Returns of Claims on the Upside and the Viability of U-Shaped Pricing Kernels*

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Abstract

When the pricing kernel is U-shaped, then expected returns of claims with payout on the upside are negative for strikes beyond a threshold, determined by the slope of the U-shaped kernel in its increasing region, and have negative partial derivative with respect to strike in the increasing region of the kernel. Using returns of (i) S&P 500 index calls, (ii) calls on major international equity indexes, (iii) digital calls, (iv) upside variance contracts, and (v) a theoretical construct that we denote as kernel call, we find broad support for the implications of U-shaped pricing kernels. A possible theoretical reconciliation of our empirical findings is explored through a model that accommodates heterogeneity in beliefs about return outcomes and short-selling.

KEY WORDS: U-shaped pricing kernels, claims on the upside, monotonically declining pricing kernels, expected returns, negative average option returns, short-selling, heterogeneity in beliefs

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1. Introduction

Starting with Lucas (1978), classical pricing kernels, which are monotonically declining in aggregate wealth, form the backbone of asset pricing theory. The theory relying on such kernels has been extensively explored and tested using security market data (e.g., Cochrane (2005)). However, recent work has presented puzzling evidence to the contrary: the pricing kernel, when plotted against the market return is not monotonically declining, but instead exhibits an upward-sloping region (e.g., Aït-Sahalia and Lo (2000), Jackwerth (2000), Rosenberg and Engle (2002), Carr, Geman, Madan, and Yor (2002), Ziegler (2007), and Shive and Shumway (2006)). To provide a possible explanation for this finding, Bakshi and Madan (2008) develop a model that incorporates both a declining region *and* an increasing region in the pricing kernel. Their theory of U-shaped kernels is based on heterogeneity in beliefs about equity return outcomes, and an increasing region in the kernel is obtained by introducing risk-averse investors, shorting the equity market.

What are the principal theoretical implications of U-shaped pricing kernels for expected returns? Which contingent claims are suitable for distinguishing between monotonically declining and U-shaped kernels? Are the theoretical restrictions imposed by U-shaped kernels supported in equity markets? The intent of this paper is to weigh the evidence for an asset pricing theory that hinges on U-shaped pricing kernels, by assessing its ability to explain various dimensions of the data. These questions are important as monotonically declining pricing kernels permeate theoretical developments in financial economics.

Addressing these issues, we first argue that only claims with payout on the upside are capable of discerning between U-shaped and monotonically declining kernels, in contrast, for example, to put options, which derive their value from the declining region, shared by both types of pricing kernels. Our Theorem 1 shows that within the class of claims with payoffs bounded by a constant, by the market return, or by the pricing kernel itself, the claims most suitable to detect the presence of U-shaped pricing kernels in the data include the *digital call*, the *call option*, and a construct that we denote *kernel call*, respectively.

To develop testable predictions, our Theorems 2, 3, and 4 establish that expected returns of claims with payout on the upside under U-shaped kernels (i) are negative for strikes beyond a threshold, determined by the slope of the U-shaped kernel in its increasing region, and (ii) have negative partial derivative with respect to strike in the increasing region of the kernel. In contrast, both expected returns and their partial derivative, are *positive* in the case of classical pricing kernels (Coval and Shumway (2001)).

In our empirical investigation, we first study S&P 500 index call returns and show that their behavior presents a stark departure from the theory of monotonically declining kernels: First, average returns of index calls are declining, not increasing, as one considers option contracts that are farther out-of-the-money (OTM hereafter). Second, the average returns on holding 3% and 5% OTM S&P 500 index calls are -2.2% and -5.8% *per month*, whereas the index average return is positive over the same sample period (see also Bondarenko (2003)). Third, we study average call returns across major international equity markets and find stronger support for negative average returns that are declining in strikes. Together, these empirical findings amount to a reversal of the textbook prediction based on positive market beta for call options (Cox and Rubinstein (1985), page 210), and constitute a basis to question premier asset pricing theory.

Integral to our body of empirical evidence, we find that average digital call returns exceed those of calls with the same strike and maturity, and average returns of digital calls and upside variance contracts are declining in strike. Such features of the data are puzzling within the theory of monotonic kernels.

Elaborating on the implications of an asset pricing model characterized by U-shaped kernel, we consider the behavior of the mimicking portfolio for the pricing kernel, relying on Hansen and Richards (1987) and Cochrane (2005). The crux of our empirical finding is that when index calls and puts are included in the set of basis assets, the mimicking portfolio for the pricing kernel is an asymmetrically U-shaped function of market returns, affirming the hypothesis that the kernel is not monotonically declining. Associated with the U-shaped valuation structure is the notion that state prices rise with wealth at high wealth levels.

To tie our findings to a possible economic channel, we investigate whether a model based on heterogeneous beliefs about return outcomes, as presented in Bakshi and Madan (2008), can facilitate our understanding of the documented features of option returns. An increasing region in the kernel is endogenously obtained in this model by introducing risk-averse investors, shorting the equity market. Building on the model and on our empirical findings, we note that OTM index calls could be to short equity exposures what OTM index puts are, on occasion, to long equity exposures, since calls and puts both provide loss protection, to short and long equity investors, respectively. In particular, the declining region of the pricing kernel reflects the risk aversion of long equity investors, which translates into demand for protection via OTM index puts (e.g., Leland (1980), Grossman and Zhou (1996), and Franke, Stapleton, and Subrahmanyam (1998)). In parallel, the increasing region of the pricing kernel may reflect the risk aversion of short equity investors, who, as we suggest, seek protection by purchasing OTM index calls. In both cases, the premium paid for loss protection results in negative expected option returns, as observed in the data.

With this economic channel in mind, we assess whether the model with heterogeneity in beliefs about return outcomes can generate plausible expected returns of contingent claims, and thus reconcile the salient features of our data. We show that, by accommodating different strength of the declining and increasing regions of the pricing kernel, the model is able to reproduce negative expected returns of OTM calls that are realistically more negative farther OTM, while also producing substantially more negative expected put returns, increasing in strike, and generating a smirk pattern of implied volatility across strikes. The model's versatility is seen in its ability to explain other aspects of the data, including returns of digital calls and upside variance contracts. Instrumental for obtaining these results is the presence of investors shorting equity, as well as long positions that dominate the short counterparts.

To corroborate the suggested economic channel, we examine the link between short-selling and index call options. Our thrust is to test empirically the theoretical connection between shorting and the increasing region of the pricing kernel on one hand, and between the slope of the kernel and expected call returns as hypothesized here on the other. In support of the theory underpinning U-shaped pricing kernels, we obtain positive relation between proxies for short-selling activity and expected call returns. We find that results obtained with new short sales as the shorting proxy are statistically significant and stronger than those obtained with short interest. Overall, we are lead to conclude that U-shaped pricing kernels provide a satisfactory framework that helps to unravel the return patterns of claims with payout on the upside.[1]

This paper is organized as follows. Section 2 develops the predictions of U-shaped kernels for expected returns of contingent claims on the upside. Section 3 is devoted to assessing the empirical case for U-shaped pricing kernels. In Section 4, we present an economy with heterogeneity in beliefs about return outcomes that is capable of supporting U-shaped kernels, and evaluate whether it can reproduce aspects of equity market data. The empirical link between the slope of the pricing kernel and index call returns is pursued in Section 5. Conclusions are in Section 6.

[1] Starting with Merton, Scholes, and Gladstein (1978), a strand of literature has studied option returns related issues, including Rubinstein (1984), Aït-Sahalia and Lo (1998, 2000), Jackwerth (2000), Buraschi and Jackwerth (2001), Coval and Shumway (2001), Dennis and Mayhew (2002), Pan (2002), Bondarenko (2003), Agarwal and Naik (2004), Bollen and Whaley (2004), Eraker (2004), Vanden (2004, 2007), Jones (2006), Benzoni, Collin-Dufresne, and Goldstein (2007), Cao and Huang (2008), Christoffersen, Jacobs, Ornthanalai, and Wang (2008), Ni (2008), Broadie, Chernov, and Johannes (2009), Cherny and Madan (2009), Constantinides, Jackwerth, and Perrakis (2009), Santa-Clara and Saretto (2009), and Santa-Clara and Yan (2009). Filling a gap, we show that the return behavior of claims on the upside is compatible with the tenets of an asset pricing model based on U-shaped pricing kernel.

2. U-shaped pricing kernels and returns of claims paying out on the upside

Let S_t be the level of the market index at time t and denote by $R \equiv \ln\left(\frac{S_{t+T}}{S_t}\right) \in \Re$, the T-period return. Assume that the market risk premium is positive, i.e., $\int R\, p[R]\, dR - \ln(1+r_f) > 0$, where $p[R]$ represents the return density under the physical probability measure and $r_f \geq 0$ is the T-period riskfree return.

2.1. Definition and attributes of a U-shaped pricing kernel

Following Harrison and Kreps (1979), Hansen and Jagannathan (1991), and Cochrane (2005), assume the existence of a pricing kernel $m[R]$ satisfying the following properties:

$$m[R] > 0, \qquad \int m[R]\, p[R]\, dR = \frac{1}{1+r_f} < +\infty, \qquad \int (m[R])^2\, p[R]\, dR < +\infty. \tag{1}$$

To rule out extreme counterexamples, assume $\int e^{\theta R}\, p[R]\, dR < +\infty$ for $|\theta| < a_0$, for some suitably chosen constant $a_0 > 1$.

When it comes to considering contingent claims written on the market return for which we have data, it suffices to work with the projection of the pricing kernel, that may admit additional state dependencies (e.g., Campbell and Cochrane (1999)), onto the space generated by the market return (e.g., Rosenberg and Engle (2002)). The price at time t of a claim paying a single cash flow $g_{t,T}[R]$ at $t+T$ is,

$$v_{t,T} = \frac{1}{1+r_f}\,\mathbb{E}^{\mathbb{P}}\left(\mathfrak{m} \times g_{t,T}[R]\right) = \frac{1}{1+r_f}\,\mathbb{E}^{\mathbb{P}}\left(\mathbb{E}^{\mathbb{P}}\left(\mathfrak{m} \times g_{t,T}[R]\middle| R\right)\right) = \mathbb{E}^{\mathbb{P}}\left(\frac{\widetilde{\mathfrak{m}}[R]}{1+r_f} \times g_{t,T}[R]\right), \tag{2}$$

where $\mathbb{E}^{\mathbb{P}}(.)$ is expectation under the physical probability measure. Moreover, $\mathfrak{m}$ represents the change of probability with the bond price as numeraire, and $\widetilde{\mathfrak{m}}[R] \equiv \mathbb{E}^{\mathbb{P}}(\mathfrak{m} \mid R)$. Thus, we need only model $\mathbb{E}^{\mathbb{P}}(\mathfrak{m} \mid R)$ without delving into state-dependent specifications of the pricing kernel. Pricing kernels dependent on market return R are implicit in the analysis of Rubinstein (1975), Brennan (1979), Hakansson (1978), Aït-Sahalia and Lo (2000), and Jackwerth (2000).

Our object of interest is characterized as follows:

Definition 1 *A pricing kernel $m[R]$ is U-shaped if:*

(a) there exists R_d such that $m'[R] \equiv \frac{dm[R]}{dR} < 0$ and $m''[R] \equiv \frac{d^2m[R]}{dR^2} > 0$ for $R < R_d$;

(b) there exists R_u such that $R_d \leq R_u$, $m'[R] > 0$ and $m''[R] > 0$ for $R > R_u$.

U-shaped pricing kernels differ from the monotonically declining counterparts, which do not exhibit the second feature (Definition 1(b)), and hence do not accommodate an increasing region. At the same time, U-shaped kernels, like monotonically declining kernels, are required to satisfy the conditions in (1).

Among the goals of this article is to assess the viability of U-shaped pricing kernels, develop their theoretical implications for expected returns, and subject them to empirical testing. Since monotonically declining kernels and U-shaped kernels are only differentiated on the upside of market returns, the appropriate financial instruments that may help discriminate between the two classes of pricing kernels need to derive their value from the upside of market returns. In contrast, U-shaped kernels are still declining in the region relevant for the pricing of put options, and hence expected put returns under U-shaped kernels may not materially differ from their counterpart under declining kernels. It is for this reason that the majority of our empirical evidence is gathered from claims paying out on the upside.

2.2. A class of claims on the upside relevant to the validity of U-shaped pricing kernels

Motivated by the above implications, denote by $g_{t,T}[R;y]$ the generic payoff to a derivative claim on the market index, with time to expiration T from period t, strike price K, and moneyness $y \equiv \frac{K}{S_t}$. Although by no means exhaustive (see Carr, Ellis, and Gupta (1998), Reiner and Rubinstein (1991), and Freedman (2005)), our efforts will be directed on the following class of claims on the upside:

$$g_{t,T}[R] \quad = \quad \begin{cases} \left(S_t\, e^R - yS_t\right)^+ & \text{Call Option} \\ \mathbf{1}_{e^R > y} & \text{Digital Call} \\ m[R]\mathbf{1}_{e^R > y} & \text{Kernel Call} \end{cases} \tag{3}$$

where $a^+ \equiv \max(0,a)$ and $\mathbf{1}_{e^R>y}$ is the indicator function of the event $\{S_{t+T} > K\}$ or, equivalently, $\{e^R > y\}$. We employ $y > 1$ throughout, as our focus is on out-of-the-money (i.e., OTM) claims on the upside.

Central to our purposes, the payoff to a digital call is in L^∞, i.e., $\sup_R \left| g_{t,T}[R] \right| < +\infty$, the payoff to a call option is in L^1, i.e., $\int \left| g_{t,T}[R] \right| p[R]\, dR < +\infty$, and, finally, consistent with equation (1), the payoff to a kernel call is in L^2, i.e., $\int \left(g_{t,T}[R]\right)^2 p[R]\, dR < +\infty$.

Let the price of a call, digital call and kernel call at t be denoted as $C_{t,T}[y]$, $D_{t,T}[y]$ and $H_{t,T}[y]$, respectively, and let the expected returns of these claims be $\mu^c[y]$, $\mu^d[y]$ and $\mu^h[y]$, so, for instance,

$$C_{t,T}[y] = \int_{\ln(y)}^{+\infty} \left(S_t\, e^R - yS_t\right)\, m[R]\, p[R]\, dR \qquad \text{and} \qquad \mu^c[y] = \frac{\int_{\ln(y)}^{+\infty} \left(S_t\, e^R - yS_t\right)\, p[R]\, dR}{C_{t,T}[y]} \;-\; 1, \tag{4}$$

where $p[R]$ is the physical return density and $m[R]$ is the pricing kernel. While call options are actively traded claims on the upside of market returns, and are thus naturally suited for this study, incorporating the other instruments provides a broader foundation for our results.

Fundamental to our choice of the set of claims defined in (3) is the following observation: when $m[R]$ is U-shaped, one expects claims paying out on the upside to have negative expected returns. This is because the pricing density dominates the physical density in the upper tail, and hence such claims have low expected payout accompanied by high price. Our analysis thus leads us to focus attention on claims with the *lowest* expected returns. When the pricing density differs from the physical density, as is generally the case, one may construct state contingent claims with a variety of attributes, among which we consider the minimization of expected returns, subject to the constraint of affordability.

Given the linearity of the objective function and the constraint of affordability, one seeks to bound the cash flows in some sensible fashion. We suggest three candidate bounds: a simple constant, a multiple of the market return and a multiple of the kernel itself. This places the cash flow under consideration in the space of bounded functions, integrable functions and square integrable functions, respectively. It is shown that the optimal claim in each of the three cases has a call option-like feature, and we thereby adopt these structures in our empirical analysis. The following theorem is intended to make the above rigorous.

Theorem 1 *Suppose $m[R]$ is U-shaped. The optimization problem is to choose $g_{t,T}[R] \geq 0$,*

$$\text{Minimize} \int g_{t,T}[R]\, p[R]\, dR, \qquad\qquad \text{Subject to} \int m[R]\, g_{t,T}[R]\, p[R]\, dR \;=\; 1. \tag{5}$$

If $g_{t,T}[R]$ is in L^∞ and bounded above by a constant $b > 0$, then the solution to (5) is a digital call. If $g_{t,T}[R]$ is in L^1 and bounded by a multiple $b > 0$ of the market return, then the optimal claim for (5) also includes a call option. Finally, if $g_{t,T}[R]$ is in L^2 and bounded by a multiple $b > 0$ of the pricing kernel, then the optimal claim is a kernel call.

