $f(t, \tau; \phi)$ will be referred to as a characteristic function throughout. Notice that we could have started with the joint characteristic function $\tilde{f}(t, \tau; \phi, \varphi) \equiv E_t^Q\{\exp(\mathrm{i}\varphi\int_t^{t+\tau} r(u)\,\mathrm{d}u + \mathrm{i}\phi X(t+\tau))\}$ which is the futures, marked-to-market, price of a claim that pays $\exp(\mathrm{i}\varphi\int_t^{t+\tau} r(u)\,\mathrm{d}u + \mathrm{i}\phi X(t+\tau))$ at time $t+\tau$. Clearly, the entity in (6) is a special case with $f(t, \tau; \phi) = \tilde{f}(t, \tau; \phi, \mathrm{i})$. As we will see, all claims contingent on $\int_t^{t+\tau} r(u)\,\mathrm{d}u$ and $X(t+\tau)$ are in the span of $\tilde{f}(t, \tau; \phi, \varphi)$, but are not necessarily spanned by $f(t, \tau; \phi)$. To allow for condensed discussion, we concentrate solely on examining the implications of the characteristic function in (6). When pricing correlation options, we study this abstraction again. Ignoring extreme counterexamples, the characteristic function is infinitely differentiable as

$$\left|\int_\Omega \exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right)(\mathrm{i}X)^n \mathrm{e}^{\mathrm{i}\phi X} q(v)\,\mathrm{d}v\right| < \infty, \quad n = 1, 2, \ldots, \infty$$

with finite algebraic moments of all orders. The characteristic function satisfies

$$\tfrac{1}{2}\mathrm{tr}[\sigma\sigma' f_{xx}] + \mu f_x - f_\tau - rf = 0 \tag{8}$$

subject to $f(t+\tau, 0; \phi) = \mathrm{e}^{\mathrm{i}\phi X(t+\tau)}$. While valuation equation (8) and its surrogate in (5) are observationally indistinguishable, the boundary condition for the characteristic function is mathematically more tractable, the former being smooth and infinitely differentiable, while the latter fails to be differentiable. As we will establish, the only challenge remaining is to analytically determine the characteristic function.

For future reference, let Re[.] denote the real part of the expression and $\mathscr{L}^1(\mathscr{C}^2)$ the space of integrable (twice continuously differentiable) functions. Formally, the payoff function $\mathscr{H}(X)$ is said to be of class $\mathscr{L}^1$ if $\int_{-\infty}^{\infty} |\mathscr{H}(X)|\,\mathrm{d}X < \infty$. Observe that the call payoff is $\mathscr{L}^1$ modulo an affine position (i.e., $\max(0, X-K) - (X-K) = \max(0, K-X)$). Motivated by such an implication, define for some constants λ_b and λ_x, universal payoffs of the type

$$\mathscr{G} \equiv \{\mathscr{H}(X) \mid \mathscr{H}(X) - \lambda_b - \lambda_x X \in \mathscr{L}^1\}, \tag{9}$$

which encompasses payoff functions of wider appeal. With this said, we now compare the span of $\mathrm{e}^{\mathrm{i}\phi X}$ and $\max(0, X-K)$ and analyze the methodology from different perspectives.

Theorem 1. *The following relations hold in arbitrage-free economies*:

(a) *For generic claim payoffs* $\mathscr{G}$, *the continuum of characteristic functions* (*indexed by* ϕ) *and the continuum of options* (*indexed by* K) *are equivalent classes of spanning securities. Thus, there exist coefficients* $w(\phi) \in \mathscr{L}^1$ *and* $z(K)$ *such that* $\mathscr{H}(X) = \lambda_b + \lambda_x X + \int_{-\infty}^{\infty} \mathrm{Re}[w(\phi)\mathrm{e}^{\mathrm{i}\phi X}]\,\mathrm{d}\phi$, *or* $\mathscr{H}(X) = \lim_{N\to\infty}\{\lambda_b^N + \lambda_x^N X + \int_0^{\infty} z^N(K) \max(0, X-K)\,\mathrm{d}K\}$ *with convergence in the* $\mathscr{L}^1$ *norm, and* λ_b^N, λ_x^N *and* $z^N(K)$ *are as displayed in* (67)–(69) *of the appendix.*

(b) *The call price in* (5) *can be unbundled into a portfolio of Arrow–Debreu securities*

$$C(t, \tau; K) = G(t, \tau)\Pi_1(t, \tau) - KB(t, \tau)\Pi_2(t, \tau), \tag{10}$$

where $B(t, \tau)$ *is the time* t *price of a discount bond with* τ *periods remaining to expiration*, $G(t, \tau)$ *represents the time* t *price of a commitment to deliver at time* $t+\tau$ *the quantity* $X(t+\tau)$ (*scaled-forward price*), *and*

$$\Pi_1(t, \tau) \equiv \frac{\int_{\mathscr{X}} \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u) X(t+\tau) q(v)\,\mathrm{d}v}{\int_{\Omega} \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u) X(t+\tau) q(v)\,\mathrm{d}v}$$

and

$$\Pi_2(t, \tau) \equiv \frac{\int_{\mathscr{X}} \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u) q(v)\,\mathrm{d}v}{\int_{\Omega} \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u) q(v)\,\mathrm{d}v}$$

are, respectively, the time t *prices of Arrow–Debreu securities.*

(c) *Each constituent security required for option valuation in* (10) *can be recovered from the characteristic function* $f(t, \tau; \phi)$ *as follows*:

- *The discount bond price and the scaled-forward price respectively obey*

$$B(t, \tau) = f(t, \tau; 0), \tag{11}$$

$$G(t, \tau) = \frac{1}{\mathrm{i}} \times f_{\phi}(t, \tau; 0), \tag{12}$$

where $f_{\phi}(t, \tau; \phi)$ *denotes the partial derivative of* $f(t, \tau; \phi)$ *with respect to* ϕ.

- *The time t price of each Arrow–Debreu security, for* $j = 1, 2$, *is*

$$\Pi_j(t, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi K} \times f_j(t, \tau; \phi)}{\mathrm{i}\phi}\right] \mathrm{d}\phi. \tag{13}$$

The characteristic functions for Arrow–Debreu securities, $f_j(t, \tau; \phi)$, *for* $j = 1, 2$, *are determined from* $f(t, \tau; \phi)$ *as made exact below*:

$$f_1(t, \tau; \phi) = \frac{1}{\mathrm{i}G(t, \tau)} \times f_{\phi}(t, \tau; \phi), \tag{14}$$

$$f_2(t, \tau; \phi) = \frac{1}{B(t, \tau)} \times f(t, \tau; \phi), \tag{15}$$

where it is understood that $f(t, \tau; \phi)$ *is available in closed form by solving the valuation equation* (8) *or the conditional expectation* (7).

The upshot that the continuum of characteristic functions and the continuum of options are equivalent classes of spanning securities in the space of $\mathscr{L}^1$ plus affine positions is perhaps not surprising: the payoff on trigonometric functions (options) can be synthesized from options (trigonometric functions). One can also envision this portion of Theorem 1 as saying that the residual $\{\mathscr{H}(X) - \lambda_b - \lambda_x X - \int_{-\infty}^{\infty} \mathrm{Re}[w(\phi)\mathrm{e}^{\mathrm{i}\phi X}]\, \mathrm{d}\phi\}$ is approximately zero in measure-theoretic sense, and likewise for options (e.g., Green and Jarrow, 1987; Nachman, 1988; Ross, 1976). If $\lambda_b = \lambda_x = 0$ and hence $\mathscr{H}(X) \in \mathscr{L}^1$, from Fourier theory, the static policy in the continuum of characteristic functions is complex valued $w(\phi) = (1/2\pi)\int_{-\infty}^{\infty} \mathscr{H}(X)\mathrm{e}^{-\mathrm{i}\phi X}\, \mathrm{d}X$. Granted that $w(\phi) = w_1(\phi) + \mathrm{i}w_2(\phi)$, the admissible trading strategy involves combining a long position $w_1(\phi)$ in $\cos(\phi X)$ and a short position $w_2(\phi)$ in $\sin(\phi X)$ for each ϕ. Substituting the Fourier coefficients into the spanning relation and exploiting the linearity of the martingale pricing rule, the arbitrage-free value of any claim, in terms of the characteristic function, then becomes $(1/2\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Re}[f(t, \tau; \phi)\, \mathscr{H}(X)\mathrm{e}^{-\mathrm{i}\phi X}]\, \mathrm{d}X\, \mathrm{d}\phi$. Otherwise, if $\lambda_b \neq 0$ and $\lambda_x \neq 0$, the claim value should be adjusted by the time t price of the underlying asset and the discount bond accordingly. Eqs. (A.9) and (A.13) of the appendix, respectively, reveal how $\mathscr{H}(X) \in \mathscr{C}^2$ and $\mathscr{H}(X) \in \mathscr{L}^1$ can be synthetically constructed from call options. In spanning the former group of claims, there exist linear combinations in the

Lebesgue continuum of strikes; however, in the latter, spanning is in the $\mathscr{L}^1$ norm.