Proof: See Appendix A. □

Since the cost of the claim is normalized to unity, the problem essentially is to minimize expected return of the claim under a U-shaped kernel. Therefore, the solution to (5) is the cashflow $g_{t,T}[R] \geq 0$ with the lowest expected return, under a physical density $p[R]$ and pricing kernel $m[R]$. The program in (5) is an optimization problem with both a linear objective and a linear constraint. However, as pointed in Jin, Xu, and Zhou (2008), this problem becomes ill-posed if one can find a non-negative cash flow with a finite price but extremely large prospective payoff in certain states. Such an outcome is disallowed with our assumption that $g_{t,T}[R]$ is bounded above. Under such an assumption a solution exists and the problem may be solved using Lagrange methods (e.g., Bertsekas (1996)). The solution depends critically on the shape of the pricing kernel.

In the context of this study, the significance of Theorem 1 is that it narrows the class of payoff functions that can enhance identification through the data if the pricing kernel is indeed U-shaped. If $m[R]$ is instead monotonically declining, the structure of the solution, as Appendix A shows, changes dramatically: for example, the expected return minimizing claim can only be a digital *put* option when $g_{t,T}[R]$ is bounded by a constant.

Even when not directly traded, the digital call payoff can be approximated through a long and short position in calls, namely: $\mathbf{1}_{S_{t+T}>K} \approx \frac{1}{2\delta K}\{(S_{t+T}-K(1-\delta))^{+}-(S_{t+T}-K(1+\delta))^{+}\}$, where $\delta > 0$ (Carr, Ellis, and Gupta (1998)), thereby facilitating the construction of digital call returns. Digitals are primitive claims, as demonstrated in Ross (1976), Breeden and Litzenberger (1978), and Ingersoll (2000). Besides, digital options on the S&P 500 index are now exchange-traded at the CBOE, providing further proof of their general importance (Freedman (2005)). In line with the tasks at hand, we will show that considering digital calls can provide additional supportive evidence for the relevance of U-shaped pricing kernels.

A natural question is how to construct the return of kernel calls when the true form of the pricing kernel is unknown? In this regard, we exploit theoretical results (i) from Hansen and Richards (1987) and Cochrane (2005) on the projection of the pricing kernel on the linear span of returns of a set of basis assets and (ii) from Ross (1976) on the spanning properties of options on the market level. This approach, which provides the mimicking portfolio for the pricing kernel, as delineated in Subsections 3.6 and 3.7, enables a model-free computation of kernel call returns. While a linear non-positive pricing kernel could give rise to arbitrage opportunities, a concern raised by Dybvig and Ingersoll (1982), our mimicking portfolios deliver the required positivity (see equation (52)).

In our empirical investigation, we additionally consider return evidence based on the claim payoff $(R-\overline{\mu})^2\mathbf{1}_{e^R>y} \in L^2$, with mean return $\overline{\mu}$, an analogue to the variance contract (e.g., Britten-Jones and

Neuberger (2000), Bakshi, Kapadia, and Madan (2003), Carr and Wu (2009), and Todorov (2009)). While this payoff is not explicitly included among the payoffs in (3), it still can be represented in terms of the payoffs of a digital and call options, using the methods of Carr and Madan (2001) and Bakshi and Madan (2000), as shown below:

$$(R-\overline{\mu})^2\,\mathbf{1}_{e^R>y} = \left(\ln\left(\frac{K}{S_t}\right)-\overline{\mu}\right)^2 \mathbf{1}_{S_{t+T}>K} + \frac{2}{K}\left(\ln\left(\frac{K}{S_t}\right)-\overline{\mu}\right)(S_{t+T}-K)^+ + \int_{\{\mathbb{K}>K\}} \varpi[\mathbb{K}](S_{t+T}-\mathbb{K})^+ d\mathbb{K}, \tag{6}$$

where $\varpi[\mathbb{K}] = \frac{2}{\mathbb{K}^2}\left(1-\ln\left(\mathbb{K}/S_t\right)+\overline{\mu}\right)$. Since the expected payoff captures variance in a rising market, we label the claim an *upside variance contract*. The merit of such a claim is that it allows us to portray yet another feature of the data and hence capture the expected return implications inherent in U-shaped kernels.

2.3. *Testable implications of U-shaped pricing kernels*

The following results establish the sign and derivative of expected returns of the claims on the upside outlined in equation (3). Since the upside variance contract shares similar properties, the related proof is omitted to avoid duplication.

Theorem 2 *If the economy supports a U-shaped pricing kernel, then the following statements are true:*

(a) Expected returns of call options on the market index level with strikes $K > S_t\, e^{R_u}$ are decreasing in the strike, where R_u is defined in Definition 1;

(b) There exists a strike price $K_c > S_t\, e^{R_u}$ such that call options with strikes higher than K_c have negative expected returns. The steeper the slope of the U-shaped pricing kernel in the increasing region, the more negative are these expected returns.

Proof: See Appendix A. □

Our Theorem 2 stands in sharp contrast to the corresponding theory based on monotonically declining kernels (e.g., Coval and Shumway (2001) and Cox and Rubinstein (1985)), which posits that the expected return of a call option written on the market is positive and exceeds the expected return of the underlying market. The contrasting implications impose testable restrictions on the behavior of call option returns.

Intuition for the results in Theorem 2 can be gained on noting that $m[R]$ in the expression for the price in (4) can be viewed as transformation of the density $p[R]$. Over the strike region corresponding to the increasing part of the U-shaped kernel, this transformation amounts to shifting probability mass towards states where the call option is more valuable (see equation (3)). Hence, the call price decreases by less than the expected payoff of the call as the strike increases, bringing expected returns to lower, and eventually to negative values. The effect is more pronounced at higher K. In what sense exaggerating the slopes of the pricing kernel translates into more negative expected returns is established in Lemma 1 of Appendix A.

Theorem 3 *If the economy supports a U-shaped pricing kernel, then the following statements are true regarding expected returns of digital calls.*

(a) Expected returns of digital calls with strikes $K > S_t\, e^{R_u}$ *are decreasing in the strike;*

(b) There exists a strike price $K_d > S_t\, e^{R_u}$ *such that digital calls with strikes higher than* K_d *have negative expected returns;*

(c) The expected return of a digital call option with strike $K > S_t\, e^{R_u}$ *exceeds the expected return of a call option with the same strike and time to expiration.*

Proof: See Appendix A. □

At the root of part (a) and part (b) of Theorem 3 is the fact that the price of the digital call falls by less than the expected payoff due to the increasing kernel and the non-decreasing payoff function of the digital. Furthermore, this effect is weaker than in the case of call options, since a smaller proportion of the price is derived from the region where the kernel takes large values, hence the validity of the statement in part (c).

Besides, it can be shown that the statement in part (c) of Theorem 3 is reversed when the pricing kernel is monotonically declining, and hence yields an additional testable hypothesis, to be exploited in discriminating between U-shaped and monotonically declining kernels. Lemma 2 of Appendix A furthermore establishes that the expected returns of digital calls are increasing in strike when the pricing kernel is monotonically declining.

Theorem 4 *If the economy supports a U-shaped pricing kernel, then the following statements are true regarding expected returns of kernel calls.*

(a) Expected returns of kernel calls with strikes $K > S_t\, e^{R_u}$ *are decreasing in the strike;*

(b) There exists a strike price $K_h > S_t\, e^{R_u}$ *such that kernel calls with strikes higher than* K_h *have negative expected returns.*

Proof: See Appendix A. □

Theorem 4 is a consequence of the shift in probability mass due to the increasing region in the pricing kernel. In addition, the price of any claim on the upside can be seen as an inner product (against the physical probability) of the claim's payoff and the pricing kernel, which is maximized when the claim is the kernel itself. Therefore, kernel calls exhibit low expected returns provided the pricing kernel grows fast enough, and hence studying kernel call returns can potentially offer an additional source of identification.

Lemma 3 of Appendix A shows that the partial derivative of the expected return of $m[R]\mathbf{1}_{e^R>y}$ with respect to strike is positive for monotonically declining kernels. Deducing whether the pricing kernel is U-shaped or monotonically declining has broader ramifications for asset pricing theory.

3. Empirical analysis of the implications of U-shaped pricing kernels

This section examines whether the predictions in Theorems 2 through 4 are consistent with various dimensions of the data and presents empirical evidence from several perspectives. While the claims on the upside are qualitatively linked in theory, our interest lies in the magnitude of the quantitative effects.

3.1. Average S&P 500 index call returns decrease in moneyness and are negative OTM

To test the theoretical predictions that put on a firmer footing our ability to distinguish between monotonically declining kernels and U-shaped kernels, we build time series of prices and returns to S&P 500 index options. Corresponding to three moneyness levels, we select calls and puts that are closest to 1%, 3%, and 5% OTM. This particular choice facilitates the construction of time-series of returns for each moneyness with no missing observations, whereby the average moneyness in the sample is 1.017%, 2.982%, and 4.944%, respectively.[2]

Following an established practice, in-the-money S&P 500 index options are discarded to mitigate possible illiquidity concerns (Bates (1991), Bakshi, Kapadia, and Madan (2003), Bollen and Whaley (2004), and Huang and Wu (2004)). The sample period is 01/1988 to 05/2007 and covers almost all available time-series data on S&P 500 index options. We do not use option data before 1988 to avoid mixing pre-crash and

[2] In principle, one could also construct price time series for index calls with strikes beyond 5% out-of-the money, by resorting to some extrapolation technique when such option prices are missing. However, we are hesitant to take this approach, since (i) $(S_{t+T} - K)^+$ will be zero for a vast majority of the observations in such time series, thereby skewing option returns towards -100%, and (ii) deep out-of-the-money index call option prices can fall below the minimum tick size in low volatility periods.

post-crash return distributions, which have been shown to differ significantly (see e.g., Rubinstein (1994), and Jackwerth (2000)). The description of the option data is provided in Appendix B.

To align the empirical approach with the postulated theory that pertains to a fixed time to expiration T, we construct time series of S&P 500 index option returns over *non-overlapping* 28-day intervals. At time t, a long position is taken in an out-of-the-money call option with moneyness y and T=28 days. The net return of the call option position is computed as,

$$r^c_{t,T}[y] \;=\; \frac{(S_t\, e^R - y\, S_t)^+}{C_{t,T}[y]} \;-\; 1, \tag{7}$$

where $(S_t\, e^R - y\, S_t)^+$ is the realized payoff of the contract at maturity, i.e., at time $t+28$ days. The procedure is repeated for each of the non-overlapping 28-day intervals, yielding a time series of index call option returns with 233 observations.

The empirical results in Table 1 on average returns of S&P 500 index calls present the first piece of evidence that is at the heart of our inquiry. First, the prediction that expected returns are uniformly positive is refuted for S&P 500 calls. Over the entire 01/1988 to 05/2007 sample period, the average return of a 3% OTM index call is -2.2% per month, while the average return of a 5% OTM call is -5.8% per month. Similar conclusions can be drawn based on the 01/1997 to 05/2007 subsample. Second, the empirical results do not support the prediction of the theory, based on monotonically declining pricing kernels, which asserts that expected returns of index calls are increasing with strike, since, in fact, returns decline in magnitude across the three call option time series as moneyness y increases.[3]

Two other points are equally important and consistent with our theory. On one hand, it is imperative to realize that the average returns of 3% and 5% OTM index calls are lower than the average return of the S&P 500 index. Specifically, over 01/1988 to 05/2007 the S&P 500 index increased on average by 0.926% per month. Thus, the failure of the classical pricing kernels to explain observed negative returns of OTM calls cannot be attributed to possibly negative average index returns. On the other hand, the near-the-money call, i.e., $y = 1\%$, exhibits *positive* average return of 7.0% per month, higher than the market return. Since expected call returns are negative for strikes in the upper tail, as shown in Theorem 2, such a finding is not only consistent with the theory of monotonically declining kernels and the empirical results reported by Coval and Shumway (2001), but can also be consistent with a U-shaped pricing kernel.

[3] Appreciate here that our focus is only on the sign of expected call returns and on the difference in call returns across moneyness, and not on the magnitude of returns. Indeed, such a focus partly mitigates Black's criticism of the empirical methodologies for estimation of expected returns (see Black (1993)).

What is the statistical significance of the results on call returns, reported in Table 1? Given option return non-normalities, we follow Bondarenko (2003), Goetzmann, Ingersoll, Spiegel, and Welch (2007), Hasanhodzic and Lo (2007), and Santa-Clara and Saretto (2009), and bootstrap the call return time series. Since the Ljung-Box test at lags up to 20 rejects autocorrelation in these time series, we perform i.i.d. bootstrap. First, for average call returns, the 90% confidence intervals obtained in $\mathcal{N} = 25,000$ bootstrap draws are shown in square brackets. In all cases these intervals include the zero, thus rejecting significance at conventional levels. Second, to test for difference in average returns across strikes, we draw $\mathcal{N}$ pairwise bootstrap samples of returns to options with strikes 1% and 3%, and 3% and 5% OTM, respectively. Then, for each pair of bootstrap samples, we calculate the difference between average returns: a negative difference indicates that deeper OTM options have higher average return in the respective bootstrap sample. The reported p-values represent the proportion of negative differences in the $\mathcal{N}$ bootstrap samples, for each pair of strikes. These p-values are 0.14 or higher for the entire sample, and 0.32 or higher for the subsample. Again, at conventional confidence levels the pairwise differences in average call returns across the three moneyness groups are insignificant.

Based on this yardstick, the statistical evidence for negative average index call returns, or call returns declining in strike is inconclusive. However, we argue later in Section 4.3 that even if insignificant on statistical grounds, negative average call returns are consistent with, and provide support for U-shaped pricing kernels. Furthermore, the evidence from Table 1 is clearly inconsistent with a monotonically declining pricing kernel.

To cast a wider perspective on the theory and statistical significance, Table 1 also shows the counterpart results for index put option returns. Average put returns (i) are uniformly negative, and (ii) tend to decrease for deeper OTM puts. We emphasize that these two properties are consistent both with the negative beta interpretation of put options and with a declining kernel, and are compatible with the evidence in Bondarenko (2003), Benzoni, Collin-Dufresne, and Goldstein (2007), and Broadie, Chernov, and Johannes (2009). Furthermore, the negative average put returns are significant at least at the 90% confidence level, with one exception in the subsample, and average returns of deeper OTM puts are significantly lower than those of 1% OTM puts. Herein lies a distinction between the negative average index call and put returns.

Note that Broadie, Chernov, and Johannes (2009) show that put option returns are not inconsistent with the Black and Scholes (1973) option model. From Rubinstein (1976) we know that the Black-Scholes model can be derived under the specification of lognormally distributed market index (under the physical measure) and a representative agent with power utility, which is subsumed within the class of monotonically declining kernels. Yet, it is the contention of this paper that extant option models, based on monotonically declining kernels, cannot generate negative average *call* option returns. Supportive simulation-based evidence is available from the authors.

Adding to the above statements, we observe in Table 1 that the documented negative average returns of index calls are far smaller in magnitude than the negative index put returns. This difference in magnitude may be due to the following reasons. First, while investors shorting the market may be hedging their positions, as we hypothesize later, by buying OTM index calls for loss protection, this demand is lower than the counterpart demand for OTM index puts, as there are fewer shorts generally. Second, call option buyers find natural supply from covered call writers, whereas no analogous supply channel exists for put buyers. Finally, market participants are known to finance long OTM index puts by writing calls, thus suppressing index call prices and limiting the severity of negative index call returns. These observations are compatible with the economic channel, explored in Section 4, which links a modeling framework that accommodates heterogeneity in beliefs about return outcomes and shorting with our findings on call returns.

To relate our findings to extant literature, note that Table 1, Panel II of Bondarenko (2003) presents summary statistics showing that, with the exception of deep OTM index calls, average returns of OTM calls are negative. Also, the bootstrapped confidence bands on average OTM call returns are wide and include zero. However, unlike this paper, the goal of Bondarenko (2003) is to understand index put returns and he fails to ascribe the call return behavior to a specific shape of the pricing kernel. The evidence discussed here is consistent with Jackwerth (2000), who does show that the risk aversion function implied by S&P 500 index option prices is partially increasing, but does not examine call returns.

Overall, Table 1 reveals a puzzling contrast between the predictions of the theory based on monotonically declining pricing kernels and certain aspects of the empirical evidence from the index options market. In particular, average call option returns *decrease* with strike, and are *negative* for deeper OTM calls. At the same time, the theory is supported by complementary pieces of empirical evidence, namely on average returns of index put options. It is the behavior of farther OTM index calls that is inconsistent with an asset pricing theory based on classical pricing kernels.

3.2. Average returns of calls are broadly negative and decrease in moneyness across international equity markets

Now we present empirical evidence from major international equity markets, the aim of which is to establish the broad nature of negative average equity index call returns. We consider call options written on the following underliers: (i) the FTSE-100 index, (ii) the Nikkei-225 index, (iii) the German DAX index, (iv) the Swiss Market index (SMI), (v) the Hang Seng index (HSI), and (vi) the Australian All Ordinaries index (ALO). The data for this exercise comes from over-the-counter option quotes from a major bank in London. Made available to us in a standard format, the call options have time to expiration of 30 days and moneyness 0%, 5% and 10% OTM. The sample covers the 10-year period 05/1995 to 05/2005, and is comparable to the studies of Bliss and Panigirtzoglou (2004), and Foresi and Wu (2005).

Three observations can be made from Table 2. First, the average returns of OTM calls are negative in 11 out of 12 OTM categories. Moreover, average call returns tend to be more negative than those of S&P 500 index calls. For instance, the monthly average return of 5% OTM calls across the 6 international equity markets is -28% versus -5.8% for the S&P 500 index calls. Second, average call returns are mostly decreasing with moneyness. Third, we find average returns of several 0% OTM calls to be positive, but there is also evidence to the contrary (the FTSE, NIKKEI, and ALO). Overall, the empirical evidence from international option markets is broadly supportive of U-shaped kernels, and strongly refutes the implications of monotonically declining pricing kernels.

Turning to bootstrap-based statistical significance, we observe that for three out of the six equity indexes, namely the FTSE, NIKKEI and ALO, the OTM average call returns are significantly negative at the 10% bootstrap significance level. An even clearer picture emerges when we assess the difference in average returns across strikes by pairwise bootstrapping returns of options with strikes 0% and 5%, and 5% and 10%, respectively. The majority of these p-values are close to, or lower than 5%, and hence the support for U-shaped pricing kernels is stronger in our evidence gathered from international equity markets.

3.3. Return patterns of digital calls defy classical pricing kernels, but favor U-shaped kernels

In order to further evaluate the promise of U-shaped kernels, we explore compatibility with additional aspects of the data and, in particular, we test the predictions of Theorem 3 on the returns of digital calls.

To proceed, we approximate the return of a digital call as:

$$r^d_{t,T}[y] \;\approx\; \frac{\left(S_t\, e^R - y(1-\delta)\, S_t\right)^+ \;-\; \left(S_t\, e^R - y\,(1+\delta) S_t\right)^+}{C_{t,T}[y(1-\delta)] \;-\; C_{t,T}[y(1+\delta)]} \;-\; 1, \tag{8}$$

where $C_{t,T}[y(1-\delta)]$ and $C_{t,T}[y(1+\delta)]$ are the time t prices of call options with moneyness $y(1-\delta) = \frac{K(1-\delta)}{S_t}$ and $y(1+\delta) = \frac{K(1+\delta)}{S_t}$, respectively, with $\delta > 0$ (e.g, Cox and Rubinstein (1985), Chriss and Ong (1995), and Carr, Ellis, and Gupta (1998)).

Table 3 presents the average returns of digital calls, approximated via S&P 500 index call spreads as in equation (8), together with bootstrapped 90% confidence intervals. Here, given data constraints, we set $\delta = 1\%$, carefully interpolating the implied volatilities (e.g., Broadie and Detemple (1996)), and consider digital calls with strikes 1% and 3% OTM, respectively (we do not consider the 5% OTM digital call, which would require excessive extrapolation for the 6% OTM call). The central point to observe is that average returns are declining, and the difference in average returns between the two digitals is negative with a bootstrapped p-value of 8.3%.

While still consistent with Theorem 3, the positive signs of the reported average digital call returns indicate that one may need to go farther OTM to observe negative returns. However, it is admittedly difficult to evaluate the expected payout of a far OTM digital call, as it equals the physical probability of $S_{t+T} > K$, and the event is rare with few outcomes in the data set. Therefore, we focus on the sharper result in part (b) of Theorem 3, and conclude that the drop in average digital return from 20.6% to 10.0%, as reported in Table 3, is in favor of the U-shaped pricing kernel.

3.4. Return restrictions across calls and digitals conform with U-shaped kernels

Concentrate now on examining the cross-asset linkages between the expected returns of calls and digitals, as stated in part (c) of Theorem 3. Specifically, the thrust is to study the implication that $\mu^d_{t,T}[y] > \mu^c_{t,T}[y]$ under U-shaped kernels. To make comparisons meaningful, we again couch the hypothesis in statistical terms by drawing $\mathcal{N} = 25,000$ pairwise bootstrap samples of returns of calls and digitals with moneyness 1% and 3% OTM, respectively. Then, for each pair of bootstrap samples, we calculate the difference between average returns.

From this perspective, we see again that the economy-wide valuation structure is compatible with a U-shaped pricing kernel and not with a monotonically declining pricing kernel. In particular, we observe that the average return of 3% OTM calls exceeds the average return of the corresponding digitals in 13% of the bootstrap samples; at the same time, the respective proportion is only 2.4% for the 1% OTM calls versus the 1% OTM digitals, furnishing statistical evidence that concurs with the wider predictions based on U-shaped pricing kernels.

3.5. Average returns of the upside variance contract are also declining in strike

To add to our discussion of the empirical relevance of U-shaped pricing kernels, we analyze returns of the upside variance contract, given by:

$$r^{v}_{t,T}[y] \;=\; \frac{(R-\overline{\mu})^{2}\,\mathbf{1}_{e^{R}>y}}{V^{u}_{t,T}[y]} \;-\; 1, \tag{9}$$

where, as in (6), the price $V^{u}_{t,T}[y]$ of the payoff $(R-\overline{\mu})^{2}\,\mathbf{1}_{e^{R}>y}$ in (9) can be synthesized as:

$$V^{u}_{t,T}[y] = \left(\ln\left(\frac{K}{S_t}\right) - \overline{\mu}\right)^{2} D[K] + \frac{2}{K}\left(\ln\left(\frac{K}{S_t}\right) - \overline{\mu}\right) C[K] + \int_{\{\mathbb{K}>K\}} \frac{2}{\mathbb{K}^{2}}\left(1-\ln\left(\mathbb{K}/S_t\right)+\overline{\mu}\right) C[\mathbb{K}]d\mathbb{K}. \tag{10}$$

$D[K]$ represents the price of a digital call option and $C[K]$ the price of a call option with strike K. $\overline{\mu}$ is set to 1% per month, matching the sample average for S&P 500 index returns. Such contracts can be useful in hedging the volatility risk embedded in out-of-the-money index call options, and their expected returns share the properties of the claims on the upside, as described in Theorems 2, 3, and 4.

The general take from the results reported in Table 4 can be summarized from two angles. At the outset, observe that the average $\sqrt{V^{u}_{t,T}[y]}$ for 1% and 3% OTM contracts is 8.67% and 8.38% (annualized), respectively. Unlike the total variance contract (e.g., Bakshi and Madan (2006) and Carr and Wu (2009)) that hedges volatility risk in options generally, these contracts have somewhat lower prices as they avoid exposure to put options. Thus, our findings highlight the disparity in the pricing of the volatility payoff in declining versus rising markets. From another angle, average returns of the upside variance contract decline in strike with low p-values, posing a hurdle to a theory based on monotonically declining kernels.

3.6. The pricing kernel projection is U-shaped when basis assets include options on the market

We next ask whether a U-shaped pricing kernel is supported by the *set* of basis assets we consider. To shed light on this question, we follow Hansen and Richards (1987), Chen and Knez (1996), and Cochrane

(2005), and study the traits of the mimicking portfolio for the pricing kernel.

Given a set of N basis assets, let Z denote the $N \times 1$ vector of gross returns, or the $N \times \mathcal{T}$ matrix of time-series observations of gross returns on these assets. The mimicking portfolio for the pricing kernel is $m^* = Z^\top \alpha$, with a vector of constant weights α, and such that $\iota_N = \mathbb{E}[Zm^*]$, for a vector of ones ι_N. It can be shown that (see Chen and Knez (1996, page 533) or Cochrane (2005, page 66)),

$$m^* \;=\; Z^\top \mathbb{E}[ZZ^\top]^{-1} \iota_N. \tag{11}$$

The line of thinking explored here is that the projection m^* given in (11) is likely to yield a better approximation to a true pricing kernel, if a collection of options on the market index are included among the basis assets. This conjecture follows from the spanning property of the option payoffs, as established in Ross (1976); it also follows from the fact that functions of the market return are in the span of the payoffs of OTM calls and puts, in the L^2-norm (e.g., Carr and Madan (2001) and Bakshi and Madan (2000)).

Directed by these theoretical connections, we take a set of basis assets which include call and put options on the market index and calculate m^* through (11). Our Z includes 28-day gross returns to (i) a riskfree asset, (ii) the market index (i.e., S&P 500), and (iii) call and put options on the S&P 500 index that are 1%, 3%, and 5% OTM. The mimicking portfolio for the pricing kernel recovered this way is based on a particular set of basis assets, a feature shared with, among others, Chapman (1997), Vanden (2004), and Brandt, Cochrane, and Santa-Clara (2006).

[Fig. 1 about here.]

Figure 1 is revealing, as it indicates the existence of U-shaped pricing kernels by appealing to the characterization in (11). Specifically, what we plot is the smoothed version of the mimicking portfolio (on the y-axis) against S&P 500 index returns (on the x-axis). The smoothed version is extracted by applying the Nadaraya-Watson kernel regression with Gaussian kernel on m^* (e.g., Pagan and Ullah (1999) and Aït-Sahalia and Duarte (2003)). We also present the 95% point-wise confidence intervals, obtained by applying the kernel regression on bootstrapped values of the pricing kernel in 25,000 bootstrap draws.

Speaking to the consistency of the underlying result, the mimicking portfolio for the pricing kernel shown in Figure 1 obeys a U-shaped pattern in the entire sample (Panel A), in each of the subsamples (Panels B and C), and also out-of-sample (Panel D). The latter mimicking portfolio for the pricing kernel is recovered by computing the weights $\alpha = \mathbb{E}[ZZ^{\top}]^{-1}\,\iota_N$ based on the initial test sample 01/1988-12/1996, and applying them to the 01/1997-05/2007 sample period.

Having call options in the set of basis assets does not necessarily assure a U-shaped mimicking portfolio for the pricing kernel: if the weights α_3, α_4, and α_5 on call option returns in (52) are all negative, then m^* can only be declining in market return. Therefore, at least one of the weights on the call returns must be positive. Indeed, this feature is confirmed in our data and it drives the increasing region in Figure 1.

The displayed m^* are asymmetrically U-shaped, meaning that the market data reflects greater weighting to downside versus upside return movements. Such an outcome is not at odds with the evidence in Table 1 that average returns of OTM index puts are substantially more negative compared to OTM index calls. Moreover, the U-shaped pricing kernel obtained in the 01/1997-5/2007 subsample is a left-shifted version of the 01/1988-12/1996 counterpart and not strictly convex.

In sum, a concrete yet complementary picture emerges from our nonparametric approach, and adds to our body of evidence that upholds the presence of U-shaped pricing kernels.

3.7. Kernel calls exemplify average returns that are decreasing in moneyness

To expand on the explicit shape of the mimicking portfolio for the pricing kernel shown in Figure 1, we finally investigate the predictions of Theorem 4 for expected returns of kernel calls. The return of this construct is given by,

$$r^h_{t,T}[y] \;=\; \frac{m[R]\mathbf{1}_{e^R>y}}{H_{t,T}[y]} \;-\; 1, \tag{12}$$

where $m[R]\mathbf{1}_{e^R>y}$ is the realized payoff at $t+28$ days, and $m[R]$ is based on (11). Appendix C outlines a procedure for computing $H_{t,T}[y]$ based on a portfolio of S&P 500 index calls and digitals.

Table 5 reports average returns for kernel calls, together with bootstrapped 90% confidence intervals, and p-values for the bootstrapped differences in average returns across strikes. The results confirm a declining pattern for average returns from 22.05% to 10.24%. Second, the reported p-values imply a difference, significant at 10%, between returns of kernel calls with moneyness 1% and 3% OTM. Both findings are in conflict with the fundamental properties of monotonically declining kernels.

The average returns in Table 5, together with Figure 1, enrich our understanding by showing that the pricing kernel contains an increasing region. Thus, our analysis brings to bear evidence that reinforces the case for an asset pricing theory based on U-shaped kernels. Moreover, our exercises suggest that it may be hard to redeem monotonically declining kernels given their broad inconsistency with the return behavior of claims paying out on the upside.

Before moving on to a possible theoretical reconciliation of the documented results, note that we do not pursue the line of inquiry based on individual equity options. While studying expected returns of individual options is important and their analysis could be embedded within this study, it should be approached carefully. First, individual equity options allow for early exercise, muddling the construction of the call payoff. Second, studying expected returns of individual equity options requires further hypotheses about their return generating process and the contribution of the systematic versus the idiosyncratic component (Duan and Wei (2009) and Ni (2008)). That is, the projection of the pricing kernel onto the space of individual equity returns introduces further complexities related to the covariation between individual equity and market returns.

4. Relating stylized results to a model with heterogeneity in beliefs

Although, in principle, one can specify exogenously a pricing kernel with the desired shape as in Constantinides (1992), one concern is whether pricing kernels with increasing regions can be the outcome of investors' optimal behavior, and whether models based on such kernels are capable of sustaining negative expected returns of claims on the upside with magnitudes resembling the empirical counterparts. The conflict is that in economies where investors have homogeneous beliefs, the risk sharing theorems (e.g., Constantinides (1982)) hold, and imply a pricing kernel that is monotonically declining in market return. Such economies cannot theoretically accommodate a positively sloped region in the pricing kernel.

Addressing the above concern, we wish to accomplish one key objective here. To do so, we resort to a model where a positively sloped region in the pricing kernel can be *endogenously* generated through heterogeneity in beliefs about return outcomes, and where investors are shorting equity. Under a parameterized version of the model, we ask whether OTM calls can have negative expected returns, decreasing in strike, and whether other claims on the upside can have expected returns, decreasing in strike, while puts simultaneously have significantly more negative expected returns, increasing in strike.

4.1. *Outline of a model accommodating heterogeneity in beliefs about return outcomes*

To describe a possible economic environment generating U-shaped kernels, we appeal to the modeling framework in Bakshi and Madan (2008), where investors exhibit heterogeneity in their beliefs about return outcomes. Such a model allows for long and short equity positions, but abstracts from option positions on the underlying equity index. Other treatments of economies with heterogeneity in beliefs can be found in Basak (2005), Basak and Croitoru (2006), Buraschi and Jiltsov (2006), Kogan, Ross, Wang, and Westerfield (2006), Blume and Easley (2007), and Gallmeyer and Hollifield (2008).