Central to the methodology, the second part of Theorem 1 asserts that the call option price can be decomposed into a portfolio of Arrow–Debreu securities. It implicitly maintains that knowing the price of four primitive securities (i.e., the matching discount bond, the scaled-forward price, and the two Arrow–Debreu securities) is equivalent to solving the option valuation problem. To briefly see the logic behind this decomposition, notice that, by the definition of state-price density, $B(t,\tau)=\int_\Omega \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)q(v)\,\mathrm{d}v$ and the scaled-forward price is $G(t,\tau)\equiv E_t^Q\{\exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)X(t+\tau)\}=\int_\Omega \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)X(t+\tau)q(v)\,\mathrm{d}v$. By appealing to the same deduction and using (3), we can rigorously represent

$$\Pi_1(t,\tau)=\frac{\int_{\mathscr{X}}\exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)X(t+\tau)q(v)\,\mathrm{d}v}{\int_\Omega \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)X(t+\tau)q(v)\,\mathrm{d}v} \tag{16}$$

$$\equiv E_t^{Q*}\{1_{\mathscr{X}}\}, \tag{17}$$

where the indicator function $1_{\mathscr{X}}$ is unity when $X(t+\tau)>K$ and zero otherwise. In deriving (17), we have utilized the Radon–Nikodym derivative

$$\frac{\mathrm{d}Q^*}{\mathrm{d}Q}=\frac{\exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)\times X(t+\tau)}{G(t,\tau)}.$$

Clearly, $\Pi_1(t,\tau)$ is the price of an Arrow–Debreu security, albeit under a transformed equivalent probability measure. By an analogous argument,

$$\Pi_2(t,\tau)=\frac{\int_{\mathscr{X}}\exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)q(v)\,\mathrm{d}v}{\int_\Omega \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)q(v)\,\mathrm{d}v} \tag{18}$$

$$\equiv E_t^{Q**}\{1_{\mathscr{X}}\} \tag{19}$$

is a well-posed Arrow–Debreu security with

$$\frac{\mathrm{d}Q^{**}}{\mathrm{d}Q}=\frac{\exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)}{B(t,\tau)}.$$

As our reliance is on elementary properties of probability density functions, the option decomposition holds for arbitrary risk structures and is valid to valuation problems in discrete time or in continuous time, a feature that can only induce broad theoretical applicability of Theorem 1.[3]

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[3] Consider a claim in a two-period, three-date model written on a nontraded underlying asset like the price of electricity at date two. Call this uncertainty x with density $q(x)$ and characteristic function $f(x;\phi)$. For simplicity, assume deterministic interest rates. Then, one can verify that the core analysis of Theorem 1 goes through. In non-Markovian, jump-diffusion, or pure-jump environments, the theoretical developments are essentially similar. Our goal is to avoid repetition, so we exclude such extended analysis. Duffie et al. (1998) provide a more technical treatment on the determination of $\Pi_1(t,\tau)$ and $\Pi_2(t,\tau)$ in the context of affine jump-diffusions.

The final part of Theorem 1 is the real driving force behind the valuation approach, however. Consistent with the task at hand, the manipulation of the characteristic function $f(t, \tau; \phi)$ simultaneously and jointly recovers the term structure of interest rates, the term structure of forward prices, and the two required Arrow–Debreu securities. While it is known that options are market completing (from Ross, 1976), the spanning properties of the characteristic function are not that transparent and not fully appreciated in their entirety. In fact, as $f(t, \tau; \phi) \equiv \int_{\Omega} \mathrm{e}^{\mathrm{i}\phi X(t+\tau)} \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u) q(v)\,\mathrm{d}v$, economically it amounts to a Fourier transform of the state-price density function and is hence basis augmenting. In particular, the resulting basis is endowed with two theoretically appealing properties: it is analytical and orthonormal (in $\mathscr{L}^2([0, 2\pi])$ and in the space of almost periodic functions. Observe that by translating or differentiating the characteristic function, one can synthesize the values of the exponential and polynomial of the underlying uncertainty. The reference measure, for example, is being transformed from, say, $q(v)$ to $X\,q(v)$ on differentiation, and to $e^X q(v)$ on translation. This is precisely the reason that $\Pi_1(\Pi_2)$ can be priced by differentiation (translation) and Fourier transformation (see also Cases 1 and 2 to follow). These simplifications are made achievable without deriving the state-price density function (which is in principle inferable). Moreover, as differentiation holds the key to constructing a polynomial basis, the superiority of characteristic functions as a primary collection of spanning securities is evident. After all, the reverse construction contains infinite series summations.

Theorem 1 should not be interpreted to mean that call options are in the span of trigonometric functions via Fourier theory. Nowhere have we established the spanning representation, i.e., $\max(X - K, 0) = \int_{-\infty}^{\infty} \mathrm{Re}[w(\phi)\mathrm{e}^{\mathrm{i}\phi X}]\,\mathrm{d}\phi$, for some $w(\phi)$. In fact, this is certainly not even possible using integrable $w(\phi)$ because the call payoff is unbounded and outside of $\mathscr{L}^1$ of Lebesgue measure. From characteristic functions, one can nonetheless build a large class of functions and also value them. Because the put option payoff is in $\mathscr{L}^1$, however, it is algebraically spanned in that there exist linear combinations in the Lebesgue continuum of transform variates ϕ. That is, $w(\phi) = (1/2\pi)\int_{-\infty}^{\infty} \max(0, K - X)\mathrm{e}^{-\mathrm{i}\phi X}\,\mathrm{d}X$. As the call payoff is $\mathscr{L}^1$ modulo $X - K$, it can therefore be tailored by investing in (i) the continuum of characteristic functions, (ii) the underlying asset, and (iii) the discount bond. The precise long position $w_1(\phi)$ in $\cos(\phi X)$ and the short position $w_2(\phi)$ in $\sin(\phi X)$ that mimic the put payoff are displayed in (60) and (61) of the appendix. By the same token, reformulating the delta security payoff as $1 - 1_{\{X<K\}}$ does not contradict the impression that Π_1 and Π_2 can be synthesized from the continuum of characteristic functions in collaboration with a discount bond (even though each security payoff violates the $\mathscr{L}^1$ requirement). Thus, Eq. (13) is a mere byproduct of spanning and pricing via characteristic functions. Such affine payoffs as the discount bond (the underlying) can be specialized from the trigonometric payoff by setting $\phi = 0$ (differentiating

and then substituting $\phi = 0$). In sum total, characteristic functions are robust spanning engines not only for payoffs in $\mathscr{L}^1$ but also in the expanded collection of $\mathscr{L}^1$ plus affine security positions.

We have admittedly bypassed a few abstract questions in our inquiry. What is the exact span of characteristic functions and options? What is the relation between the algebraic span of options and characteristic functions? Under what circumstances is one span larger or smaller than its counterpart? For example, $X^2 \in \mathscr{C}^2$ (and $e^{e^x} \in \mathscr{C}^2$) is in the algebraic span of options but not so for characteristic functions using $\mathscr{L}^1$. Yet if attention is restricted to some compact interval $(-\ell, \ell)$, then from Fourier theory, X^2 will also be in the span of characteristic functions as well. Stated differently, a wider net of securities can be spanned by the characteristic function on compact intervals. In particular, claims that belong to $\mathscr{L}^1(Q)$ (i.e., $\int_{-\infty}^{\infty} |\mathscr{H}(X)|\, q(X) dX < \infty$) can be approximated in the $\mathscr{L}^1(Q)$ norm by claims possessing compact support. Thus, contingent claims satisfying this working criterion can be spanned and priced as depicted above. Evidently, $X^2\, e^{i\phi X}$ and successive characteristic function derivatives are not an $\mathscr{L}^1$ object for all ϕ, but can be valued anyway from characteristic function (6).

That the characteristic function and consequently all contingent claims in its basis can be priced by solving a single valuation equation (partial or integro-differential) is methodologically important.[4] The reader will recall that the traditional approach to contingent claim/option valuation pioneered by Cox et al. (1985), hereafter CIR, and Merton (1973) involves developing a fundamental valuation equation such as the one posited in (5). While circumventing the need to solve for the state-price density, this approach nonetheless demands a correct candidate conjecture. By analogy, the conjecture is in the family of (10). Substituting the conjecture into the fundamental equation generally produces at most four additional valuation equations or the corresponding conditional expectations, i.e., one each for $G(t, \tau)$, $B(t, \tau)$, Π_1, and Π_2. But claim valuation is not yet entirely complete as one must now again, by trial and

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[4] It has been pointed to us that the scaled-forward price, $G(t, \tau)$, is predetermined for a broad class of option contracts. As a general rule, this notion is flawed. Options on futures (under random volatility and random interest rates/convenience yields) are an obvious counterexample. Likewise, for the entire family of interest rate options and nonstandard contingent claims, the scaled forward price is anything but known a priori. To guide consensus, it is demonstrated in the later example exercises that the scaled-forward price embedded in the average interest-rate and knock-out options with payoff, say, $\prod_{n=1}^{N} \max(0, s(t + n\,\Delta t) - K)$, are hard to conjecture. While a large literature exists on the term structure of interest rates, the spirit of the above remarks equally applies to discount bond prices (although to a lesser extent). Nonetheless, a systematic way to determine the scaled-forward price and the discount bond price is desirable and warranted. In particular, the characteristic function of the stochastic discount factor can serve a similarly useful role in dynamic equilibrium economies. Details are omitted here.

error, conjecture a solution to each of the four valuation equations.[5] While the four-step valuation methodology is technically correct and has lead to numerous theoretical advances and model refinements, it is, nevertheless, cumbersome and imposes a tight constraint on the valuation structure: if one component valuation is unsolved, it gridlocks contingent claims valuation. In contrast, the availability of the characteristic function renders claims valuation complete in the same single step (and hence weeds out solving complex valuation equations).

On a related theme, notice that affiliated with each constituent security valuation is also a set of ordinary differential equations (after a solution is conjectured). Adopting the spanning and pricing strategy will also eliminate the need to solve a large set of ordinary differential equations. Translating and differentiating a smooth function is clearly trivial relative to solving additional valuation equations (PDE or integro-differential) or additional ordinary differential equations. But keep in mind that our simplifications do not apply to the characteristic function of the state-price density which must be available in closed form by solving the valuation equation (8) or the conditional expectation (7). However, due to the characteristic function's exponential boundary condition, this quantity is easier to solve in general than is the option price directly.