In accordance with Theorem 1 in Bakshi and Madan (2008) and with our goal to study a parameterized version of the model, we adopt pricing kernels of the type:

$$m[R] = \sum_{\omega} \mathbb{N}_{\omega}[R], \tag{13}$$

where $\omega \in (0,1)$ represents the optimal fraction of wealth invested/disinvested in equity, and,

$$\begin{aligned} \mathbb{N}_{\omega}[R] \;\; = \;\; & \xi^L[R]\left(\omega \times e^R + (1-\omega) \times (1+r_f)\right)^{-\gamma} \Psi^L_{\omega} \;\; + \\ & \xi^S[R]\left(-\omega \times e^R + (1+\omega) \times (1+r_f)\right)^{-\gamma} (1-\Psi^L_{\omega}). \end{aligned} \tag{14}$$

In equation (14), $\left(\omega \times e^R + (1-\omega) \times (1+r_f)\right)^{-\gamma}$ and $\left(-\omega \times e^R + (1+\omega) \times (1+r_f)\right)^{-\gamma}$ are respectively the marginal rate of substitution of long and short equity investors, characterized by a given optimal fraction ω invested/disinvested in equity. r_f is the riskfree return and the risk aversion coefficient of the investors is γ. The change-of-measure densities $\xi^L[R]$ and $\xi^S[R]$ are parameterized by $(\beta^L_0, \beta^L, \mu_*, \beta^S_0, \beta^S, \mu_{**})$, and specified below:

$$\xi^L[R] \; = \; \beta^L_0 \; + \; (1-\beta^L_0)\left(\frac{\exp(\beta^L(R-\mu_*))}{1+\exp(\beta^L\,(R-\mu_*))}\right), \qquad 0 < \beta^L_0 < 1, \qquad \beta^L > 0, \tag{15}$$

$$\xi^S[R] \; = \; \beta^S_0 \; + \; (1-\beta^S_0)\left(\frac{\exp(-\beta^S(R-\mu_{**}))}{1+\exp(-\beta^S\,(R-\mu_{**}))}\right), \qquad 0 < \beta^S_0 < 1, \qquad \beta^S > 0. \tag{16}$$

The densities $\xi^L[R]$ and $\xi^S[R]$ in (15)-(16) capture the return beliefs of long and short equity investors, respectively. In particular, $\xi^L[R]$ is an increasing function, while $\xi^S[R]$ is a decreasing function, implying that long and short equity investors invest and disinvest equity consistent with their return beliefs. The coefficients μ_* and μ_{**} determine the return regions in which the measure changes are active. Exact details on the optimization problems solved by long and short equity investors are in Bakshi and Madan (2008).

Closing the description of the model, Ψ^L_ω denotes the proportion of investors in a cohort that are long equity a fraction ω of their wealth, and $1-\Psi^L_\omega$ denotes the proportion of investors that are short equity and disinvesting a fraction ω of their wealth. In economies populated by investors with heterogeneity in beliefs, the pricing kernel is a change-of-measure tilted function of the marginal rate of substitution of long and short equity investors, as specified in (13). When some investors are shorting equity in this framework, the pricing kernel can admit a positively sloped region.

To connect the theoretical framework in (13) to our empirical findings, we hypothesize that OTM index calls serve short equity positions in the same way as OTM index puts serve long equity positions, insofar as calls and puts provide loss protection. In particular, the declining region of the kernel reflects the risk aversion to downside market moves of investors long equity, which translates into demand for protection via OTM index puts. In parallel, the increasing region of the kernel reflects the risk aversion to upside market moves of investors shorting equity, who seek protection by purchasing OTM index calls. In both cases, the premium paid for loss protection results in negative expected returns of OTM call and put options.

Guided by this economic intuition, we first investigate in Subsections 4.2 and 4.3 the generic implications of the model for expected returns and their dispersion respectively, and then in Section 5 the interlinkages between the strength of the increasing region of the pricing kernel, aggregate shorting activity (as measured by both new short sales and short interest), and call options.

4.2. Can a parameterized version of the model explain stylized features of the returns data?

This subsection evaluates whether the framework with heterogeneity in beliefs about return outcomes, which endogenously generates a U-shaped pricing kernel, can reasonably reproduce the observed average returns of (i) calls, (ii) puts, (iii) digital calls, and (iv) upside variance contracts. In the vein of recent studies (e.g., Liu, Pan, and Wang (2005), Benzoni, Collin-Dufresne, and Goldstein (2007), and Santa-Clara and Yan (2009)), our motivation is to gauge the model's ability to reconcile observed economic phenomena.

Keeping parsimony of parametrization in mind, we consider three cohorts of investors, each cohort containing both long and short investors, differentiated by their beliefs about return outcomes. Each long/short investor in a cohort optimally invests/disinvests the same fraction ω of their wealth in equity. To compute the optimal equity positions, we employ $p[R]$ suitably truncated Normal with mean 1% and standard deviation 5%, similar to those for index returns over $T = 28$ days, and we assume $R \in [-\ln(2), \ln(2)]$ to keep $\mathbb{N}_\omega[R]$ in equation (14) finite. Throughout, we keep fixed $\gamma = 12$, $r_f = 0$, and $\beta^L = \beta^S = 50$, while varying the remaining two parameters in each change-of-measure density, separately for long and short investors.

The optimal equity investment/disinvestment across cohorts ranges between 50% and 70% of cohort wealth, as seen by the values of ω in Panel A of Table 6. Ψ^L_ω are chosen to ensure that long equity positions dominate the short counterparts, while the initial wealth of cohorts can be fixed so that total equity demand matches supply outstanding. In this setting the pricing kernel $m[R] = \sum_\omega \mathbb{N}_\omega[R]$ is an asymmetric U-shaped function of market return, where the slope over the increasing region critically depends on the strength of investors shorting equity.

We appraise the model by looking at its ability to generate plausible expected returns of the contingent claims. In particular, we compute numerically expected returns under $p[R]$, using the pricing kernel $m[R]$, and present the results in Panel B of Table 6.

Accommodating different strengths of the declining and increasing region of the pricing kernel through our parametrization, the model is able to qualitatively mimic the features of S&P 500 index option returns reported in Table 1. Model-based call returns decrease in strike, whereby OTM calls have negative expected returns (as in Theorem 2), while calls closest to at-the-money have positive expected returns. What can also be inferred is that puts have substantially more negative expected returns, increasing in strike. For instance, the 3% OTM call (put) in our sample has average return of -2.2% (-58.6%), whereas the model-based expected call (put) return at the same moneyness is -1.3% (-55.4%). The essential point to be made here is that the model shows flexibility to match the empirical counterparts of *both* call and put expected returns.

A question worth asking is whether the model generates volatility smirks, and here we employ model-determined option prices to infer the corresponding Black-Scholes implied volatility as a function of option moneyness ($y = K/S$). As seen in Figure 2, model-generated implied volatility declines with y over $y \in [0.94, 1.06]$, which is typical, as documented in Rubinstein (1994) and Pan (2002). Therefore, a model with heterogeneity in beliefs can replicate the behavior of both (i) the expected returns of calls and puts, and (ii) option prices across moneyness, as reflected in the Black-Scholes volatility smirk. In this sense, we obtain corroboration of model adequacy from distinct perspectives. Finally, a volatility smirk, as produced by the model, can be conceptually consistent with negative call option returns and the possible overpricing of calls, as long as the pricing density dominates the physical density in the upper tail, which could occur if the volatility implied in OTM call prices is higher than physical volatility.

[Fig. 2 about here.]

What about the model's ability to reproduce average returns beyond those of calls and puts? In this regard, we observe that while the average sample returns for 1% and 3% OTM digital calls are 20.6% and 10.0%, the model-generated expected return are 15.1% and 12.4% respectively, as reported below:

% OTM	1%	3%	5%	
Expected digital call returns	15.1	12.4	7.0	
Expected upside variance contract returns	-7.7	-8.1	-10.5	
Risk-neutral volatility (28-day)				6.3
Risk-neutral skewness (28-day)				-0.64

Thus, our exercise suggests that a model based on heterogeneity in beliefs can be successful in simultaneously explaining expected returns of digital calls. Additionally, we observe that model-based expected returns of the upside variance contract decrease in strike, an attribute that is in accord with U-shaped pricing kernels, with magnitudes corresponding to those reported in Table 4.

We further note that the model is consistent with the risk-neutral volatility and risk-neutral skewness embedded in index options. Specifically, the model-generated monthly risk-neutral volatility is akin to that reported in Jiang and Tian (2005), Bakshi and Madan (2006), Carr and Wu (2009), and Chernov (2007). Comparable to Jackwerth and Rubinstein (1996), the short-maturity risk-neutral return distribution deduced from the model is substantially negatively skewed.

To summarize, features of the data that appear puzzling under monotonically declining pricing kernels, e.g., negative or declining in strike average returns of index calls and digitals, do not pose a hurdle within a suitably parameterized version of a model that is also able to match the observed highly negative returns of index puts and at the same time endogenously generates a U-shaped pricing kernel.

Before proceeding, note that in the classic approach, an exogenously imposed aggregate dividend process and investor preferences lead to an equilibrium pricing kernel and equity risk premium (as in Lucas (1978)). In a similar fashion, the fundamentals in this economy are the heterogeneity of beliefs among investors, and they lead to an endogenous pricing kernel when coupled with the preferences and derived portfolio positions of optimizing investors. In our parametrization, the equity premium is supported by the pricing kernel, i.e., the Euler equation is satisfied, and in this sense the equity premium represents a proper compensation for risk, as measured by the covariance of returns with the endogenous pricing kernel.

4.3. Bootstrap intervals for model-based average call option returns are wide

Within the context of the economy described in Subsection 4.2, we next construct bootstrap confidence intervals for average call returns. The purpose is to show that the statistical ambiguity in the sign of average returns of claims on the upside, as observed in actual data and discussed in the empirical evaluation in Section 3, can be an intrinsic feature of economies, where valuations are governed by U-shaped pricing kernels.

Matching the length of our option-return time series, we draw $\mathcal{N} = 25,000$ sets of length $\mathcal{T} = 233$ of random variates. Then we calculate the prices of OTM options corresponding to the U-shaped kernel in (14). With these prices and with each set of $\mathcal{T}$ random variates, we calculate $\mathcal{N}$ average returns of calls and puts that are 1%, 3% and 5% OTM.

Panel B of Table 6 reports the 90% confidence intervals (in square brackets) for average returns across the $\mathcal{N}$ time series of length $\mathcal{T}$. The conclusion that comes out is that confidence intervals on the average returns of the claims on the upside under the U-shaped kernel are not tight. In particular, none of the confidence intervals renders significant at 10% the negative estimates of the average returns of calls. Therefore, negative, even when insignificant, average returns of OTM claims on the upside can be consistent with U-shaped pricing kernels.

The overall message is that detecting a U-shaped kernel in returns data is subtle and hence the emphasis on identification by appealing to the claims in (3). On one hand, negative returns of claims on the upside, even if insignificant, reveal evidence against monotonically declining pricing kernels. On the other, average put option returns, as seen in Table 6 are strongly negative, as in the case of monotonically declining pricing kernels, suggesting that put options are inadequate for detecting a U-shaped pricing kernel.

5. Empirical link between the pricing kernel slope, shorting, and call returns

The effect of the slope of the pricing kernel on call options could be tested directly, by regressing call prices or call returns on some measure of the slope of the kernel over its increasing region. It follows from the theory in Section 2 that as the pricing kernel gets more positively sloped over the increasing region, then call prices increase, and expected call returns decrease (Lemma 1, Appendix A). However, despite the simplicity of this prediction, testing the theory is hindered in practice by empirical realities. First, the true parametric form of the pricing kernel is admittedly unknown. Second, lacking such an identification, empirical proxies for the slope of the pricing kernel in its increasing region are likely to be misspecified.

Both drawbacks of the direct estimation strategy could be avoided by exploiting the economic insight on the relation between the slope of the pricing kernel in its increasing region, and short-selling, as argued in this paper and as suggested in Bakshi and Madan (2008). In our approach, we surrogate the slope of the kernel through aggregate shorting, denoted by X_t, and test for its relation with S&P 500 index call prices. In this way we (i) use only observed variables and (ii) emphasize one economic determinant of the relation between index call options and the pricing kernel slope in the increasing region.

5.1. Impact of new short sales and short interest on call prices

Motivated by the above reasoning, the thrust is to test the validity of the theoretical hypothesis using the following specification:

$$\Delta\left(\frac{C_{t,T}[y]}{S_t}\right) \;=\; \eta_0 \,+\, \eta_x \Delta X_t \,+\, \varepsilon_t, \tag{17}$$

where Δ denotes first difference, and $C_{t,T}[y]$ is the price of an index call option with 28 days to expiration and moneyness y. The index call price is scaled by the index level to account for variation of the index over time. Results when the dependent variable is $\Delta C_{t,T}[y]$ are similar, and omitted to avoid repetition.

In the specification (17), we anticipate a positive η_x which would indicate that a stronger short-selling disposition tends to increase call prices.[4] We recognize that while the theory in Section 2 is based on call option expected returns, the relations tested in (17) are stated in terms of call option prices. Such a choice of the dependent variable implies the assumption that the physical density of 28-day market index returns only changes slowly through time, hence changes in expected call returns are, to first order, driven by changes in call prices (i.e., the denominator in (4)).

Some investors choose to short sell equity due to heterogeneity in beliefs about future equity returns (in the model of Bakshi and Madan (2008)), and we test the hypothesis that they hedge the risk of upside movements using long OTM call positions (i.e., $\eta_x > 0$). In our tests, X_t is proxied by both (i) log of new short sales by NYSE members and the public over the week ending at each date t on which $C_{t,T}[y]$ are observed, and (ii) log of short interest at the NYSE. The data source is the NYSE Members Reports and

[4] Since, in the tested hypothesis, the slope of the pricing kernel is connected to shorting activity, our approach is linked, among others, to treatments in Figlewski and Webb (1993), DAvolio (2002), Garleanu, Pedersen, and Poteshman (2009), Geczy, Musto, and Reed (2002), Asquith, Pathak, and Ritter (2005), and Cohen, Diether, and Malloy (2007).

Barrons. Short interest on Spiders is also used from 04/1993 to 05/2007, whereas new short sales data on Spiders is unavailable. While we anticipate that new short sales are more directly linked to current market environment, we recognize that short interest can also bear potentially relevant information due to closed call positions, which can exert downward pressure on call prices and influence the left-hand side of (17).

What is the empirical distribution of the variables used to estimate the regression model (17)? To answer this question, we tabulate the frequency of observations in intervals defined by multiples (ς) of volatility. For instance, for mean $\mu_{\Delta X}$ and standard deviation $\sigma_{\Delta X}$ of ΔX, the interval end-points are given as $\mu_{\Delta X} + \varsigma \times \sigma_{\Delta X}$. With this understanding, we report the frequency for ΔX and $\Delta(C/S)$ for 3% OTM calls, along with the standard Normal variate benchmark:

	ς < -3	$\varsigma \in$ $[-3,-2)$	$\varsigma \in$ $[-2,1)$	$\varsigma \in$ $[-1,0)$	$\varsigma \in$ $(0,1]$	$\varsigma \in$ $(1,2]$	$\varsigma \in$ $(2,3]$	ς > 3
Normal variate	0.0013	0.0215	0.1359	0.3413	0.3413	0.1359	0.0215	0.0013
ΔX (new short sales)	0.0043	0.0216	0.0948	0.3922	0.3491	0.1078	0.0302	0.0000
ΔX (short interest)	0.0129	0.0000	0.069	0.4224	0.4095	0.0690	0.0129	0.0043
$\Delta(C/S)$ for 3% OTM	0.0086	0.0216	0.0689	0.4138	0.3966	0.0603	0.0216	0.0086

Accordingly, care must be taken in estimating the relation in (17), as inference can be plagued by non-normality. Maintaining the flavor of this result, the Jarque-Bera test statistics shows that the hypothesis of normality for ε (reported in Table 7), $\Delta(C/S)$, and ΔX are all rejected. Hence, to allay non-normality concerns in small samples, we follow, among others, Cochrane and Piazzesi (2005) and perform a bootstrap to compute the p-values, carefully accounting for heteroskedasticity of the residuals, as suggested by Davidson and Flachaire (2008).

Table 7 reports the estimates of η_x and the one-sided p-values for the null hypothesis that $\eta_x = 0$, calculated using both heteroskedasticity-consistent estimator and the bootstrap, and reveals several features of the tested relation.