While sharing with Heston (1993) the feature that each pure security price is reverse-engineered from the respective characteristic function, the treatment here differs fundamentally. In Heston, the two characteristic functions are, for instance, obtained mostly by solving two separate valuation equations and by conjecturing their solutions; see Eqs. (12) and (22) in Heston. Under our technique, their recovery is through the characteristic function of the state-price density. Our economic analysis makes explicit how the two characteristic functions are intrinsically linked in that the first characteristic function is either a translate or a derivative of its counterpart. More specifically, existing works tend to blur the recursive structure of option valuation; seldom have they tapped into the unifying spanning concept. In this regard, there are crucial lessons to examining claims with payoffs that are variants of the original one in Eq. (1). Under the Heston framework, the entire set of valuation equations must be resolved all over again (including a conjecture for the original valuation equation). This is, however, not the case under our derivative valuation approach. Knowing the characteristic function of the state-price density will

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[5] In their model (see also Constantinides, 1992; Longstaff and Schwartz, 1992), the discount bond price and the European option written on it satisfy the same fundamental partial differential equation (PDE). For the option formula to be internally consistent with the valuation PDE, the (two) noncentral chi-squared distributed probabilities and the (two) discount bond prices will satisfy unique valuation equations of their own (with a distinct boundary condition). Each component valuation security price is then explicitly computed by solving the relevant expectation.

automatically determine the intrinsic value of the cash-flow streams, as is demonstrated below.

Case 1. Let the claim payoff be $\max(0, X^2(t+\tau) - K)$ with exercise region $X(t+\tau) > \sqrt{K}$. Despite this nonlinear transformation, the claim price still satisfies (10) with (as before) $B(t, \tau) = f(t, \tau; 0)$ and $f_2(t, \tau; \phi) = [1/B(t, \tau)]\, f(t, \tau; \phi)$. In accordance with Theorem 1,

$$G(t, \tau) = \frac{1}{\mathrm{i}^2} f_{\phi\phi}(t, \tau; 0), \tag{20}$$

$$f_1(t, \tau; \phi) = \frac{1}{\mathrm{i}^2 G(t, \tau)} f_{\phi\phi}(t, \tau; \phi), \tag{21}$$

where $f_{\phi\phi}(t, \tau; \phi)$ denotes the second-order partial derivative of $f(t, \tau; \phi)$ with respect to ϕ and

$$\Pi_j(t, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi\sqrt{K}} \times f_j(t, \tau; \phi)}{\mathrm{i}\phi}\right] \mathrm{d}\phi, \quad \text{for } j = 1, 2.$$

The option claims on successive (higher) algebraic moments and other (integer) polynomials can be priced correspondingly.

Case 2. Alter $C(t+\tau, 0; K) = \max(0, \mathrm{e}^{X(t+\tau)} - K)$. Here, $B(t, \tau) = f(t, \tau; 0)$ and $f_2(t, \tau; \phi) = [1/B(t, \tilde{\tau})] f(t, \tau; \phi)$ with

$$G(t, \tau) = f(t, \tau; -\mathrm{i}), \tag{22}$$

$$f_1(t, \tau; \phi) = \frac{1}{G(t, \tau)} f(t, \tau; \phi - \mathrm{i}) \tag{23}$$

and

$$\Pi_j(t, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi \ln[K]} \times f_j(t, \tau; \phi)}{\mathrm{i}\phi}\right] \mathrm{d}\phi \quad \text{for } j = 1, 2. \tag{24}$$

If we set $X(t+\tau) = \ln[S(t+\tau)]$, then (22)–(24) accommodate, as a special parametric case, most equity option models with $f_1(t, \tau, \phi)$ and $f_2(t, \tau; \phi)$ translated (such as the ones in Bakshi et al., 1997; Duffie et al., 1998; Heston, 1993; Hull and White, 1987; Scott, 1997; Stein and Stein, 1991).

Case 3. To see the comprehensive nature of the approach, consider an arbitrary option-like payoff $\max(0, H[X(t+\tau)] - K)$ for some (differentiable) $H[X] > 0$. This contract, however, mandates the knowledge of $M(t, \tau; \phi) \equiv \int_\Omega \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)\, \mathrm{e}^{\mathrm{i}\phi X(t+\tau)} H[X(t+\tau)]\, q(v)\,\mathrm{d}v$. Appendix substantiates how

$M(t, \tau; \phi)$ and the price of the call is inferable from $f(t, \tau; \phi)$:

$$G(t, \tau) = M(t, \tau; 0), \tag{25}$$

$$\Pi_1(t, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi H^{-1}[K]} \times M(t, \tau; \phi)}{\mathrm{i}\phi \times M(t, \tau; 0)}\right]\mathrm{d}\phi \tag{26}$$

and

$$\Pi_2(t, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi H^{-1}[K]} \times f(t, \tau; \phi)}{\mathrm{i}\phi \times f(t, \tau; 0)}\right]\mathrm{d}\phi, \tag{27}$$

where $B(t, \tau) = f(t, \tau; 0)$ and

$$M(t, \tau; \phi) = \bar{f}(t, \tau; \phi) \times H(X_0) + \sum_{n=1}^{\infty} \frac{\mathscr{A}_n \times \mathscr{B}_n}{\mathrm{i}^n \times n!} \tag{28}$$

with

$$\mathscr{A}_n \equiv \frac{\partial^n H}{\partial X^n}(X_0), \quad \mathscr{B}_n \equiv \frac{\partial^n \bar{f}(t, \tau; \phi)}{\partial \phi^n} \quad \text{and} \quad \bar{f}(t, \tau; \phi) \equiv \mathrm{e}^{-\mathrm{i}\phi X_0} f(t, \tau; \phi)$$

as a stand-in for the characteristic function of the translated uncertainty $X(t+\tau) - X_0$ (for some constant X_0). To fix ideas, suppose $H[X] = X(t+\tau)^\delta$ for $-\infty < \delta < \infty$. Then, all fractional power claims can be priced explicitly in closed form, as can other similarly rich arbitrary claims on $H[X]$ with $K = 0$.

Regardless of how our alternative approach is interpreted, it brings out a valuation aspect of immense practical interest. That is, it fills in the gap by making explicit the technical conditions under which the two probability elements can be members of similar, or distinct, parametric classes. For a general class of cases, Π_1 and Π_2 are connected in a precise way, and this quantitative relationship is characterized best in terms of the transforms of the state-price densities. Specifically, this consists of situations in which the payoff function on which the option is written, e.g., a positive function, $H[X]$, is functionally related to a monotone transformation of the underlying uncertainty, say $h[X]$. Hypothesize

$$H[X] = \sum_{n=1}^{N} \alpha_n h[X]^n + \sum_{j=1}^{J} \beta_j \, \mathrm{e}^{\gamma_j h[X]}. \tag{29}$$

In this case, $f(t, \tau; \phi) \equiv \int_\Omega \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)\, \mathrm{e}^{\mathrm{i}\phi h[X]} q(v)\,\mathrm{d}v$, and $f_2(t, \tau; \phi) = f(t, \tau; \phi)/f(t, \tau; 0)$, which allows us to deduce the general restriction

$$f_1(t, \tau; \phi) = \frac{B(t, \tau)}{\mathrm{i}G(t, \tau)}\left\{\sum_{n=1}^{N} \frac{\alpha_n}{\mathrm{i}^n} \times \frac{\partial^n f_2(t, \tau; \phi)}{\partial \phi^n} + \sum_{j=1}^{J} \beta_j f_2(t, \tau; \phi - \mathrm{i}\gamma_j)\right\}, \tag{30}$$

where α_n, β_j, and γ_j are arbitrary constants with

$$G(t,\tau) = B(t,\tau)\left\{ \sum_{n=1}^{N} \frac{\alpha_n}{\mathrm{i}^n} \times \frac{\partial^n f_2(t,\tau;0)}{\partial \phi^n} + \sum_{j=1}^{J} \beta_j f_2(t,\tau;-\mathrm{i}\gamma_j) \right\}. \tag{31}$$

In particular, when the option claim is on the exponential of the uncertainty, i.e., $h[X] = \ln(X)$, $\alpha_n = 0$ for all n, $\beta_1 = \gamma_1 = 1$, and $\beta_j = 0$ for $j > 1$, then the characteristic function corresponding to Π_1 and Π_2 are translates of one another. This is why Π_1 and its counterpart inherit the same parametric class. This property, for instance, induces probability structures in CIR (which are noncentral chi-squared) and stochastic volatility equity option models (the characteristic functions are each exponential affine) that fall internally in the same family of distributions. But when the claim is contingent on the level of the uncertainty (or its powers and polynomials), the first characteristic function is obtained by differentiation and the second by translation. Derivatives so priced are, thus, composed of Arrow–Debreu securities that are generally dissimilar in their probability-theoretic foundations. To reverse the situation, keep $H[X]$ unrestricted but specialize $h[X] = X$. Eqs. (26)–(28) of Case 3 reillustrate the same dichotomy: the transform of Π_1 is analytically linked to its counterpart as its consecutive derivative (by replacing $f(t,\tau;\phi)$ with $B(t,\tau)f_2(t,\tau;\phi)$). For properties such as bounds on deltas and state prices in one-dimensional diffusion economies, see Grundy and Wiener (1996).

At a conceptual level, Cases 1 and 3 highlight a subtle, yet crucial, attribute of the valuation paradigm: the delta claim Π_2 is comparatively easier to determine than Π_1. In other words, when Π_1 and Π_2 lie in different parametric classes, it is trickier to guess solutions to $f_1(t,\tau)$ and $G(t,\tau)$ and hence to the composite option problem. Consequently, when pricing nontraditional and exotic derivatives, our simplification is more about determining Π_1 and $G(t,\tau)$ rather than Π_2. We revisit this theme when pricing average-rate interest rate claims. Having said this, we move on to the pricing of specific contingent claims. Each option-like security is novel and shares a common denominator: no closed form solutions have yet been discovered (to our knowledge), even though the characteristic function (of the remaining uncertainty) is straightforward. These derivative securities are all intended to capture the essence and richness of the spanning-induced simplification.

3. Average-rate interest rate options

Inspired by the preceding analysis, the remainder of this section documents how a broad class of path-dependent claims can be valued using our methodology. To maintain sharp focus, adopt a payoff structure that is average-interest-rate contingent. Set the initial date for the averaging interval to be time 0 (with no loss of generality) and specify the payoff on the average-rate call as $C(t+\tau, 0; K) = \max(0, [1/(t+\tau)]\int_0^{t+\tau} r(u)\,\mathrm{d}u - K)$ where the time t call option price is denoted by $C(t, \tau; K)$. To avoid free lunches,

$$C(t, \tau; K) = E_t^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right)\right.$$
$$\left.\times\left[\frac{1}{t+\tau}\int_t^{t+\tau} r(u)\,\mathrm{d}u + \frac{A(t)}{t+\tau} - K\right]\times 1_{\mathscr{E}}\right\}, \tag{32}$$

where $1_{\mathscr{E}}$ stands for an indicator variable that is unity when the call is exercised (and zero otherwise), $\mathscr{E} \equiv \{\int_t^{t+\tau} r(u)\,\mathrm{d}u > [t+\tau]K - A(t)\}$, and $A(t) \equiv \int_0^t r(u)\,\mathrm{d}u$. Therefore,

$$\mathrm{d}A(t) = r(t)\,\mathrm{d}t \tag{33}$$

from Leibnitz's differentiation rule.