First, a rise in new short sales is unambiguously associated with a rise in $C_{t,T}[y]/S_t$. The explanatory power of the regressions, measured by adjusted R-squared, ranges between 1.5% and 3.2%. Nonetheless, in our specification involving new short sales, the estimates η_x are positive and significant. Certain demand for OTM index calls thus appears to be due to investors controlling for market risk embedded in NYSE new short sales.

Second, we also obtain positive regression coefficients with short interest, but the effect is statistically insignificant. Over shorter sub-samples, we again obtain insignificance, which suggests the robustness of the feature. Furthermore, we include both new short sales and short interest in a multivariate regression, and find that the significance of new short sales is preserved, whereas short interest remains insignificant.

To additionally examine this finding, we also employ log short interest on Spiders to measure X_t and are still unable to statistically reject $\eta_x = 0$. As seen in Figure 3, the short interest on Spiders and at the NYSE share common variation, with a correlation coefficient of 0.91 in levels, and 0.22 in terms of percentage changes. The results from short interest on NYSE and Spiders are mutually consistent.

[Fig. 3 about here.]

The results suggest that new short sales help reveal the postulated link between call prices and short-selling in our sample, whereas the relation between short interest and call prices may be affected by closed short positions. Since short interest (i) increases due to new short sales and (ii) decreases due to closed short positions, to explain the evidence in Panel A and B of Table 7 one needs an explanation for the role of closed short positions.

5.2. Interpretation and further discussion

The observed disparity in the results obtained with new short sales and with short interest appears to counter the intuition that if investors cancel their short equity positions, then they no longer need to hold the long OTM call options to hedge the risk of upside equity movement and hence would sell these calls. When acting this way, investors would push the call prices down at the same time when short interest goes down, thus yielding a link between call prices and short interest. However, short equity investors may not be closing their call option hedges for a few reasons. If they cancel the short equity position to take profit, the cost of the hedge has already been taken into account, and the now deeper OTM call may be kept, given the option bid-ask spread and that investors may no longer perceive a further downward potential in the equity. In the reverse, if the short equity position is canceled to cut losses, the call may not be sold immediately, but kept to maturity to avoid paying the option bid-ask spread and, possibly, to enhance profitability.

To further probe the suggestion that the major impact on call options may be coming from new short sales, and that closing of previous short positions may have little perceptible price impact, we appeal to data on short-maturity S&P 500 call option volume and open interest (from OptionMetrics, over 1/1996 to 5/2007, averaged daily), which allows us to discern a pattern across moneyness groups:

	Average call trading volume (mln.)			Average call open interest (mln.)		
Days to maturity	**15-19**	**20-24**	**25-29**	**15-19**	**20-24**	**25-29**
$[0, 2\%)$ OTM	10.0	8.9	10.1	67.3	61.6	56.4
$[2, 4\%)$ OTM	7.5	7.9	8.1	65.6	62.9	51.9
$[4, 6\%)$ OTM	3.4	4.7	8.2	50.0	54.1	42.8

While trading volume remains relatively stable as time to maturity decreases below one month for call options closer to at-the-money, volume falls sharply for deeper OTM calls. At the same time, call option open interest tends to increase for each moneyness group as maturity declines. This pattern admits the interpretation that trading in calls with maturity between one month and two weeks predominantly results in establishing new option positions, rather than closing existing option positions, for the latter would be associated with flat or decreasing call open interest. The effect is particularly strong for deeper OTM call options. This empirical observation tallies with our suggestion that mechanisms are in place that deter the closing of short-term OTM index calls, and provides a possible explanation for the stronger relation that we find in the data between new short sales and call option prices.

Before closing, we comment on the appropriateness of using index call prices as a response variable in the empirical implementation. In particular, who are the short-sellers, and would they choose index calls to hedge their risk on the upside? The model's framework applies to investors shorting equity, who are exposed to upside equity moves and seek tail-risk protection. This broad description could include, among other players, investors shorting equity index futures, Dedicated Short hedge funds or Long/Short hedge funds that are net short. As far as such players, like their counterparts with long equity exposure, manage portfolios composed of large numbers of stocks, they would seek protection for the portfolio as a whole. In this respect, OTM index options appear to be a natural hedging instrument (for anecdotal evidence, see, for example, WSJ 11/20/2008), and using them in our empirical exercise would adequately reflect the economic intuition underlying the model.[5] On the other hand, when a smaller set of shorted stocks is involved, call options on individual stocks may provide an alternative way to hedge the short positions, as the anecdotal evidence suggests (see WSJ 08/08/2008). The possibility that stock-specific

[5]Some precedence exists from option markets that investors generally adopt index options to protect downward movements, the evidence for which can be found in the dichotomy between the substantially left-skewed pricing (risk-neutral) distributions of the market index and the individual counterparts (e.g., Toft and Prucyk (1997), Bakshi, Kapadia, and Madan (2003), and Bollen and Whaley (2004)). This observed feature of pricing distributions is counterintuitive since the market index is a weighted average of the individual components, and hence the market return distribution should be more symmetric due to the Central Limit Theorem. In summary, individuals buy some, but not full, protection and when they buy protection, they often hedge the market component of risk. Evidence on hedging long equity positions can also be inferred from the pronounced risk-neutral index volatilities compared to the historical, that is, the negative market volatility risk premium.

risk in short equity positions, besides market risk, is being covered may provide further support to our economic rationale, and remains consistent with U-shaped kernels. In addition, when single-name calls are bought to protect short equity positions, the sellers of these calls may be hedging the exposure in large option books by buying index calls, providing a link between the shorting of individual stocks and the prices of index calls.

That shorting might be hedged with OTM calls can be further illustrated, albeit indirectly, by considering the return behavior of the Dedicated Short-Selling index (from CSFB/Tremont, starting in 1994, denoted z_t^{DS}). Given that negative deltas are intrinsic to the baseline strategy of Dedicated Short-Selling hedge funds, this suggestion is pursued from two different perspectives. First, we regress z_t^{DS} on returns of the market (denoted z_t^m) and returns of 3% OTM call options (denoted $z_t^{c,3\%\ otm}$). Second, in an alternative specification, we regress z_t^{DS} on returns of near-the-money calls (denoted $z_t^{c,atm}$) and returns of 3% OTM calls.[6] In both cases, we anticipate a positive regression coefficient on the returns of OTM call options (heteroskedasticity consistent one-sided p-values are shown in square brackets):

$$
\begin{array}{llllllll}
z_t^{DS} = & 0.006 & -1.036\, z_t^m & +0.004\, z_t^{c,3\%\ otm}, & \overline{R}^2\text{=70.0\%}, & \text{DW=1.74}, & \chi^2(2)\text{=192 with } p\text{-value=0.000}, \\
 & [0.005] & [0.000] & [0.042] & & & \\
z_t^{DS} = & 0.002 & -0.084\, z_t^{c,atm} & +0.037\, z_t^{c,3\%\ otm}, & \overline{R}^2\text{=39.5\%}, & \text{DW=2.02}, & \chi^2(2)\text{=129 with } p\text{-value=0.000}. \\
 & [0.012] & [0.000] & [0.000] & & &
\end{array}
$$

For our purposes, there are three points to note: First, we observe a negative coefficient on the returns of the market and the near-the-money calls, reflecting their negative deltas. Second, accounting for the first-order effect of negative delta, the returns of OTM calls exhibit a positive coefficient, which suggests that the negative deltas are being hedged. Finally, the $\chi^2(2)$ test rejects the lack of joint parameter significance at conventional levels.

The takeaway is that the predictions of the theory appear consistent with the data, and the responses of the index call option variables to changes in short-selling conform with the model. The estimates of η_x in (17) are economically sensible: ceteris paribus, increased short selling strengthens the price impact of short equity investors, thus leading to a more strongly increasing region in the pricing kernel. Absent investors shorting equity, the pricing kernel would be monotonically declining and inconsistent with the

[6]As in the study of Agarwal and Naik (2004), we construct the monthly holding-period return of a call option (near-the-money or OTM) by buying an option at the beginning of the month and selling it at the beginning of the next month. When bought (sold), the call option has a maturity of approximately 46 (16) days. The returns of 5% OTM calls have correlation of 0.95 with the returns of 3% OTM calls, and lead to similar results that are omitted.

observed call returns. To reiterate, the economic role of index put options, from the perspective of long equity investors seeking loss protection, is mirrored by the role of index call options from the perspective of investors that are shorting equity. It is this economic channel that renders the average index call option returns negative.[7]

6. Conclusions

Focusing on claims with payout on the upside, it is shown that their average returns contradict the implications of downward-sloping pricing kernels, but can be consistent with those of U-shaped pricing kernels. We draw this conclusion by studying various dimensions of the data, including (i) S&P 500 index calls, (ii) calls on major international equity indexes, (iii) digital calls, (iv) upside variance contracts, and (v) a theoretical construct that we denote as kernel calls. In particular, U-shaped pricing kernels are able to explain the observed negative average returns of OTM index calls, average returns of digital calls and upside variance contracts declining in strike, and highly negative OTM put returns, increasing in strike. Our inquiry suggests that U-shaped pricing kernels present an empirically appealing alternative to monotonically declining counterparts.

To investigate the theoretical underpinnings of U-shaped pricing kernels, we explore the relation between short-selling, slope of the kernel and call options. Here, we adopt a model based on heterogeneity in beliefs about return outcomes, whereby the increasing region in the kernel is due to risk-averse investors, shorting the equity market. We show that the model can generate plausible expected returns of contingent claims, and thus reconcile the salient features of our data. The theoretical connections between short-selling, the slope of the pricing kernel and expected call option returns are tested empirically, and we obtain statistically significant relation between new shorting and expected call returns. The accumulated evidence suggests that U-shaped pricing kernels help to unravel the return patterns of claims with payout on the upside.

[7]To be empirically relevant, the U-shape in the pricing kernel should be manifest over the market return interval of $\pm 5\%$, in order to explain the observed 28-day OTM option returns in our sample. This requirement imposes additional constraints on candidate functional forms for U-shaped pricing kernels, and on the associated economic assumptions. For example, a pricing kernel that reflects a dislike for negative skewness is modeled as quadratic in market return by Harvey and Siddique (2000), which has the potential to generate a symmetric U-shape. However, to the extent that such a kernel does not exhibit an increasing region over the relevant range of market returns, it will have difficulty reproducing negative OTM index call option returns.

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Appendix A: Proof of Results

Proof of Theorem 1. Let $q[R] \equiv \frac{m[R]\,p[R]}{\int m[R]\,p[R]\,dR}$ be the return pricing density, and without any loss of generality let $S_t = 1$ and $r_f = 0$. Hence, write $\frac{q[S_{t+T}]}{p[S_{t+T}]} = m[S_{t+T}]$.

Consider the equivalent program for the claim payout in L^∞, which is:

$$\text{Minimize} \int g_{t,T}[S_{t+T}]\,p[S_{t+T}]\,dS_{t+T}, \tag{18}$$

$$\text{Subject to} \int g_{t,T}[S_{t+T}]\,q[S_{t+T}]\,dS_{t+T} = 1, \tag{19}$$

$$0 \;\leq\; g_{t,T}[S_{t+T}] \;\leq\; b. \tag{20}$$

With the understanding that the elementary inequality constraint (20) is incorporated in the Kuhn-Tucker condition (22), we form the Lagrangian,

$$\mathcal{L} \;=\; \int g_{t,T}[S_{t+T}]\,p[S_{t+T}]\,dS_{t+T} \;+\; \lambda\left(1 \;-\; \int g_{t,T}[S_{t+T}]\,q[S_{t+T}]\,dS_{t+T}\right). \tag{21}$$

The Kuhn-Tucker condition is,

$$\frac{\partial \mathcal{L}}{\partial g_{t,T}[S_{t+T}]} \;=\; \begin{cases} p[S_{t+T}] - \lambda q[S_{t+T}] \leq 0, & \text{If } \; g_{t,T}[S_{t+T}] = b > 0, \\ p[S_{t+T}] - \lambda q[S_{t+T}] \geq 0 & \text{If } \; g_{t,T}[S_{t+T}] = 0. \end{cases} \tag{22}$$

Since the program has linear objective with linear constraint, the solution, when $0 \;\leq\; g_{t,T}[S_{t+T}] \;\leq\; b$, is a digital cash flow that takes the value $b > 0$ on the set below:

$$p[S_{t+T}] \;\leq\; \lambda q[S_{t+T}], \tag{23}$$

and zero otherwise. Denote the set $\mathcal{D} \equiv \{S_{t+T} \,\big|\, p[S_{t+T}] - \lambda q[S_{t+T}] \leq 0\}$. We note that the constraint (19) must be satisfied by the solution to the program, therefore,

$$1 \;=\; b \int \mathbf{1}_{\mathcal{D}}\, q[S_{t+T}]\, dS_{t+T}, \tag{24}$$

and hence λ depends on constant b, and can be accordingly determined to meet this constraint.

Proceeding, suppose $q[S_{t+T}]/p[S_{t+T}]$ is monotonically increasing in the upper tail. We know that because of convexity of the kernel, $q[S_{t+T}]/p[S_{t+T}] \to +\infty$ as $S_{t+T} \to +\infty$, hence the set $\mathcal{D}$ contains intervals of the form $[\widehat{k},+\infty)$, where $\widehat{k}>0$. Hence, $g_{t,T}[S_{t+T}] = b\,\mathbf{1}_{\mathcal{D}}$, and the digital call option is an optimal claim.

The set $\mathcal{D}$ formalizes values of S_{t+T} such that $q[S_{t+T}]/p[S_{t+T}] \geq \frac{1}{\lambda}$, and, as such, determines the expected return of the digital call. To appreciate the properties of the solution, note that,

$$1+\mathbb{E}^{\mathbb{P}}\left(r^d\right) \;=\; \frac{\int \mathbf{1}_{\mathcal{D}}\, p[S_{t+T}]\, dS_{t+T}}{\int \mathbf{1}_{\mathcal{D}}\, q[S_{t+T}]\, dS_{t+T}}, \tag{25}$$

$$< \;\lambda \qquad\qquad \left(\text{since } \; b\int \mathbf{1}_{\mathcal{D}}\, p[S_{t+T}]\, dS_{t+T} - \lambda b \int \mathbf{1}_{\mathcal{D}}\, q[S_{t+T}]\, dS_{t+T} \leq 0\right), \tag{26}$$

and hence negative expected returns of the digital call correspond to $\lambda < 1$. In the set-up of the problem, if λ is low, the set $\mathcal{D}$ is small and the budget constraint is easily satisfied. To find the λ, one increases λ from a small value till the budget constraint is just satisfied. This value of λ must be less than 1 if the budget constraint is violated for $\lambda = 1$. For negative expected return and a solution for $\lambda < 1$, one requires that b be large enough to insure that the budget constraint is violated at $\lambda = 1$.

In sum, for a solution supporting $\lambda < 1$ we have $q[S_{t+T}]/p[S_{t+T}] = m[S_{t+T}] \geq 1/\lambda > 1$ in the increasing region of the kernel. In fact, kernels of the type $m[R] = \phi\, e^{-\gamma R} + (1-\phi)e^{\gamma R}$ satisfy $m[R] > 1$ in the increasing region at some $R > 0$.

When the measure change $q[S_{t+T}]/p[S_{t+T}]$ is monotone decreasing in S_{t+T}, i.e., $\left(\frac{q[S_{t+T}]}{p[S_{t+T}]}\right)' < 0$, then the set $p[S_{t+T}] \leq \lambda q[S_{t+T}]$ has the form $(0,\bar{k}]$ and the desired $g_{t,T}[S_{t+T}]$ is a digital put.