For its theoretical tractability, assume that the law of motion for the spot interest rate, r, is governed by the single-factor CIR-type square-root process (θ, κ, and σ are all positive constants):

$$\mathrm{d}r(t) = \kappa(\theta - r(t))\,\mathrm{d}t + \sigma\sqrt{r(t)}\,\mathrm{d}\omega_r(t). \tag{34}$$

Because $(r(t), A(t))$ form a Markov system from (33) and (34), standard steps produce the valuation PDE for the average-rate call below (subject to the call payoff):

$$\tfrac{1}{2}\sigma^2 r C_{rr} + \kappa(\theta - r)C_r - C_\tau - rC + rC_A = 0, \tag{35}$$

where the subscripts C_r and C_{rr} respectively represent, for instance, the first- and second-order partial derivative with respect to r. If an alternative single-factor or multiple-factor interest rate model is used as a benchmark instead, the characteristic function will be slightly more difficult to solve analytically as in Constantinides (1992), Longstaff (1989), or in the Markovian jump-diffusion mixture class. However, the main thrust of this section is invariant to the choice of interest rate model. Two additional points are worth mentioning. First, as the evolution of $r(t)$ is under the martingale measure, the interest rate factor risk premium is already reflected in the drift, $\kappa(\theta - r)$. With $A(t)$ deterministic, no risk compensation is required for this state variable. Second, by virtue of its dependence on $A(t)$, the option price is path dependent. As a consequence, the valuation equation (35) differs from its now famous counterparts (e.g., CIR; Constantinides, 1992; Longstaff and Schwartz, 1992). Essentially, the induced path dependence has made valuation intractable and no (complete) analytical characterizations have yet been proposed (for single or multifactor interest rate processes). Bakshi and Madan (1997), Chacko and Das (1997), Geman and Yor (1993), Ju (1997), and Zhang (1998), among others, document on how average-rate derivatives are routinely adopted by practitioners to manage interest rate and commodity price risk. The incremental contribution of this paper will be noted shortly.

Despite the hurdles in solving (35), directed by Theorem 1, the characteristic function is the sole building block for spanning and pricing all average-rate contingent claims. To articulate this point in sufficient detail, it is first verified in the appendix that

$$
\begin{aligned}
f(t,\tau;\phi) &\equiv \mathrm{E}_t^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right)\times\exp\left(\mathrm{i}\phi\int_t^{t+\tau} r(u)\,\mathrm{d}u\right)\right\}\\
&= \exp[-\mathscr{M}(\tau;\phi)-\mathscr{N}(\tau;\phi)\,r(t)], \qquad (36)
\end{aligned}
$$

where $\mathscr{M}(\tau;\phi)$ and $\mathscr{N}(\tau;\phi)$ are defined in (A.32) and (A.33) of the appendix. Next, relying on a parallel theoretical development and (36), the solution to (35) is as (recursively) posited below:

Proposition 1. *The call option price on average interest takes the form*

$$
C(t,\tau;K)=\frac{G(t,\tau)}{t+\tau}\,\Pi_1(t,\tau)-\left\{K-\frac{A(t)}{t+\tau}\right\}B(t,\tau)\,\Pi_2(t,\tau), \qquad (37)
$$

where $B(t,\tau)=f(t,\tau;0)=\exp[-\mathscr{M}(\tau;0)-\mathscr{N}(\tau;0)\,r(t)]$ with $f(t,\tau;\phi)$ *as postulated in* (36) *and the time t scaled-forward price is*

$$
\begin{aligned}
G(t,\tau) &= \frac{1}{\mathrm{i}}\times f_{\phi}(t,\tau;0)\\
&= \lim_{\phi\to 0}\frac{1}{\mathrm{i}}\times\exp[-\mathscr{M}(\tau;\phi)-\mathscr{N}(\tau;\phi)r(t)]\times\left\{-\frac{\partial\mathscr{M}}{\partial\phi}-\frac{\partial\mathscr{N}}{\partial\phi}r(t)\right\}
\end{aligned}
\qquad (38)
$$

for some (easily computable) functions $\partial\mathscr{M}/\partial\phi$ *and* $\partial\mathscr{N}/\partial\phi$. *The time t price of delta securities, for* $j=1,2$, *is*

$$
\Pi_j(t,\tau)=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi[(t+\tau)K-A(t)]}\times f_j(t,\tau;\phi)}{\mathrm{i}\phi}\right]\mathrm{d}\phi, \qquad (39)
$$

with the first characteristic function determined from

$$
\begin{aligned}
f_1(t,\tau;\phi) &= \frac{1}{\mathrm{i}G(t,\tau)}\times f_{\phi}(t,\tau;\phi)\\
&= \frac{1}{\mathrm{i}G(t,\tau)}\times\exp[-\mathscr{M}(\tau;\phi)-\mathscr{N}(\tau;\phi)r(t)]\left\{-\frac{\partial\mathscr{M}}{\partial\phi}-\frac{\partial\mathscr{N}}{\partial\phi}r(t)\right\},
\end{aligned}
\qquad (40)
$$

and the second characteristic function is

$$f_2(t, \tau; \phi) = \frac{1}{B(t, \tau)} \exp[-\mathcal{M}(\tau; \phi) - \mathcal{N}(\tau; \phi) r(t)]. \tag{41}$$

Formula (37) constitutes an exact closed-form solution to the option on average interest.[6] It brings into forefront the valuation role of the characteristic function; by its translation and differentiation, all the underlying primitive securities can be recovered. To see how the spanning and pricing engine works in practice and to assess the extent of simplification it induces, take the first Arrow–Debreu security. Write the valuation PDE/Backward-equation (see (A.28)) as

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 \Pi_1}{\partial r^2} + \left\{ \kappa(\theta - r) + \sigma^2 r \frac{1}{G}\frac{\partial G}{\partial r} \right\} \frac{\partial \Pi_1}{\partial r} - \frac{\partial \Pi_1}{\partial \tau} + r \frac{\partial \Pi_1}{\partial A} = 0. \tag{42}$$

At first glance, it appears that no closed-form representation is possible for this class of PDEs. But as made precise in (40), differentiating the characteristic function and standardizing the resulting entity by $G(t, \tau)$ (which makes it a characteristic function for a probability) and using the inverse Fourier transformation pins down the solution to valuation equation (42). This step can be consistently implemented so long as the characteristic function is analytical. But the two modes of analysis are not strictly equivalent: one requires a simple differentiation step and the other requires intricate conjecturing abilities. Proceeding similarly, the scaled-forward price is in compliance with $\frac{1}{2}\sigma^2 r G_{rr} + \kappa(\theta - r) G_r - G_\tau - rG = -rB(t, \tau)$, and its closed-form formulation in (38) stems virtually from the same mechanism. In words, differentiating the characteristic function and evaluating the resulting expression at zero will replicate the conditional expectation for this vanishing contingent claim (notice that the price is monotonically declining with the passage of time). The pricing of the put can be achieved from put-call parity: $\{K - A(t)/(t + \tau)\} B(t, \tau) [1 - \Pi_2(t, \tau)] - (G(t, \tau)/(t + \tau))[1 - \Pi_1(t, \tau)]$.[7]

---

[6] Under the premise that r is a Bessel process and $r(u)$ are independent for all u, the average-rate call can be priced via Geman and Yor (1993) since Bessel processes are stable under additivity. Unfortunately, none of the existing processes fall into the viability set. When r is governed according to (34), it is obviously Bessel, although there is an autocorrelation problem. If the option is on a basket of securities $\sum_{j=1}^{J} \alpha_j x_j(t + \tau)$ for some loading α_j with the $x_j(t)$ all independent Bessel, then there are no valuation difficulties via either our approach or Geman and Yor.

[7] Ju analytically derives the Fourier transform of $\int_t^{t+\tau} r(u)\,\mathrm{d}u$ and then recovers the state-price density function via inverse transformation in his Eq. (24). His solution for pricing average-rate claims is numerical in nature; ours is superior and mathematically more elegant because of spanning. Nonetheless, the primary message is that the state-price density is generally redundant for derivative asset valuation if the characteristic function is known (see also Carr and Madan (1999)).

Proposition 1 imposes a stringent restriction on the pure securities Π_1 and Π_2. This is primarily so since $f_2(t, \tau; \phi)$ is exponential-affine, but $f_1(t, \tau; \phi)$ is surely outside of that class. Furthermore, $\Pi_2(t, \tau)$ is noncentral chi-squared distributed (it satisfies the Kolmogorov-backward equation for noncentral chi-squared variables, as in the CIR bond option formula). Thus, the average-rate option valuation problem is potentially one application in which Π_1 and Π_2 are not in the same parametric family of distribution functions. In attacking the exact same problem, Chacko and Das (1997) are unable to analytically characterize $G(t, \tau)$ and $f_1(t, \tau)$. As a result, their general focus is confined to the pricing of the Digital, i.e., $\Pi_2(t, \tau)$. On balance, conjecturing ad hoc solutions to option valuation problems tends to obscure the tight linkage between each Arrow–Debreu security and between $G(t, \tau)$ and $f_1(t, \tau)$.

The principal comparative statics findings are not at odds with what intuition suggests: the average-rate call is (i) increasing in $A(t)$ (the prior path dependence), and (ii) convex and increasing in $r(t)$. Numerical analysis indicates that higher interest rate primitives (i.e., θ, κ, and σ) all lead to a higher call price. By shorting $\mathcal{N}(t, \tau; 0)C_r$ units of the discount bond and going long $C(t, \tau) + \mathcal{N}(t, \tau; 0)\, C_r B(t, \tau)$ in cash (which makes the overall position self-financed), the call can be dynamically replicated. Because the partial derivative C_r is in analytical form, the closed-form characterization facilitates the execution of delta-neutral hedges.

Our approach can be adapted to price assorted payoff structures. To synthesize this dimension of the technique, take the option on the average yield as the basis. Maintaining the CIR interest rate dynamics, the yield to maturity $R(t, \tilde{\tau}) = \mathcal{M}(\tilde{\tau}; 0)/\tilde{\tau} + [\mathcal{N}(\tilde{\tau}; 0)/\tilde{\tau}]r(t)$. Let $\bar{C}(t, \tau)$ denote the call option price on this average yield with expiration τ in periods from time t. Then

$$\bar{C}(t, \tau; K) = \frac{\mathcal{N}(\tilde{\tau}; 0)}{(t+\tau)\tilde{\tau}} G(t, \tau)\bar{\Pi}_1(t, \tau) - \bar{K}\frac{B(t, \tau)}{(t+\tau)\tilde{\tau}}\bar{\Pi}_2(t, \tau), \tag{43}$$

where $\bar{K} \equiv \tilde{\tau}[t+\tau]K - [t+\tau]\mathcal{M}(\tilde{\tau}; 0) - \mathcal{N}(\tilde{\tau}; 0)A(t)$, and the risk-neutralized probabilities are

$$\bar{\Pi}_j(t, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{1}{\mathrm{i}\phi} \mathrm{e}^{-\mathrm{i}\phi\bar{K}/\mathcal{N}(\tilde{\tau};\, 0)} f_j(t, \tau; \phi)\right] \mathrm{d}\phi, \quad j = 1, 2, \tag{44}$$

where $G(t, \tau)$, $f_1(t, \tau; \phi)$, and $f_2(t, \tau; \phi)$ respectively are as given in (38) and (40)–(41). Option claims on higher algebraic moments and fractional powers and polynomials can also be accommodated in the same single step. In summary, by constructing the appropriate characteristic function, any interest rate derivative can be priced in closed form.

4. Correlation options

The preceding contingent claims are all written on a single underlying asset (and with a one-dimensional exercise region). However, valuation applications of special interest to financial economist often have payoffs dependent on two assets. To see how such claims can be priced within our framework, let $s(t)$ and $p(t)$ be the time t price of the two securities. Specify a generic payoff of the type

$$C(t+\tau,\, 0;\, K_s, K_p) = \max\left(0,\, \frac{s(t+\tau)}{s(t)} - K_s\right) \times \max\left(0,\, \frac{p(t+\tau)}{p(t)} - K_p\right), \tag{45}$$

where $C(t+\tau,\, 0;\, K_s, K_p)$ is the price of the correlation option at time $(t+\tau)$ and the respective strike prices are denoted by K_s and K_p. Since

$$1_{\{s(t+\tau)/s(t) > K_s\}} \times 1_{\{p(t+\tau)/p(t) > K_p\}} = \max(0,\, 1_{\{s(t+\tau)/s(t) > K_s\}} + 1_{\{p(t+\tau)/p(t) > K_p\}} - 1),$$

it is noteworthy, from Nachman (1988) and Ross (1976), that correlation options are market completing. The key result of this section is stated next.

Proposition 2. *Let* $s(t)$ *and* $p(t)$ *each be governed by the continuous-time stochastic processes* (*under the risk-neutral measure*) *below*:

$$\frac{\mathrm{d}s(t)}{s(t)} = r\,\mathrm{d}t + \sigma_s\sqrt{v(t)}\,\mathrm{d}\omega_s(t),$$

$$\frac{\mathrm{d}p(t)}{p(t)} = r\,\mathrm{d}t + \sigma_p\sqrt{v(t)}\,\mathrm{d}\omega_p(t),$$

$$\mathrm{d}v(t) = [\alpha - \beta v(t)]\,\mathrm{d}t + \sigma\sqrt{v(t)}\,\mathrm{d}\omega_v(t),$$

where $\mathrm{Cov}_t(\omega_s(t),\, \omega_p(t)) \equiv \eta$, $\mathrm{Cov}_t(\omega_s(t),\, \omega_v(t)) \equiv \rho_1$, $\mathrm{Cov}_t(\omega_p(t),\, \omega_v(t)) \equiv \rho_2$, *and* r *is the constant interest rate. For this problem, the joint characteristic function is*

$$f(t,\, \tau;\, \phi,\, \varphi) = \mathrm{E}_t^Q\left\{\exp\left(-r\tau + \mathrm{i}\phi \ln\left[\frac{s(t+\tau)}{s(t)}\right] + \mathrm{i}\varphi \ln\left[\frac{p(t+\tau)}{p(t)}\right]\right)\right\} \tag{46}$$

$$= \exp[\mathscr{Y}(t,\, \tau;\, \phi,\, \varphi) + \mathscr{Z}(t,\, \tau;\, \phi,\, \varphi)v(t)], \tag{47}$$

where $\mathscr{Y}(t,\, \tau;\, \phi,\, \varphi)$ and $\mathscr{Z}(t,\, \tau;\, \phi,\, \varphi)$ *are displayed in* (A.35) *and* (A.36) *of the appendix. Then*

$$C(t,\, \tau) = f(t,\, \tau;\, -\mathrm{i},\, -\mathrm{i})\Pi_1(t,\, \tau) - K_s\Pi_2(t,\, \tau) - K_p\Pi_3(t,\, \tau) + K_sK_p\,\mathrm{e}^{-r\tau}\Pi_4(t,\, \tau), \tag{48}$$

where

$$\Pi_j(t,\ \tau) \equiv \mathrm{Prob}\left(\left\{\ln\left[\frac{s(t+\tau)}{s(t)}\right] > \ln[K_s]\right\} \cap \left\{\ln\left[\frac{p(t+\tau)}{p(t)}\right] > \ln[K_p]\right\}\right)$$

(*under mutually equivalent probability measures*) *with, for* $j = 1, \ldots, 4$,

$$\begin{aligned}\Pi_j(t,\ \tau) = & \frac{1}{4} + \frac{1}{2\pi}\int_0^\infty \mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi \ln[K_s]} f_j(t,\ \tau;\ \phi,\ 0)}{\mathrm{i}\phi}\right] \mathrm{d}\phi \\ & + \frac{1}{2\pi}\int_0^\infty \mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\varphi \ln[K_p]} f_j(t,\ \tau;\ 0,\ \varphi)}{\mathrm{i}\varphi}\right] \mathrm{d}\varphi \\ & - \frac{1}{2\pi^2}\int_0^\infty \int_0^\infty \left\{\mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi \ln[K_s] - \mathrm{i}\varphi \ln[K_p]} f_j(t,\ \tau;\ \phi,\ \varphi)}{\phi\varphi}\right]\right. \\ & \left. - \mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi \ln[K_s] + \mathrm{i}\varphi \ln[K_p]} f_j(t,\ \tau;\ \phi,\ -\varphi)}{\phi\varphi}\right]\right\} \mathrm{d}\phi\, \mathrm{d}\varphi.\end{aligned}$$

The corresponding characteristic functions are

$$\begin{aligned} f_1(t,\ \tau;\ \phi,\ \varphi) &= \frac{f(t,\ \tau;\ \phi - \mathrm{i},\ \varphi - \mathrm{i})}{f(t,\ \tau;\ -\mathrm{i},\ -\mathrm{i})}, \\ f_2(t,\ \tau;\ \phi,\ \varphi) &= f(t,\ \tau;\ \phi,\ \varphi - \mathrm{i}), \\ f_3(t,\ \tau;\ \phi,\ \varphi) &= f(t,\ \tau;\ \phi - \mathrm{i},\ \varphi), \\ f_4(t,\ \tau;\ \phi,\ \varphi) &= \mathrm{e}^{r\tau} f(t,\ \tau;\ \phi,\ \varphi). \end{aligned}$$

The price of the put option with payoff $C(t+\tau,\ 0;\ K_s, K_p) = \max(0, K_s - s(t+\tau)/s(t)) \times \max(0, K_p - p(t+\tau)/p(t))$ *can be deduced from put-call parity for correlation options.*

In extending existing treatments, our work provides at least three additional contributions. First, we offer an exact closed-form solution for correlation options under stochastic volatility. By setting $\alpha = \beta = \sigma = 0$ and using L'Hopital's rule, the valuation formula in (40) converges to its counterpart under geometric Brownian motion. Under these restrictions, the return characteristic functions in (46) and (47) are precisely those of the bivariate normal distribution. Having a more general valuation formula, with (i) shocks to each asset correlated with volatility shocks (i.e., $\rho_1 \neq 0$ and $\rho_2 \neq 0$) and (ii) the modeling of a more plausible and time-varying correlation structure between assets, should help close the gap in understanding and predicting how these claims respond in a non-lognormal setting. The general stochastic structure considered in Proposition 2 is consequently more consistent with such real-life applications as contingent securities on currency bonds, commodity-linked bonds and cross-exchange rates (and where market forces induce stochastically varying asset price comovements).

Second, by the spanning property of the joint characteristic function, valuation again has a one-step flavor. That is, the closed-form expression for $f(t, \tau; \phi, \varphi)$ guarantees the simultaneous recovery of all the (four) characteristic functions and one scaled-forward price. As a consequence, the requirement that the counterpart valuation equations be explicitly evaluated (i.e., Stulz, 1982) has been bypassed even under this two-dimensional exercise region setup. Observe that the assumption of time-invariant interest rates and zero convenience yields is for the sake of expositional convenience only. In such general settings and for other two-dimensional contract structures, the traditional approach will be far more demanding (at most eight valuation equations in total). So, the simplifications via the integrative spanning concept are likely to be substantive there as well. For instance, option valuation in the maximum or the minimum of two asset class (e.g., Stulz, 1982) also hinges on the joint characteristic function (46). As a result, their valuations can be reconciled internally within Proposition 2. In the same spirit, payoff variants of (45) with $s(t+\tau)/s(t)$ rising (declining) and $p(t+\tau)/p(t)$ declining (rising), $\max(0, s(t+\tau))/s(t) - K_s) \times \max(0, K_p - p(t+\tau)/p(t))$ (or, $\max(0, K_s - s(t+\tau)/s(t)) \times \max(0, p(t+\tau)/p(t) - K_p)$) involves option-pricing under negative cross-correlation, which is also within the scope of the present model.[8] In fact, the pricing of any claim on the joint uncertainty is immediate from (46) to (47).