As an alternative to (20), suppose now we consider claims bounded below by zero and above by $b\,S_{t+T}$ and hence $g_{t,T}[S_{t+T}] \in L^1$, and also claims bounded below by zero and above by $b\,\frac{q[S_{t+T}]}{p[S_{t+T}]}$, and hence $g_{t,T}[S_{t+T}] \in L^2$, for $b > 0$. Then the $g_{t,T}[S_{t+T}]$ with the lowest expected return has the structure,

$$g_{t,T}[S_{t+T}] \;=\; \begin{cases} b\,S_{t+T}, & \text{when } \; p[S_{t+T}] - \lambda q[S_{t+T}] \leq 0, \\ b\,\frac{q[S_{t+T}]}{p[S_{t+T}]}, & \text{when } \; p[S_{t+T}] - \lambda q[S_{t+T}] \leq 0, \end{cases} \tag{27}$$

and zero otherwise. The proof follows analogously as the payoff $b\,S_{t+T}\,\mathbf{1}_{\mathcal{D}}$ can be decomposed into that of a digital and a call option. ■

Proof of Theorem 2. In the proofs that follow, we use $\mu^c[K]$ and $\mu^c[y]$ interchangeably. Expected call returns corresponding to strike price K, from equation (4) are:

$$\mu^c[K] \;\equiv\; \mathbb{E}^{\mathbb{P}}\left(r^c_{t,T}[K]\right) \;=\; \frac{\int_{\ln(K/S_t)}^{+\infty}(S_t\, e^R - K)\, p[R]\, dR}{\int_{\ln(K/S_t)}^{+\infty}(S_t\, e^R - K)\, m[R]\, p[R]\, dR} \;-\; 1. \tag{28}$$

Based on Leibniz integral rule,

$$\frac{\partial \mathbb{E}^{\mathbb{P}}\left(r^c_{t,T}[K]\right)}{\partial K} = \frac{-\int_{\ln(K/S_t)}^{+\infty}(S_t\,e^R-K)\,m[R]\,\widehat{p}[R]\,dR+\left(\int_{\ln(K/S_t)}^{+\infty}(S_t\,e^R-K)\,\widehat{p}[R]\,dR\right)\left(\int_{\ln(K/S_t)}^{+\infty}m[R]\,\widehat{p}[R]\,dR\right)}{\left\{\int_{\ln(K/S_t)}^{+\infty}(S_t\,e^R-K)\,m[R]\,\widehat{p}[R]\,dR\right\}^2}, \tag{29}$$

where $\widehat{p}[R] = \frac{p[R]}{\int_{\ln(K/S_t)}^{+\infty} p[R]\,dR}$ is the density of the truncated variable $\left\{R \middle| R > \ln(K/S_t)\right\}$.

Note now that the numerator in (29) is the negative of the covariance of the option payoff and the pricing kernel under the $\widehat{p}[R]$ measure (see also Coval and Shumway (2001)). Appealing to the definition of the U-shaped kernel, this covariance is positive for $K > S_t\,e^{R_u}$, hence the derivative of expected call returns with respect to K is negative for strikes $K > S_t\,e^{R_u}$ and expected call returns decline in the strike price for this range of strikes.

The remainder follows from (28) and the definition of a U-shaped kernel. If there exists an $R_0 > R_u$ such that $m[R_0] = 1$, then $m[R]p[R] > p[R]$ for all $R > R_0$, and hence $\mathbb{E}^{\mathbb{P}}\left(r^c_{t,T}[K]\right) < 0$ for all strikes $K > S_t\,e^{R_0}$. To see that such an R_0 indeed exists, note that, first, $m[R] < 1$ for some values of R, since $\int m[R]\,p[R]\,dR < 1$ from (1), and second, $m''[R] > 0$ by assumption, ruling out the case $m[R] < 1$ for all $R > R_u$. Finally, we establish the following statement.

Lemma 1 *The steeper the slope of the U-shaped pricing kernel in the increasing region, the more negative are the expected call returns.*

Proof: Suppose we have negative expected returns to the claim payout $g[R] = (S_t e^R - K)^+$. Therefore,

$$\int g[R]\,p[R]\,dR < \int g[R]\,m[R]\,p[R]\,dR, \qquad \text{or equivalently,} \qquad \int g[R]\,(m[R]-1)\;p[R]\,dR \;>\; 0. \tag{30}$$

Consider now a perturbation to the pricing kernel from $m[R]$ to $\widetilde{m}[R]$ that raises the positive slopes and reduces the negative slopes by a multiple. In particular consider for $\widehat{\delta} > 0$,

$$\widetilde{m}'[R] = (1+\widehat{\delta})\,m'[R]. \tag{31}$$

Let R_0 be such that $m[R_0] = 1$. For a continuous measure change that is nontrivial such an R_0 always exists. From basic manipulation, we now have that

$$\widetilde{m}[R] - \widetilde{m}[R_0] \;=\; \int_{R_0}^{R} \widetilde{m}'[u]\,du = (1+\widehat{\delta})\,(m[R]-1)\,. \tag{32}$$

Taking without of loss of generality $\widetilde{m}[R_0] = 1$, we may write

$$\widetilde{m}[R] - 1 = m[R] - 1 + \widehat{\delta}\,(m[R] - 1)\,, \qquad \text{or equivalently,} \qquad \widetilde{m}[R] \;=\; m[R] + \widehat{\delta}\,(m[R] - 1). \tag{33}$$

Note that $\widetilde{m}[R]$ is a pricing kernel as it integrates to unity and is positive for

$$\widehat{\delta} < \inf_R \frac{m[R]}{(1 - m[R])^+}. \tag{34}$$

The difference in price between the kernel $\widetilde{m}[R]$ and $m[R]$ is

$$\int g[R]\,(\widetilde{m}[R] - m[R])\; p[R]\,dR \;=\; \widehat{\delta} \int g[R]\,(m[R] - 1)\; p[R]\,dR \tag{35}$$

and this is positive for all claims with a negative expected return satisfying (30). Hence an exaggeration of the slopes of the pricing kernel leads to a higher price for claims with a negative expected return and hence to an even more negative expected return. ■

Proof of Theorem 3. Again, the expected return of a digital call is, $\mu^d[K] = \frac{\int_{\ln(K/S_t)}^{+\infty} p[R]dR}{\int_{\ln(K/S_t)}^{+\infty} m[R]p[R]dR} - 1$. Since $m[R]$ is increasing for $R > R_u$, then for strikes $K > S_t\, e^{R_u}$,

$$m[\ln(K/S_t)] \int_{\ln(K/S_t)}^{+\infty} p[R]\,dR \;<\; \int_{\ln(K/S_t)}^{+\infty} m[R]\,p[R]\,dR. \tag{36}$$

Therefore, we may determine that,

$$\frac{\partial \mu^d[K]}{\partial K} = \frac{-p[\ln(K/S_t)] \int_{\ln(K/S_t)}^{+\infty} m[R]\,p[R]\,dR \;+\; m[\ln(K/S_t)]\,p[\ln(K/S_t)] \int_{\ln(K/S_t)}^{+\infty} p[R]\,dR}{\left(\int_{\ln(K/S_t)}^{+\infty} m[R]\,p[R]\,dR\right)^2} \;<\; 0, \tag{37}$$

hence expected returns of digital calls decline in the strike price for $K > S_t\, e^{R_u}$.

Analogous to the proof of Theorem 2, there exists an $R_0 > R_u$ such that $m[R_0] = 1$, $m[R]p[R] > p[R]$ for all $R > R_0$, and hence $\mathbb{E}^{\mathbb{P}}\left(r_{t,T}^d[K]\right) < 0$ for $K > S_t e^{R_0}$.

Finally, it follows from Theorem 2 and from (29) that for strikes $K > S_t\, e^{R_u}$,

$$\int_{\ln(K/S_t)}^{+\infty} (S_t\, e^R - K)\, \widehat{p}[R]\,dR \int_{\ln(K/S_t)}^{+\infty} m[R]\, \widehat{p}[R]\,dR < \int_{\ln(K/S_t)}^{+\infty} (S_t\, e^R - K)\, m[R]\, \widehat{p}[R]\,dR. \tag{38}$$

Since $\widehat{p}[R]=\frac{p[R]}{\int_{\ln(K/S_t)}^{+\infty}p[R]\,dR}$, therefore,

$$\frac{\int_{\ln(K/S_t)}^{+\infty}(S_t\,e^R-K)\,p[R]\,dR}{\int_{\ln(K/S_t)}^{+\infty}(S_t\,e^R-K)\,m[R]\,p[R]\,dR}<\frac{\int_{\ln(K/S_t)}^{+\infty}p[R]\,dR}{\int_{\ln(K/S_t)}^{+\infty}m[R]\,p[R]\,dR}, \tag{39}$$

which is equivalent to,

$$\mathbb{E}^{\mathbb{P}}\left(r_{t,T}^{c}[K]\right)\;<\;\mathbb{E}^{\mathbb{P}}\left(r_{t,T}^{d}[K]\right). \tag{40}$$

This is the final step of the proof. ■

Lemma 2 *Suppose $m[R]$ is monotonically declining, i.e., $(q[S_{t+T}]/p[S_{t+T}])'<0$. Then, the expected return of the digital under monotonically declining pricing kernels satisfies $\left(\mu^d[K]\right)'>0$.*

Proof: For brevity, set $S_t=1$. Via a standard argument,

$$\begin{aligned}1+\mu^d[K] &= \frac{1}{\int_K^{+\infty}q[S_{t+T}]\,dS_{t+T}}\int_K^{+\infty}\frac{p[S_{t+T}]}{q[S_{t+T}]}\,q[S_{t+T}]\,dS_{t+T} && (41)\\ &> \frac{p[K]}{q[K]}, && (42)\end{aligned}$$

which translates into,

$$\frac{q[K]}{\int_K^{+\infty}q[S_{t+T}]\,dS_{t+T}}>\frac{p[K]}{\int_K^{+\infty}p[S_{t+T}]\,dS_{t+T}}. \tag{43}$$

Combining (43) and applying Leibniz rule to (41), we arrive at

$$\begin{aligned}\frac{d\ln(\mu^d[K])}{dK} &= \frac{q[K]}{\int_K^{+\infty}q[S_{t+T}]\,dS_{t+T}}-\frac{p[K]}{\int_K^{+\infty}p[S_{t+T}]\,dS_{t+T}}, && (44)\\ &> 0. && \text{(from equation (43))}.\end{aligned}$$

Thus, $\left(\mu^d[K]\right)'>0$ in the case of monotonically declining pricing kernels. ■

Proof of Theorem 4. We define the expected return of a kernel call as,

$$\mathbb{E}^{\mathbb{P}}\left(r_{t,T}^{h}[K]\right)=\frac{\int_{\ln(K/S_t)}^{+\infty}m[R]p[R]dR}{\int_{\ln(K/S_t)}^{+\infty}(m[R])^2\,p[R]\,dR}\;-\;1. \tag{45}$$

The existence of both integrals in (45) is ensured by the conditions in equation (1), and in particular

$q[R] = m[R]p[R]$ can be regarded as a new density, after normalization. Hence we can also write,

$$\mathbb{E}^{\mathbb{P}}\left(r^h_{t,T}[K]\right) = \frac{\int_{\ln(K/S_t)}^{+\infty} q[R]\,dR}{\int_{\ln(K/S_t)}^{+\infty} m[R]\,q[R]\,dR} - 1, \tag{46}$$

which is equivalent to the expression for the expected return of the digital call. Therefore, the rest follows from the proof of Theorem 3. ■

Lemma 3 *The expected return of* $m[R]\mathbf{1}_{e^R>y}$ *is decreasing in strike when the pricing kernel is monotonically declining.*

Proof: For brevity, set $S_t = 1$. As before, $q[S_{t+T}] \equiv \frac{m[S_{t+T}]\,p[S_{t+T}]}{\int m[S_{t+T}]\,p[S_{t+T}]\,dS_{t+T}}$ is the risk-neutral (pricing) density, and from the definition of the expected return of $m[R]\mathbf{1}_{e^R>y}$,

$$1+\mu^h[K] = \frac{\int_K^{+\infty} m[S_{t+T}]\,p[S_{t+T}]\,dS_{t+T}}{\int_K^{+\infty} m[S_{t+T}]\,q[S_{t+T}]\,dS_{t+T}} = \frac{\int_K^{+\infty}\left(\frac{q[S_{t+T}]}{p[S_{t+T}]}\right) p[S_{t+T}]\,dS_{t+T}}{\int_K^{+\infty}\left(\frac{q[S_{t+T}]}{p[S_{t+T}]}\right) q[S_{t+T}]\,dS_{t+T}} = \frac{\int_K^{+\infty} q[S_{t+T}]\,dS_{t+T}}{\int_K^{+\infty}\left(\frac{q[S_{t+T}]}{p[S_{t+T}]}\right) q[S_{t+T}]\,dS_{t+T}}. \tag{47}$$

Applying the logarithmic operator and taking the partial derivative with respect to K, yields,

$$\frac{1}{(1+\mu^h[K])}\frac{\partial \mu^h[K]}{\partial K} = -q[K]\underbrace{\left(\frac{1}{\int_K^{+\infty} q[S_{t+T}]\,dS_{t+T}} - \frac{\left(\frac{q[K]}{p[K]}\right)}{\int_K^{+\infty}\left(\frac{q[S_{t+T}]}{p[S_{t+T}]}\right) q[S_{t+T}]\,dS_{t+T}}\right)}_{<0 \text{ from equation } (51)}, \tag{48}$$

$$> 0. \tag{49}$$

Equation (49) is satisfied since $\left(\frac{q[S_{t+T}]}{p[S_{t+T}]}\right)' < 0$ when the pricing kernel is monotonically declining,

$$\frac{1}{1+\mu^h[K]} = \frac{\int_K^{+\infty}\left(\frac{q[S_{t+T}]}{p[S_{t+T}]}\right) q[S_{t+T}]\,dS_{t+T}}{\int_K^{\infty} q[S_{t+T}]\,dS_{t+T}}, \text{ or, equivalently,} \tag{50}$$

$$\frac{\int_K^{+\infty}\left(\frac{q[S_{t+T}]}{p[S_{t+T}]}\right) q[S_{t+T}]\,dS_{t+T}}{\int_K^{+\infty} q[S_{t+T}]\,dS_{t+T}} < \frac{q[K]}{p[K]}, \tag{51}$$

concluding the proof. ■

Appendix B: Option Data on the S&P 500 Index

The option data between January 1988 and December 1996 is from the Berkeley Options Database, while the data between January 1997 and May 2007 is from OptionMetrics. In-the-money S&P 500 index calls and puts are omitted and only options with 28 days to expiration are selected.

To the remaining data, we apply two standard exclusionary criteria. First, we discard options with zero traded volume, hence we avoid using matrix prices for options which have open interest, but are not traded. Second, we discard options which allow for arbitrage across strikes.

In about 10 instances, there is no option with moneyness within close proximity of 1%, 3% or 5% OTM respectively. In these cases, we follow Bates (1991) and Broadie and Detemple (1996), among others, and interpolate between the Black-Scholes implied volatilities of the two available options with strikes straddling the missing strike, or otherwise extrapolate linearly from the implied volatilities of all available OTM options with strikes lower than the missing strike. We are careful to verify that the option prices obtained in this way do not allow for arbitrage and are above the minimum tick size. We do not construct return time series for call options with moneyness beyond 5% OTM: such time series would require excessive extrapolation and the use of options with prices below the minimum tick size.

Appendix C: Price of $m[R]\mathbf{1}_{e^R>y}$ based on the mimicking portfolio for the pricing kernel

Recall that Z denotes the $N \times 1$ vector, or the $N \times \mathcal{T}$ matrix of time-series observations of gross returns of the basis assets. Let Z_t stand for the $N \times 1$ gross returns at time t over $T = 28$ days. In our setup, the mimicking portfolio for the pricing kernel at time t over $T = 28$ days can be represented as,

$$\begin{aligned} m_t^* \;=\; Z_t^\top \alpha \;&=\; \alpha_1 + \alpha_2 \frac{S_{t+T}}{S_t} + \sum_{i=1}^{3} \alpha_{i+2} \frac{(S_{t+T} - K_i)^+}{C_{t,T}[K_i]} + \sum_{i=4}^{6} \alpha_{i+2} \frac{(K_i - S_{t+T})^+}{P_{t,T}[K_i]}, \\ &=\; \alpha_1 + \alpha_2 e^R + \sum_{i=1}^{3} \alpha_{i+2} \frac{(S_t e^R - K_i)^+}{C_{t,T}[K_i]} + \sum_{i=4}^{6} \alpha_{i+2} \frac{(K_i - S_t e^R)^+}{P_{t,T}[K_i]}, \end{aligned} \tag{52}$$

where $\alpha_1, \ldots, \alpha_8$ are computed via $\mathbb{E}[ZZ^\top]^{-1} \iota_N$. The call strikes $K_1 < K_2 < K_3$ and the put strikes $K_4 < K_5 < K_6$ correspond to moneyness of 1%, 3%, and 5% OTM respectively.