Lastly, in deriving the Arrow–Debreu security prices $\Pi_j(t, \tau)$, for $j = 1, \ldots, 4$, we have introduced the inversion formula for their determination (when the exercise region is a bivariate vector). Recall that the probability Π_4 is the time t price of the delta security (under the risk-neutral measure) that pays one dollar when $\ln[s(t+\tau)] - \ln[s(t)] > \ln[K_s]$ and $\ln[p(t+\tau)] - \ln[p(t)] > \ln[K_p]$, and zero otherwise. Other probabilities have similar intuitive interpretations. In proposing our solutions for the delta securities, we have adapted a result originally due to Shephard (1991) (details are in the appendix) which has allowed us to extend the one-dimensional Fourier inversion methodology (as in Kendall and Stuart, 1977; Lukacs, 1960) to the pricing of options written on two assets. Moreover, because $s(t)$ and $p(t)$ are proportional stochastic processes and the option payoff is exponential-affine in the joint uncertainty, all the characteristic functions are translated from the joint characteristic function. Therefore, unlike the previous example the probabilities, Π_j are in the same parametric class. It

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[8] Some authors such as Zhang (1998) have described the correlation (call) option contract to mean the following payoff: $\max(0, s(t+\tau)/s(t) - K_s)$ if $p(t+\tau)/p(t) > K_p$ and vice versa. Restricting the middle two terms in (48) to zero and setting $K_p = 1$ in the fourth term will give the price of such a contract.

remains to be emphasized that although the determination of each probability demands a bivariate numerical integration, it presents no implementation difficulties. In reality, the probabilities and hence the option prices can be obtained with high speed.

5. A class of discretely monitored knock-out options

For this final application, consider a discretely monitored knock-out call option with a contractually determined payoff (for ease of exposition, some of the notation has changed):

$$C(t+N\,\Delta t,\,0) = 1_{\{s(t+\Delta t)>K\}} \times 1_{\{s(t+2\Delta t)>K\}} \times \cdots \times 1_{\{s(t+N\,\Delta t)>K\}} \\ \times \max(0, s(t+N\Delta t)-K), \tag{49}$$

where $s(t)$ is the time t price of the spot asset. If at any time prior to expiration, the spot price goes below a prespecified barrier K, the call option is knocked out and hence worthless. For simplicity, assume that the spot price evolves according to a log-normal process:

$$\ln[s(t+\Delta t)] - \ln[s(t)] = (r - \tfrac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\,\varepsilon(t+\Delta t), \quad s(0)>0, \tag{50}$$

where r and σ are constants and $\varepsilon(t)$ is a standard normal variate for all t. Thus, the characteristic function of the remaining uncertainty with density $q(s(t+\Delta t), \ldots, s(t+N\Delta t))$ is

$$f(\phi_1,\ldots,\phi_N) = \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-r\,N\Delta t}\exp\left(\sum_{n=1}^{N} \mathrm{i}\phi_n \ln[S(t+n\Delta t)]\right) \\ q(s(t+\Delta t),\ldots,s(t+N\Delta t))ds(t+\Delta t)\cdots ds(t+N\Delta t) \\ = \exp\left[-rN\,\Delta t + \left(r - \frac{1}{2}\sigma^2\right)\Delta t \sum_{j=1}^{N}\sum_{n=j}^{N} \mathrm{i}\phi_n \right. \\ \left. + \frac{1}{2}\sigma^2\Delta t \sum_{j=1}^{N}\left(\sum_{n=j}^{T} \mathrm{i}\phi_t\right)^2 + \sum_{n=1}^{N} \mathrm{i}\phi_n \ln[s(t)]\right].$$

Using a similar sequence of steps as in Theorem 1, we have the call price

$$C(t, N\Delta t;\, K) = f(0,\ldots,0,-\mathrm{i})\,\Pi_1(t, N\Delta t) - K\,\mathrm{e}^{-rN\Delta t}\,\Pi_2(t, N\,\Delta t) \tag{51}$$

where the characteristic functions corresponding to the risk-neutral probabilities are, respectively,

$$f_1(\phi_1,\ldots,\phi_N) = \frac{f(\phi_1,\ldots,\phi_{N-1},\phi_N-\mathrm{i})}{f(0,\ldots,0,-\mathrm{i})}, \tag{52}$$

$$f_2(\phi_1,\ldots,\phi_N) = \mathrm{e}^{rN\,\Delta t} f(\phi_1,\ldots,\phi_N) \tag{53}$$

and the N-dimensional Fourier inversion formula in Shephard (1991) can be adapted to arrive at the probabilities Π_1 and Π_2. This N-day, or N-week, formula is recursive and easily programmable in standard packages. For the twin contract, $\prod_{n=1}^{N} \max(0, s(t + n\,\Delta t) - K)$, the structure of valuation is only slightly more complex (but uses the same joint-characteristic function). Since formulas for the probabilities are not compact, they are omitted to save on space and to stress the spanning focal point. One can complement the log-normal assumption by a stochastic process with either Poisson jump arrival rates and lognormal/gamma distributed jump intensities or pure-jump processes under generalized Lévy measures with, say, infinite arrival rates. Although each enhancement will lead to distributions that dominate the log-normal pricing distribution on several fronts, their modeling aspects can be rather involved (especially with the Lévy measure), and hence these extensions are largely ignored in the development of (51).

6. Conclusions

It is known that the value of the call option recovers the corresponding put option price without actually solving its valuation equation. It is, however, not yet fully understood that the valuation equation for the characteristic function is sufficient to recover all underlying primitive claim prices. As the main idea is spanning, this remark is valid whether the underlying is an option on the underlying uncertainty, its powers, its exponential, or virtually any other function of the uncertainty. Our work also provides a way to formalize and unify the valuation of the term structure of interest rates, the valuation of the term structure of forward and futures prices, and the valuation of Arrow–Debreu securities. When derivative securities with higher-dimensional exercise regions are considered, the characteristic function basis provides superior analytical tractability (as in correlation and discretely monitored knock-out options).

Our work can be extended along several dimensions. First, our methodology can be used to revisit contingent claims valuation in the context of Heath et al. (1992). Adopting characteristic function-based methods could alleviate the burden of derivative-security valuation in their models. Second, it can be used to rethink American option valuation under more plausible stochastic dynamics. Here, one could determine the characteristic function of the optimal stopping problem which could jointly deliver the European option price and the early exercise premium. Third, as is done in Bakshi et al. (1999), our theoretical results can be adapted to design option positions mimicking the risk-neutral skewness and kurtosis (or the entire density function). Finally, the methodology can be employed to price exotic options with complex boundary conditions and under stochastic volatility (e.g., barriers and lookbacks). All of these extensions are left to a future scrutiny.

Appendix A

Proof of spanning equivalence in part (*a*) of Theorem 1. Recall $\mathrm{e}^{\mathrm{i}\phi X} = \cos(\phi X) + \mathrm{i}\sin(\phi X)$. Thus, the proof entails comparing the span of trigonometric functions with those of call options. The proof is divided into four parts for clarity.

A.1. Spanning claims in $\mathscr{L}^1$ via characteristic functions

For now, fix $\lambda_b = \lambda_x = 0$ and let $\mathscr{H}(X) \in \mathscr{L}^1$ be the claim payoff under consideration. Suppressing time arguments on X, define trading strategies as complex valued policies $w(\phi) \in \mathscr{L}^1$ such that

$$\mathscr{H}(X) = \int_{-\infty}^{\infty} \mathrm{Re}[w(\phi)\mathrm{e}^{\mathrm{i}\phi X}]\,\mathrm{d}\phi \tag{A.1}$$

which is just the cash flow attained by the strategy with

$$w(\phi) = w_1(\phi) + \mathrm{i}w_2(\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{H}(X)\mathrm{e}^{-\mathrm{i}\phi X}\,\mathrm{d}X \tag{A.2}$$

by the mathematics of inverse Fourier transformation. Thus, if the portfolio policy implicit in (A.1) and (A.2) is adopted, then Fourier theory for $\mathscr{L}^1$ asserts that (A.1) holds exactly (Goldberg, 1965, Chapter 1). So, $\mathscr{H}(X)$ is in the algebraic span of trigonometric functions.

Disentangling $w(\phi)$ into its real and imaginary components as in (A.2) and substituting into Eq. (A.1) produces

$$\mathscr{H}(X) = \int_{-\infty}^{\infty} \mathrm{Re}[(w_1(\phi) + \mathrm{i}w_2(\phi))\{\cos(\phi X) + \mathrm{i}\sin(\phi X)\}]\,\mathrm{d}\phi \tag{A.3}$$

$$= \int_{-\infty}^{\infty} [w_1(\phi)\cos(\phi X) - w_2(\phi)\sin(\phi X)]\,\mathrm{d}\phi \tag{A.4}$$

which formalizes how the continuum of long positions, $w_1(\phi)$ in $\cos(\phi X)$, and short positions, $w_2(\phi)$ in $\sin(\phi X)$, can conceive any $\mathscr{H}(X) \in \mathscr{L}^1$. Simplifying (A.2),

$$w_1(\phi) + \mathrm{i}\,w_2(\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{H}(X)\cos(\phi X)\,\mathrm{d}X - \mathrm{i}\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{H}(X)\sin(\phi X)\,\mathrm{d}X, \tag{A.5}$$

determines $w_1(\phi)$ and $w_2(\phi)$ in terms of the payoff function to be spanned and the sine and the cosine. This completes the description of how to span claims in $\mathscr{L}^1$ via trigonometric functions.