In the kernel call return, $r^h_{t,T}[y] = \frac{m[R]\mathbf{1}_{e^R>y}}{H_{t,T}[y]} - 1$, the realized payoff $m[R]\mathbf{1}_{e^R>y}$, or equivalently, $m[R]\,\mathbf{1}_{S_{t+T}>K}$ is computable under our proxy to $m[R]$ in (52). $H_{t,T}[y]$, or equivalently, $H_{t,T}[K]$, equals $e^{-r_f}\mathbb{E}^{\mathbb{Q}}[m[R]\mathbf{1}_{S_{t+T}>K}]$, where $\mathbb{E}^{\mathbb{Q}}[.]$ is expectation under the risk-neutral (pricing) measure and $e^{-r_f} \approx 1/(1+r_f)$.

Let $C[K]$ be the price of the call with strike K and observe that the price of the digital $D[K] \equiv$

$e^{-r_f}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{S_{t+T}>K}] \approx \frac{C[K(1-\delta)]-C[K(1+\delta)]}{2\delta K}$. The price of a kernel call with strike K_1 can be approximated as:

$$H_{t,T}[K_1] \;\approx\; \alpha_1 D[K_1] \,+\, \alpha_2 \frac{C[K_1]}{S_t} \,+\, \alpha_2 \left(\frac{K_1}{S_t}\right) D[K_1] \,+\, \alpha_3 \,+\, \alpha_4 \,+\, \alpha_5 \tag{53}$$

since

$$\begin{aligned}
e^{-r_f}\mathbb{E}^{\mathbb{Q}}[\alpha_1 \mathbf{1}_{S_{t+T}>K_1}] &\approx \alpha_1 \frac{C[K_1(1-\delta)]-C[K_1(1+\delta)]}{2\delta K_1} && (54)\\
e^{-r_f}\mathbb{E}^{\mathbb{Q}}[\alpha_2 \frac{S_{t+T}}{S_t} \mathbf{1}_{S_{t+T}>K_1}] &\approx \alpha_2 \frac{C[K_1]}{S_t} + \alpha_2 \left(\frac{K_1}{S_t}\right)\left(\frac{C[K_1(1-\delta)]-C[K_1(1+\delta)]}{2\delta K_1}\right) && (55)\\
e^{-r_f}\mathbb{E}^{\mathbb{Q}}[\alpha_{i+2} \frac{(S_t e^R - K_i)^+}{C_{t,T}[K_i]} \mathbf{1}_{S_{t+T}>K_1}] &= \alpha_{i+2} \qquad i=1,2,3, && (56)\\
e^{-r_f}\mathbb{E}^{\mathbb{Q}}[\alpha_{i+2} \frac{(K_i - S_t e^R)^+}{P_{t,T}[K_i]} \mathbf{1}_{S_{t+T}>K_1}] &= 0 \qquad i=4,5,6. && (57)
\end{aligned}$$

In a similar way,

$$\begin{aligned}
H_{t,T}[K_2] \;\approx\; & \alpha_1 D[K_2] \,+\, \alpha_2 \frac{C[K_2]}{S_t} \,+\, \alpha_2 \left(\frac{K_2}{S_t}\right) D[K_2] \\
& + \alpha_3 \left(\frac{K_2-K_1}{C[K_1]}\right) D[K_2] \,+\, \alpha_3 \frac{C[K_2]}{C[K_1]} \,+\, \alpha_4 \,+\, \alpha_5,
\end{aligned} \tag{58}$$

since

$$\begin{aligned}
e^{-r_f}\mathbb{E}^{\mathbb{Q}}[\alpha_1 \mathbf{1}_{S_{t+T}>K_2}] &\approx \alpha_1 \left(\frac{C[K_2(1-\delta)]-C[K_2(1+\delta)]}{2\,\delta K_2}\right) && (59)\\
e^{-r_f}\mathbb{E}^{\mathbb{Q}}[\alpha_2 \frac{S_{t+T}}{S_t} \mathbf{1}_{S_{t+T}>K_2}] &\approx \alpha_2 \frac{C[K_2]}{S_t} + \alpha_2 \left(\frac{K_2}{S_t}\right)\left(\frac{C[K_2(1-\delta)]-C[K_2(1+\delta)]}{2\,\delta K_2}\right) && (60)\\
e^{-r_f}\mathbb{E}^{\mathbb{Q}}[\alpha_{i+2} \frac{(S_t e^R - K_1)^+}{C_{t,T}[K_1]} \mathbf{1}_{S_{t+T}>K_2}] &\approx \alpha_3 \left(\frac{K_2-K_1}{C[K_1]}\right)\left(\frac{C[K_2(1-\delta)]-C[K_2(1+\delta)]}{2\,\delta K_2}\right) + \alpha_3 \frac{C[K_2]}{C[K_1]} && \\
e^{-r_f}\mathbb{E}^{\mathbb{Q}}[\alpha_{i+2} \frac{(S_t e^R - K_i)^+}{C_{t,T}[K_i]} \mathbf{1}_{S_{t+T}>K_2}] &= \alpha_{i+2} \qquad i=2,3, && (61)\\
e^{-r_f}\mathbb{E}^{\mathbb{Q}}[\alpha_{i+2} \frac{(K_i - S_t e^R)^+}{P_{t,T}[K_i]} \mathbf{1}_{S_{t+T}>K_2}] &= 0 \qquad i=4,5,6. && (62)
\end{aligned}$$

Thus, using the mimicking portfolio for the pricing kernel, we can approximate the price of a kernel call through a portfolio of call options and digitals.

For reasons already articulated, we do not construct the 6% OTM call series (needed for 5% OTM digital price), hence only average returns corresponding to 1% and 3% OTM are shown in Table 5. ■

Table 1
Average 28-day returns of S&P 500 index options
Call and put option returns are calculated as:

$$r^c_{t,T}[y] = \frac{\left(S_t\, e^R - y\, S_t\right)^+}{C_{t,T}[y]} \;-\; 1, \qquad\qquad r^p_{t,T}[y] = \frac{\left(y\, S_t - S_t\, e^R\right)^+}{P_{t,T}[y]} \;-\; 1,$$

where S_t is the market index price at time t and $R = \ln(S_{t+T}/S_t)$ is the T-period return of the market index. $C_{t,T}[y]$ and $P_{t,T}[y]$ are time t prices of call and put options with time to expiration T and moneyness $y = \frac{K}{S_t}$. Reported are average returns of call and put options on the S&P 500 index over 01/1988-05/2007 (233 observations), with strikes 1%, 3% and 5% OTM (i.e., y is 1.01, 1.03 and 1.05 for calls, and 0.99, 0.97 and 0.95 for puts), and with $T = 28$ days. For each moneyness, we show average and standard deviation of option returns (in percent) over non-overlapping 28-day intervals, together with 90% confidence intervals for average returns, obtained in $\mathcal{N} = 25,000$ bootstrap draws from our sample of option returns (in square brackets). To test for difference in average returns across strikes, we first draw $\mathcal{N}$ pairwise bootstrap samples of returns to options with strikes 1% and 3%, to 3% and 5% OTM, respectively. Then, for each pair of bootstrap samples, we calculate the difference between average returns: a negative difference indicates that deeper OTM options have higher average returns in the respective bootstrap sample. The reported p-values are the proportion of negative differences in the $\mathcal{N}$ bootstrap samples, for each pair of strikes.

		OTM option returns			Difference between averages: (bootstrapped p-values)	
	% OTM	**1%**	**3%**	**5%**	**1 vs. 3%**	**3 vs. 5%**
		A. Calls				
01/1988 - 05/2007	Average (28 day)	7.0	-2.2	-5.8	0.14	0.33
	90% Confidence	[-9,23]	[-27,25]	[-42,36]		
	Standard Deviation	150.3	239.1	368.3		
01/1997 - 05/2007	Average (28 day)	2.4	-1.6	-5.1	0.32	0.38
	90% Confidence	[-18,24]	[-31,30]	[-45,39]		
	Standard Deviation	143.8	209.1	287.5		
		B. Puts				
01/1988 - 05/2007	Average (28 day)	-46.4	-58.6	-64.2	0.01	0.17
	90% Confidence	[-61,-31]	[-75,-39]	[-85,-39]		
	Standard Deviation	142.9	170.3	219.1		
01/1997 - 05/2007	Average (28 day)	-24.7	-39.5	-49.5	0.01	0.15
	90% Confidence	[-48,2]	[-68,-7]	[-83,-8]		
	Standard Deviation	171.6	207.4	259.9		

Table 2

Average returns of calls on international equity indexes

Reported are average returns of call options on six international equity indexes over 05/1995-05/2005 ($\mathcal{T}$=120), with strikes of 0%, 5% and 10% OTM (i.e., $y = \frac{K}{S_t}$ equal to 1.00, 1.05 and 1.10), and with time to expiration of $T = 30$ days. The indexes are FTSE (FTSE-100), NIKKEI (Nikkei-225 Stock Average), DAX (German Stock Index), SMI (Swiss Market Index), HSI (Hang Seng Index), and, ALO (Australian All Ordinaries Index). For each moneyness, we present average call returns (in percent) over non-overlapping 30-day intervals, together with 90% confidence intervals, obtained in $\mathcal{N} = 25,000$ bootstrap draws from the respective sample of option returns (in square brackets). To test for difference in average returns across strikes, we first draw $\mathcal{N}$ pairwise bootstrap samples of returns to options with strikes 0% and 5%, to 5% and 10% OTM, respectively. Then, for each pair of bootstrap samples, we calculate the difference between average returns: a negative difference indicates that deeper OTM options have higher average returns in the respective bootstrap sample. The reported p-values are the proportion of negative differences in the $\mathcal{N}$ bootstrap samples, for each pair of strikes.

		Call option returns			Difference between averages: (bootstrapped p-values)	
	% OTM	**0%**	**5%**	**10%**	**0 vs. 5%**	**5 vs. 10%**
UK (FTSE)	Average (30 days)	-22.3	-69.0	-95.5	0.00	0.00
	90% Confidence	[-36,-8]	[-84,-51]	[-99,-90]		
Japan (NIKKEI)	Average (30 days)	-16.6	-42.7	-63.3	0.01	0.10
	90% Confidence	[-33,1]	[-68,-13]	[-94,-17]		
Germany (DAX)	Average (30 days)	18.1	30.3	-3.1	0.69	0.24
	90% Confidence	[-3,40]	[-18,87]	[-87,101]		
Switzerland (SMI)	Average (30 days)	7.1	-3.9	-49.2	0.27	0.07
	90% Confidence	[-12,27]	[-45,42]	[-95,12]		
Hong Kong (HSI)	Average (30 days)	9.9	-4.5	-30.0	0.12	0.08
	90% Confidence	[-11,31]	[-37,31]	[-72,17]		
Australia (ALO)	Average (30 days)	-9.5	-76.8	-100	0.00	0.00
	90% Confidence	[-25,6]	[-89,-63]	[-100,-100]		

Table 3

Average returns of digital calls

Average returns of digital calls, with time to expiration of $T = 28$ days, are calculated using options on the S&P 500 index over 01/1988-05/2007. Reported are average returns to expiration, together with 90% confidence intervals for average returns obtained in $\mathcal{N} = 25,000$ bootstrap draws (in square brackets). The return of a digital call is approximated as,

$$r^d_{t,T}[y] \approx \frac{\left(S_t e^R - y(1-\delta) S_t\right)^+ - \left(S_t e^R - y(1+\delta) S_t\right)^+}{C_{t,T}[y(1-\delta)] - C_{t,T}[y(1+\delta)]} - 1,$$

where $C_{t,T}[y(1-\delta)]$ and $C_{t,T}[y(1+\delta)]$ are the time t prices of S&P 500 call options with moneyness $y(1-\delta) = \frac{K(1-\delta)}{S_t}$ and $y(1+\delta) = \frac{K(1+\delta)}{S_t}$, respectively, with $\delta = 1\%$. To test for difference in average returns across strikes, we first draw $\mathcal{N}$ pairwise bootstrap samples of returns to claims with strikes 1% and 3% OTM. Then, for each pair of bootstrap samples, we calculate the difference between average returns: a negative difference indicates that deeper OTM digital call have higher average returns in the respective bootstrap sample. The reported bootstrap p-values are the proportion of negative differences in the $\mathcal{N}$ bootstrap samples, for each pair of strikes. We do not report average return for the digital call with moneyness $y = 5\%$ OTM as it would involve excessive extrapolation for the 6% OTM call.

		Digital call returns		Difference between averages (bootstrapped p-values)
	% OTM	**1%**	**3%**	**1 vs. 3%**
01/1988 - 05/2007	Average (28 day)	20.60	10.0	0.083
	90% Confidence	[9,32]	[-8,28]	
01/1997 - 05/2007	Average (28 day)	12.60	0.10	0.086
	90% Confidence	[-3,28]	[-21,22]	

Table 4

Average returns of the upside variance contract on the S&P 500 index

The return of an upside variance contract with strike K and time to expiration T is,

$$r^{v}_{t,T}[y] \;=\; \frac{(R-\overline{\mu})^2\,\mathbf{1}_{e^R>y}}{V^{u}_{t,T}[y]} \;-\; 1,$$

where the price $V^{u}_{t,T}[y]$ can be obtained as,

$$V^{u}_{t,T}[y] = \left(\ln\left(\frac{K}{S_t}\right) - \overline{\mu}\right)^2 D[K] + \frac{2}{K}\left(\ln\left(\frac{K}{S_t}\right) - \overline{\mu}\right) C[K] + \int_{\{\mathbb{K}>K\}} \frac{2}{\mathbb{K}^2}\left(1-\ln\left(\mathbb{K}/S_t\right)+\overline{\mu}\right) C[\mathbb{K}]d\mathbb{K}.$$

$D[K]$ represents the price of a digital call option and $C[K]$ the price of a call option with strike K. Based on options on the S&P 500 index, we report average returns (in percent) over non-overlapping 28-day intervals, together with 90% confidence intervals, obtained in $\mathcal{N} = 25,000$ bootstrap draws (in square brackets). To test for difference in average returns across strikes, we first draw $\mathcal{N}$ pairwise bootstrap samples of returns to contracts with strikes 1% and 3% OTM. Then, for each pair of bootstrap samples, we calculate the difference between average returns: a negative difference indicates that deeper OTM claims have higher average returns in the respective bootstrap sample. The reported p-values are the proportion of negative differences in the $\mathcal{N}$ bootstrap samples. Results are based on $\overline{\mu} = 1\%$, that corresponds to the monthly sample average for S&P 500 index returns. Reported average $\sqrt{V^{u}_{t,T}[y]}$ for the 28-day contract are in percent.