A.2. Spanning call options via characteristic functions

The payoff on the call option is not in $\mathscr{L}^1$. Exploiting the identity $\max(0, X - K) = \max(0, K - X) + X - K$ and Fourier theory applied to the put payoff (since $\max(0, K - X) \in \mathscr{L}^1$) yields

$$\max(0, X - K) = \lambda_b + \lambda_x X + \int_{-\infty}^{\infty} \mathrm{Re}[w(\phi) \mathrm{e}^{\mathrm{i}\phi X}] \, \mathrm{d}\phi \tag{A.6}$$

where $w(\phi) = w_1(\phi) + \mathrm{i} w_2(\phi) = (1/2\pi)\int_{-\infty}^{\infty} \max(0, K - X) \mathrm{e}^{-\mathrm{i}\phi X} \, \mathrm{d}X$, $\lambda_b = -K$, and $\lambda_x = 1$. The exact composition of $w_1(\phi)$ and $w_2(\phi)$ remains to be shown. From standard integration steps

$$\begin{aligned} w_1(\phi) &= \frac{1}{2\pi} \int_0^K (K - X) \cos(\phi X) \, \mathrm{d}X \\ &= \frac{1}{2\pi\phi^2} [1 - \cos(\phi K)] \end{aligned} \tag{A.7}$$

and again from Eq. (A.5)

$$\begin{aligned} w_2(\phi) &= -\frac{1}{2\pi} \int_0^K (K - X) \sin(\phi X) \, \mathrm{d}X \\ &= -\frac{1}{2\pi} \left[\frac{K}{\phi} - \frac{\sin(\phi K)}{\phi^2} \right] \end{aligned} \tag{A.8}$$

which are the legitimate long and short positions in the cosine and the sine to span calls from characteristic functions (augmented by the discount bond and the underlying asset).

A.3. Spanning characteristic functions via call options

From Theorem 1 of Carr and Madan (1997), the following spanning representation holds for $\mathscr{H}(X) \in \mathscr{C}^2$ (the space of twice continuously differentiable functions):

$$\begin{aligned} \mathscr{H}(X) = \mathscr{H}(X_0) + \mathscr{H}_X(X_0)(X - X_0) + \int_0^{X_0} \mathscr{H}_{XX}(K) \max(0, K - X) \, \mathrm{d}K \\ + \int_{X_0}^{\infty} \mathscr{H}_{XX}(K) \max(0, X - K) \, \mathrm{d}K \end{aligned} \tag{A.9}$$

for some constant X_0; $\mathscr{H}_X (\mathscr{H}_{XX})$ stands for the first (second) order partial derivative of the claim payoff with respect to X. Or, substituting $X_0 = 0$ and

$\mathscr{H}(X) = \cos(\phi X)$ delivers

$$\cos(\phi X) = \lambda_b + \int_0^\infty z(K) \max(0, X - K)\,\mathrm{d}K \tag{A.10}$$

for $\lambda_b = 1$ and $z(K) = -\phi^2 \cos(\phi K)$. Similarly, letting $\mathscr{H}(X) = \sin(\phi X)$ and $X_0 = 0$, we recover

$$\sin(\phi X) = \lambda_x X + \int_0^\infty z(K) \max(0, X - K)\,\mathrm{d}K \tag{A.11}$$

for $\lambda_x = 1$ and $z(K) = -\phi^2 \sin(\phi K)$. Thus, we have the result that trigonometric functions, and hence characteristic functions, can be effectively synthesized from a continuum of call options.

A.4. Spanning claims in $\mathscr{L}^1$ via call options

For the purpose of spanning claims $\mathscr{H}(X) \in \mathscr{L}^1$ through call options, we adopt the result due to Wiener (Goldberg, 1965, pp. 32–33) that $\mathscr{H}(X)$ can be constructed from a finite portfolio of $\bar{\mathscr{H}}(X) \in \mathscr{C}^2$. In particular,

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \left| \mathscr{H}(X) - \sum_{j=1}^{J^N} a_j^N \, \bar{\mathscr{H}}_N(X + b_j^N) \right| \mathrm{d}X \to 0 \tag{A.12}$$

for complex sequence a_j^N and real b_j^N for $j = 1, \ldots, J^N$. To accomplish (A.12), observe that the standard Gaussian density has (i) a well-defined Fourier transform, (ii) at least $\mathscr{C}^2$, and (iii) satisfies all the technical regularity conditions in Theorem 10C of Goldberg. Thus, by taking $\bar{\mathscr{H}}(X) = (1/\sqrt{2\pi})\mathrm{e}^{-X^2/2}$, one can span any integrable function by a linear combination of translated standard Gaussian densities. Implementing these steps in the present context and using (A.9) to span $\bar{\mathscr{H}}(X)$, we can conclude

$$\mathscr{H}(X) = \lim_{N\to\infty} \left\{ \lambda_b^N + \lambda_x^N X + \int_0^\infty z^N(K) \max(0, X - K)\,\mathrm{d}K \right\} \tag{A.13}$$

in the $\mathscr{L}^1$ norm with

$$\lambda_b^N \equiv \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{J^N} a_j^N \mathrm{e}^{-b_j^{N^2}/2}, \tag{A.14}$$

$$\lambda_x^N \equiv -\frac{1}{\sqrt{2\pi}} \sum_{j=1}^{J^N} a_j^N b_j^N \mathrm{e}^{-b_j^{N^2}/2}, \tag{A.15}$$

$$z^N(K) \equiv \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{J^N} a_j^N \mathrm{e}^{-(b_j^N + K)^2/2} [(b_j^N + K)^2 - 1] \tag{A.16}$$

and the theorem is proved.

Thus, in summary, we have proved that (i) continuums of options and continuums of trigonometric functions are equivalent classes of spanning securities for $\mathscr{L}^1$ with convergence in the $\mathscr{L}^1$ norm, and (ii) the construction of cash flows has a similar integral representation with the resulting classes of functions not mutually orthogonal.

Proof of Part (b) and Part (c) of Theorem 1. The proof relies on the fundamental properties of probability density functions and characteristic functions. By collapsing the integral in (3), it follows that

$$C(t,\tau;K) = G(t,\tau)\Pi_1(t,\tau) - KB(t,\tau)\Pi_2(t,\tau), \tag{A.17}$$

where the scaled forward price and the discount bond price are

$$G(t,\tau) \equiv \int_\Omega \exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right) X(t+\tau) q(v)\,\mathrm{d}v \tag{A.18}$$

and

$$B(t,\tau) \equiv \int_\Omega \exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right) q(v)\,\mathrm{d}v, \tag{A.19}$$

recalling that $q(v)$ denotes the risk-neutral density of $v \equiv (\int_t^{t+\tau} r(u)\,\mathrm{d}u, X(t+\tau))$ and $\mathscr{X}$ and Ω respectively stand for the sets $\{X(t+\tau) > K\}$ and $\{X(t+\tau) > 0\}$. Continuing with the same style of reasoning,

$$\Pi_1(t,\tau) \equiv \frac{\int_{\mathscr{X}} \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)\,X(t+\tau)\,q(v)\,\mathrm{d}v}{\int_\Omega \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)\,X(t+\tau)\,q(v)\,\mathrm{d}v} \tag{A.20}$$

and

$$\Pi_2(t,\tau) \equiv \frac{\int_{\mathscr{X}} \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)\,q(v)\,\mathrm{d}v}{\int_\Omega \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)\,q(v)\,\mathrm{d}v}, \tag{A.21}$$

which are valid probabilities since $\Pi_j \in (0,1)$ for $j = 1, 2$.

To substantiate the relation between each constituent security in Eq. (5) and the characteristic function, $f(t,\tau;\phi)$, define

$$f(t,\tau;\phi) \equiv \int_\Omega \exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right) \mathrm{e}^{\mathrm{i}\phi X(t+\tau)} q(v)\,\mathrm{d}v, \tag{A.22}$$

which is the Fourier transform of the state-price density. Setting $\phi = 0$ and rearranging,

$$f(t,\tau;0) = \int_\Omega \exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right) q(v)\,\mathrm{d}v \equiv B(t,\tau)$$

which coincides with (11) of Theorem 1. Correspondingly, differentiating both sides of Eq. (A.22) with respect to ϕ,

$$f_\phi(t,\ \tau;\ \phi)=\mathrm{i}\int_\Omega \exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right)\times X(t+\tau)\times \mathrm{e}^{\mathrm{i}\phi X(t+\tau)}\,q(v)\,\mathrm{d}v. \tag{A.23}$$

Using (A.23) and evaluating $f_\phi(t,\ \tau;\ \phi)$ at $\phi=0$ justifies the assertion in (12).

Now generate the characteristic function of the first Arrow–Debreu function, $f_1(t,\ \tau)$, from Eq. (A.20), as (e.g., Lukacs, 1960; Kendall and Stuart, 1977):

$$f_1(t,\ \tau;\ \phi)=\frac{\int_\Omega \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)\,X(t+\tau)\mathrm{e}^{\mathrm{i}\,\phi\,X(t+\tau)}\,q(v)\,\mathrm{d}v}{\int_\Omega \exp(-\int_t^{t+\tau} r(u)\,\mathrm{d}u)\,X(t+\tau)\,q(v)\,\mathrm{d}v}, \tag{A.24}$$

$$\equiv\frac{f_\phi(t,\ \tau;\ \phi)}{f_\phi(t,\ \tau;\ 0)} \tag{A.25}$$

with the aid of (A.23). By similarly manipulating the primitive characteristic function, the derivation of $f_2(t,\ \tau;\ \phi)$ is quite clear-cut.

Proof of (25)–(28) in Case 3. Let $X(t+\tau)-X_0$ denote translated uncertainty for some constant X_0 and $\bar{f}(t,\ \tau;\ \phi)$ its characteristic function. Then $\bar{f}(t,\ \tau;\ \phi)=\mathrm{e}^{\mathrm{i}\phi X_0}f(t,\ \tau;\ \phi)$. Now,

$$M(t,\ \tau;\ \phi)\equiv\int_\Omega \exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right)H(X)\mathrm{e}^{\mathrm{i}\phi X}\,q(v)\,\mathrm{d}v. \tag{A.26}$$

Taking a Taylor series of $H(X)$ around X_0 and reformulating each component security as in Theorem 1 confirms (25)–(28).

Proof of the path-dependent interest rate option formula in Proposition 1. Proceeding in the same spirit as Theorem 1, now let $v\equiv\int_t^{t+\tau} r(u)\,\mathrm{d}u$ with density $q(v)$ and $\mathscr{E}\equiv\{v>[t+\tau]K-A(t)\}$. By definition,

$$C(t,\ \tau)=\int_{\mathscr{E}} \mathrm{e}^{-v}\left[\frac{v}{t+\tau}+\frac{A(t)}{t+\tau}-K\right]q(v)\,\mathrm{d}v. \tag{A.27}$$

Decomposing this conditional expectation, we get (37) with

$$G(t,\ \tau)\equiv\int_0^\infty \mathrm{e}^{-v}\,v\,q(v)\,\mathrm{d}v,\quad B(t,\ \tau)\equiv\int_0^\infty \mathrm{e}^{-v}\,q(v)\,\mathrm{d}v,$$

$$\Pi_1(t,\ \tau)\equiv\frac{\int_{\mathscr{E}} \mathrm{e}^{-v}\,v\,q(v)\,\mathrm{d}v}{\int_0^\infty \mathrm{e}^{-v}\,v\,q(v)\,\mathrm{d}v}$$

and

$$\Pi_2(t,\ \tau)\equiv\frac{\int_{\mathscr{E}} \mathrm{e}^{-v}\,q(v)\,\mathrm{d}v}{\int_0^\infty \mathrm{e}^{-v}\,q(v)\,\mathrm{d}v}.$$

So the corresponding characteristic functions must be

$$f_1(t,\tau;\phi)=\frac{1}{G(t,\tau)}\int_0^\infty \mathrm{e}^{-v}\, v\mathrm{e}^{\mathrm{i}\phi v}\, q(v)\,\mathrm{d}v \tag{A.28}$$

and

$$f_2(t,\tau;\phi)=\frac{1}{B(t,\tau)}\int_0^\infty \mathrm{e}^{-v}\mathrm{e}^{\mathrm{i}\phi v}\, q(v)\,\mathrm{d}v, \tag{A.29}$$

and the characteristic function of the state-price density is

$$f(t,\tau;\phi)=\int_0^\infty \mathrm{e}^{-v}\mathrm{e}^{\mathrm{i}\phi v}\, q(v)\,\mathrm{d}v \tag{A.30}$$

$$=\mathrm{E}_t^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)\,\mathrm{d}u\right)\times\exp\left(\mathrm{i}\phi\int_t^{t+\tau} r(u)\,\mathrm{d}u\right)\right\}. \tag{A.31}$$

Differentiating and translating (A.30) confirms (40) and (41) of Proposition 1. Finally, solving (A.30) fills in the missing structural link displayed in (36) with $\gamma(\phi)\equiv\sqrt{\kappa^2-2(\mathrm{i}\phi-1)\sigma^2}$ and

$$\mathscr{M}(\tau;\phi)\equiv\frac{\kappa\theta}{\sigma^2}\left[(\gamma-\kappa)\tau+2\ln\left(1-\frac{(\gamma-\kappa)(1-\mathrm{e}^{-\gamma\tau})}{2\gamma}\right)\right], \tag{A.32}$$

$$\mathscr{N}(\tau;\phi)\equiv\frac{2(1-\mathrm{i}\phi)(1-\mathrm{e}^{-\gamma\tau})}{2\gamma-(\gamma-\kappa)(1-\mathrm{e}^{-\gamma\tau})}. \tag{A.33}$$

Proof of Proposition 2. For parsimony of presentation, let $S\equiv\ln[s(t+\tau)/s(t)]$ and $P\equiv\ln[p(t+\tau)/p(t)]$. Omitting time arguments, write the joint density function as $q(S,P)$. The joint characteristic function is

$$f(t,\tau;\phi,\varphi)\equiv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-r\tau+\mathrm{i}\phi S+\mathrm{i}\varphi P}\, q(S,P)\,\mathrm{d}S\,\mathrm{d}P. \tag{A.34}$$

Directly solving this conditional expectation delivers (47) with

$$\begin{aligned}\mathscr{Y}(t,\tau;\phi,\varphi)\equiv{}&[\mathrm{i}\phi+\mathrm{i}\varphi-1]\,r\,\tau\\ &-\frac{\alpha}{\sigma^2}\left[2\ln\left(1-\frac{[\vartheta-\Gamma](1-\mathrm{e}^{-\vartheta\tau})}{2\vartheta}\right)-[\vartheta-\Gamma]\tau\right]\end{aligned} \tag{A.35}$$

and

$$\mathscr{X}(t,\tau;\phi,\varphi)\equiv\frac{2\zeta(1-\mathrm{e}^{-\vartheta\tau})}{2\vartheta-[\vartheta-\Gamma](1-\mathrm{e}^{-\vartheta\tau})}, \tag{A.36}$$

defining $\Gamma(\phi,\varphi)\equiv\beta-\mathrm{i}\,\phi\,\sigma_s\,\sigma\,\rho_1-\mathrm{i}\,\varphi\,\sigma_p\,\sigma\,\rho_2$, $\vartheta(\phi,\varphi)\equiv\sqrt{\Gamma^2-2\,\sigma^2\,\zeta}$, and $\zeta(\phi,\varphi)\equiv-\frac{1}{2}\mathrm{i}\,\phi\,\sigma_s^2-\frac{1}{2}\mathrm{i}\,\varphi\,\sigma_p^2-\frac{1}{2}\phi^2\sigma_s^2-\frac{1}{2}\varphi^2\sigma_p^2-\phi\,\varphi\,\sigma_s\,\sigma_p\,\eta$.

Returning to the correlation option problem, rewrite the conditional expectation (45) as

$$\begin{aligned}C(t,\ \tau;\ K_s, K_p) = &\int_{\ln[K_p]}^{\infty}\int_{\ln[K_s]}^{\infty}\{\mathrm{e}^{-r\tau+S+P} - K_s\mathrm{e}^{-r\tau+P}\\ &- K_p\mathrm{e}^{-r\tau+S} + K_sK_p\mathrm{e}^{-r\tau}\}\, q(S,P)\,\mathrm{d}S\,\mathrm{d}P\\ =&f(t,\ \tau;\ -\mathrm{i},-\mathrm{i})\,\Pi_1(t,\ \tau) - K_s\Pi_2(t,\ \tau) - K_p\,\Pi_3(t,\ \tau)\\ &+ K_sK_p\mathrm{e}^{-r\tau}\,\Pi_4(t,\ \tau).\end{aligned}$$

The claim in (48) follows by observing that

$$f(t,\ \tau;-\mathrm{i},-\mathrm{i}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-r\tau+S+P}\,q(S,P)\,\mathrm{d}S\,\mathrm{d}P, \tag{A.37}$$

$$\Pi_1(t,\ \tau) = \frac{\int_{\ln[K_p]}^{\infty}\int_{\ln[K_s]}^{\infty}\mathrm{e}^{-r\tau+S+P}\,q(S,P)\,\mathrm{d}S\,\mathrm{d}P}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-r\tau+S+P}\,q(S,P)\,\mathrm{d}S\,\mathrm{d}P}, \tag{A.38}$$

and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-r\tau+S}\,q(S,P)\,\mathrm{d}S\,\mathrm{d}P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-r\tau+P}\,q(S,P)\,\mathrm{d}S\,\mathrm{d}P = 1$ (the martingale restriction).

The sole task remaining is to get each probability. Again, focus on the probability $\Pi_1(t,\ \tau)$. By appealing to standard probability theory, one can verify that its characteristic function is

$$\begin{aligned}f_1(t,\ \tau;\ \phi,\ \varphi) &= \frac{1}{f(t,\ \tau;-\mathrm{i},-\mathrm{i})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-r\tau+S+P}\times\mathrm{e}^{\mathrm{i}\phi S+\mathrm{i}\,\varphi\,P}\,q(S,P)\,\mathrm{d}S\,\mathrm{d}P\\ &\equiv\frac{f(t,\ \tau;\ \phi-\mathrm{i},\ \varphi-\mathrm{i})}{f(t,\ \tau;-\mathrm{i},-\mathrm{i})},\end{aligned}$$

which affirms that $f_1(t,\tau)$ can be expressed in terms of the joint characteristic function.

Adapting Theorem 5 in Shephard (1991) to the present two-dimensional problem, for any distribution function $F(S,P;\ a,b)$ with joint characteristic function $f(S,P;\ \phi,\ \varphi)$, the following can be asserted:

$$\begin{aligned}F(S,P;\ a,b) = &-\tfrac{1}{4}+\tfrac{1}{2}F(S;\ a)+\tfrac{1}{2}F(P;\ b)\\ &-\frac{1}{2\pi^2}\int_0^{\infty}\int_0^{\infty}\left(\mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi a-\mathrm{i}\varphi b}f(S,P;\ \phi,\ \varphi)}{\phi\varphi}\right]\right.\\ &\left.-\mathrm{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\phi a+\mathrm{i}\varphi b}f(S,P;\ \phi,-\varphi)}{\phi\varphi}\right]\right)\mathrm{d}\phi\,\mathrm{d}\varphi,\end{aligned} \tag{A.39}$$

and the marginal distributions for S and P are, respectively, given by

$$F(S;\, a) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi a} f(S;\, \phi,\, 0)}{i\phi}\right] d\phi \tag{A.40}$$

and

$$F(P;\, b) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\varphi b} f(P;\, 0,\, \varphi)}{i\varphi}\right] d\varphi. \tag{A.41}$$

Armed by the joint and marginal distributions in (A.39)–(A.41), the probability $\Pi_1(t,\, \tau)$ can now be constructed as follows:

$$\Pi_1(t,\, \tau) = 1 - F(S;\, \ln[K_s]) - F(P;\, \ln[K_p]) + F(S, P;\, \ln[K_s],\, \ln[K_p]).$$

Rearranging verifies the Fourier-inversion formula displayed in Proposition 2.

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