		Upside variance contract		Difference between averages (bootstrapped p-values)
	% OTM	**1%**	**3%**	**1 vs. 3%**
01/1988 - 05/2007	Average $r^{v}_{t,T}[y]$ (28 day)	-9.7	-10.5	0.015
	90% Confidence	[-25,20]	[-29,16]	
	Average $\sqrt{V^{u}_{t,T}[y]}$	2.40	2.32	
01/1997 - 05/2007	Average $r^{v}_{t,T}[y]$ (28 day)	-8.9	-12.1	0.001
	90% Confidence	[-27,32]	[-34,21]	
	Average $\sqrt{V^{u}_{t,T}[y]}$	2.63	2.62	

Table 5

Average returns of kernel calls

The return of an kernel call is $r^h_{t,T}[y] = \frac{m[R]\mathbf{1}_{e^R > y}}{H_{t,T}[y]} - 1$, where $m[R]$ is proxied by the mimicking portfolio for the pricing kernel displayed in (52). When $y = 1\%$, we compute $H_{t,T}[y]$ as derived in (53) and when $y = 3\%$, we compute $H_{t,T}[y]$ as derived in (58), with $\delta = 1\%$. For each y, the payoff $m[R]\mathbf{1}_{e^R > y}$ is based on (52) with weights calculated as $\mathbb{E}[ZZ^\top]^{-1}\iota_N$. Average returns of kernel calls are calculated using options on the S&P 500 index with time to expiration of $T = 28$ days. We report average returns to expiration (in percent) over non-overlapping 28-day intervals, together with 90% confidence intervals for average returns, obtained in $\mathcal{N} = 25,000$ bootstrap draws (in square brackets). To test for difference in average returns across strikes, we first draw $\mathcal{N}$ pairwise bootstrap samples of returns to claims with strikes 1% and 3% OTM for kernel calls. Then, for each pair of bootstrap samples, we calculate the difference between average returns: a negative difference indicates that deeper OTM claims have higher average returns in the respective bootstrap sample. The reported p-values are the proportion of negative differences in the $\mathcal{N}$ bootstrap samples, for each pair of strikes.

		Kernel call returns		Difference between averages (bootstrapped p-values)
	% OTM	**1%**	**3%**	**1 vs. 3%**
01/1988 - 05/2007	Average (28 day)	22.05	10.24	0.098
	90% Confidence	[10,34]	[-9,31]	
01/1997 - 05/2007	Average (28 day)	13.23	-1.39	0.066
	90% Confidence	[-2,29]	[-23,21]	

Table 6

Expected returns from a model accounting for heterogeneity in beliefs about return outcomes

Investors in this economy have power utility and are heterogeneous in their beliefs about return outcomes. We adopt pricing kernel $m[R] = \sum_\omega \mathbb{N}_\omega[R]$, where ω represents the optimal fraction of wealth invested/disinvested in equity, and

$$\mathbb{N}_\omega[R] \;=\; \xi^L[R]\left(\omega \times e^R + (1-\omega)\times(1+r_f)\right)^{-\gamma}\Psi^L_\omega + \xi^S[R]\left(-\omega \times e^R + (1+\omega)\times(1+r_f)\right)^{-\gamma}(1-\Psi^L_\omega)$$

with risk aversion γ, riskfree return r_f, proportion of investors long equity of size ω given by Ψ^L_ω, and where,

$$\xi^L[R] \;=\; \beta^L_0 \;+\; (1-\beta^L_0)\left(\frac{\exp(\beta^L(R-\mu_*))}{1+\exp(\beta^L\,(R-\mu_*))}\right), \qquad 0<\beta^L_0<1, \qquad \beta^L>0,$$

$$\xi^S[R] \;=\; \beta^S_0 \;+\; (1-\beta^S_0)\left(\frac{\exp(-\beta^S(R-\mu_{**}))}{1+\exp(-\beta^S\,(R-\mu_{**}))}\right), \qquad 0<\beta^S_0<1, \qquad \beta^S>0.$$

The change-of-measure densities $\xi^L[R]$ and $\xi^S[R]$ capture the return beliefs of long and short equity investors, respectively. We construct three cohorts of investors, differentiated by their return beliefs and containing both long and short investors, whereby each long/short investor in a cohort optimally invests/disinvests the same fraction ω of their wealth in equity. Throughout $\gamma = 12$, $r_f = 0$ and $\beta^L = \beta^S = 50$. We employ $p[R]$ that is suitably truncated Normal with mean of 1% and standard deviation of 5%, corresponding to S&P 500 index returns over $T = 28$ days, and we assume $R \in [-\ln(2), \ln(2)]$ to keep $\mathbb{N}_\omega[R]$ in equation (14) finite. The aggregate pricing kernel $m[R]$ is constructed using the parameters in Panel A across cohorts. The risk-neutral (pricing) density is accordingly, $q[R] \equiv \frac{m[R]\,p[R]}{\int m[R]\,p[R]\,dR}$. Under $p[R]$ we draw $\mathcal{N} = 25{,}000$ sets of truncated Normal variates with mean 1% and standard deviation 5% (similar to those for S&P 500 index 28-day returns data). Each set is of length $\mathcal{T} = 233$, matching the length of our 28-day option return time series. With model-based prices, and with each set of $\mathcal{T}$ random variates we calculate $\mathcal{N}$ average returns of each claim and at each moneyness. We report *averages* of the $\mathcal{N}$ average returns (in percent), as well as 90% confidence intervals for these averages (in square brackets).

Panel A: Parametric specification across cohorts

Cohort	β^L_0	β^S_0	μ_*	μ_{**}	ω	Ψ^L_ω
1	0.0021	0.0087	0.25	-0.15	0.50	0.90
2	0.0001	0.0007	0.30	-0.20	0.60	0.80
3	0.0001	0.0006	0.30	-0.20	0.70	0.70

Panel B: Model generated expected returns for calls and puts and bootstrap confidence intervals

		Claim returns			Difference across strikes: (Bootstrapped p-value)	
	% OTM	**1%**	**3%**	**5%**	**1 vs. 3%**	**3 vs. 5%**
Calls:	Average	4.4	-1.3	-9.2	6.2	5.3
	90% Confidence	[-12, 22]	[-21, 20]	[-34, 17]		
Puts:	Average	-44.4	-55.4	-66.5	0.0	0.0
	90% Confidence	[-56, -33]	[-68, -42]	[-80, -52]		

Table 7

Regression analysis of OTM call option dynamics

Reported results are based on the following empirical specification:

$$\Delta\left(\frac{C_{t,T}[y]}{S_t}\right) = \eta_0 + \eta_x \Delta X_t + \varepsilon_t,$$

where Δ denotes first difference, and $C_{t,T}[y]$ is the price of a call option with 28 days to expiration and moneyness y. The proxy for X_t in Panel A is log of new short sales by NYSE members and the public over the week ending at each date t on which $C_{t,T}[y]$ are observed, while it is log short interest at the NYSE in Panel B. The data source is the NYSE Members Reports and Barron's. The asymptotic p-values are based on heteroskedasticity consistent robust errors, and the bootstrap p-values are based on the procedure suggested by Davidson and Flachaire (2008) and computed using 25,000 bootstrap draws. Reported Adj-R^2 represents the adjusted R^2 statistic, DW is the Durbin-Watson statistic, and the row marked Jarque-Bera shows the p-values for the test of normality for ε_t (low p-values imply rejection of normality).

	Panel A: New short sales			Panel B: Short interest		
% OTM	**1%**	**3%**	**5%**	**1%**	**3%**	**5%**
η_0	-0.0001	-0.0000	-0.0000	-0.0001	-0.0000	-0.0000
p-values (one-sided):						
robust asymptotic	0.328	0.330	0.362	0.368	0.340	0.386
bootstrap	0.329	0.329	0.356	0.369	0.334	0.379
η_x	0.0034	0.0019	0.0017	0.0013	0.0017	0.0009
p-values (one-sided):						
robust asymptotic	0.007	0.047	0.023	0.395	0.317	0.370
bootstrap	0.003	0.037	0.021	0.415	0.338	0.391
Adj-R^2	3.2%	1.5%	2.1%	-0.4%	-0.4%	-0.4%
DW	2.50	2.47	2.50	2.50	2.46	2.50
Jarque-Bera, ε_t	0.00	0.00	0.00	0.00	0.00	0.00

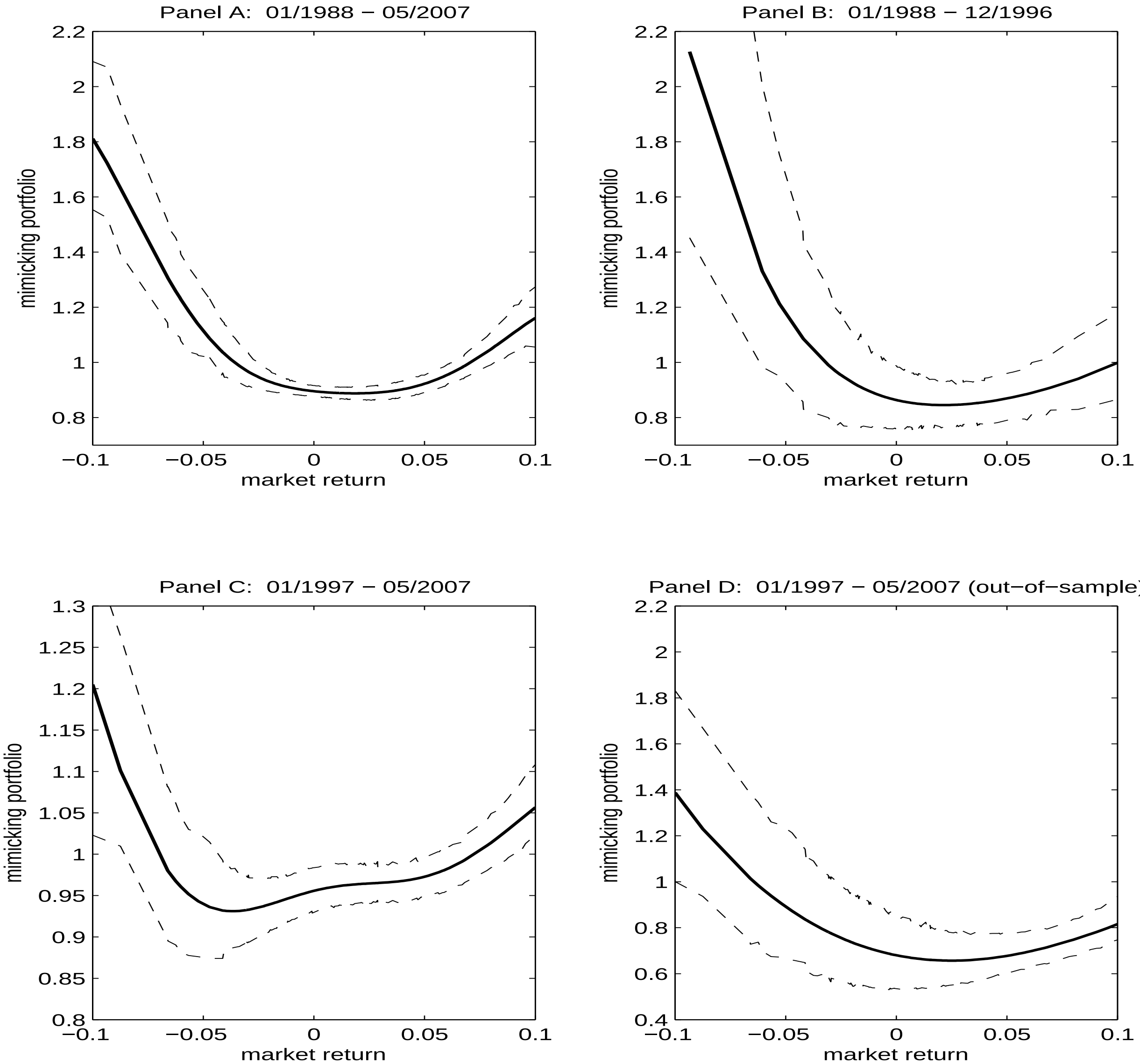

Fig. 1. **Mimicking portfolio for the pricing kernel** (m^*)

Plotted is the mimicking portfolio for the pricing kernel (Cochrane (2005), Chapter 4, equation 4.1) versus the market return, where, as in (11), $m^* = Z^\top \mathbb{E}[ZZ^\top]^{-1}\iota_N$. Z denotes the $N \times 1$ vector, or the $N \times \mathcal{T}$ matrix of time-series observations of gross returns of the basis assets. Here Z is taken to be the 28-day gross returns of (i) a riskfree asset, (ii) the market index (i.e., S&P 500), and (iii) call and put options on the S&P 500 index that are 1%, 3%, and 5% OTM. The plots display smoothed versions of m^*, which are obtained via Nadaraya-Watson kernel regression with Gaussian kernel. The m^* are calculated using (i) the entire sample period 01/1988-05/2007, (ii) the 01/1988-12/1996 subsample, and (iii) the 01/1997-05/2007 subsample. Finally, the out-of-sample m^* is calculated using returns of the basis assets over 01/1997-05/2007, with weights $\alpha = \mathbb{E}[ZZ^\top]^{-1}\iota_N$ based on returns of the basis assets over 01/1988-12/1996. Dashed curves show 95% point-wise confidence intervals, obtained by applying the kernel regression on bootstrapped values of m^* in 25,000 bootstrapped draws.

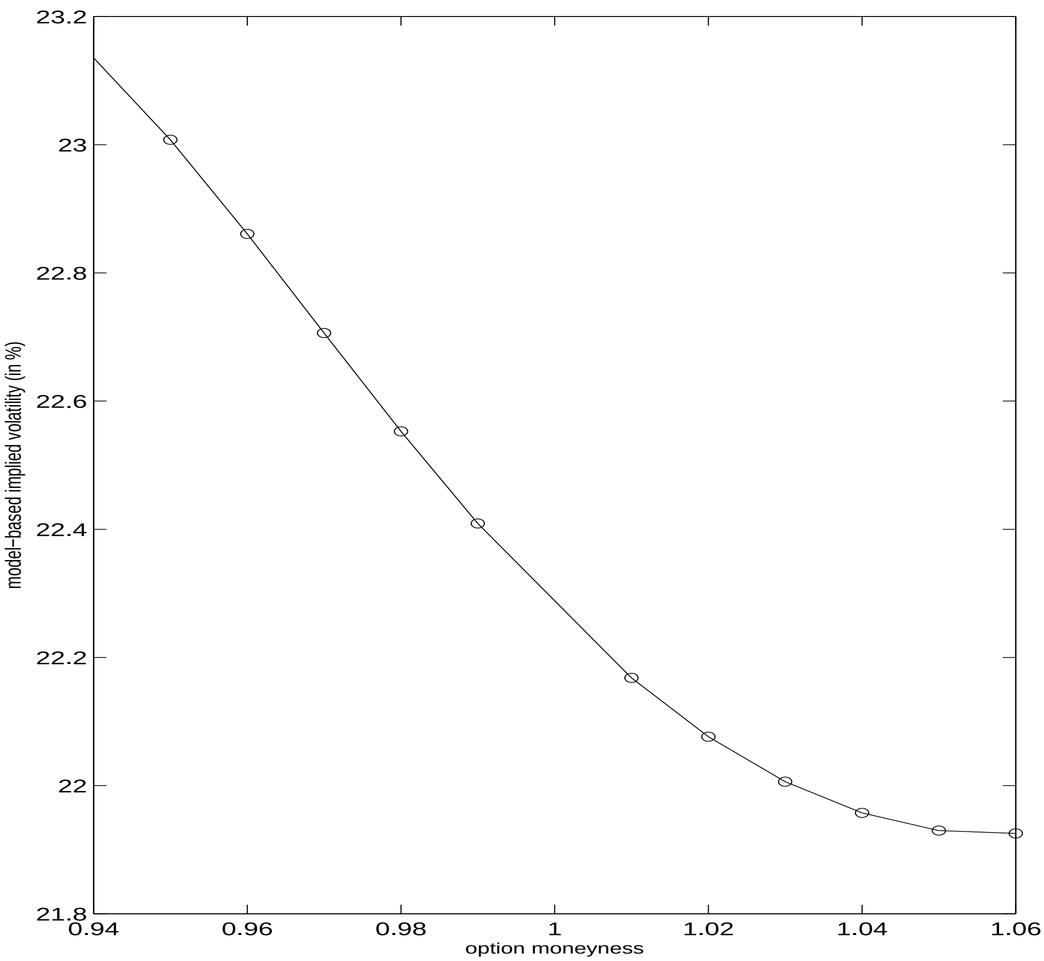

Fig. 2. **Model-based implied volatility**

Plotted is implied volatility versus the option moneyness, $y = K/S$. Implied volatility is the volatility that equates the model price of the option to the Black-Scholes value. Parameters used to construct the model option price are as displayed in Table 6. The option maturity is assumed to be 28 days.

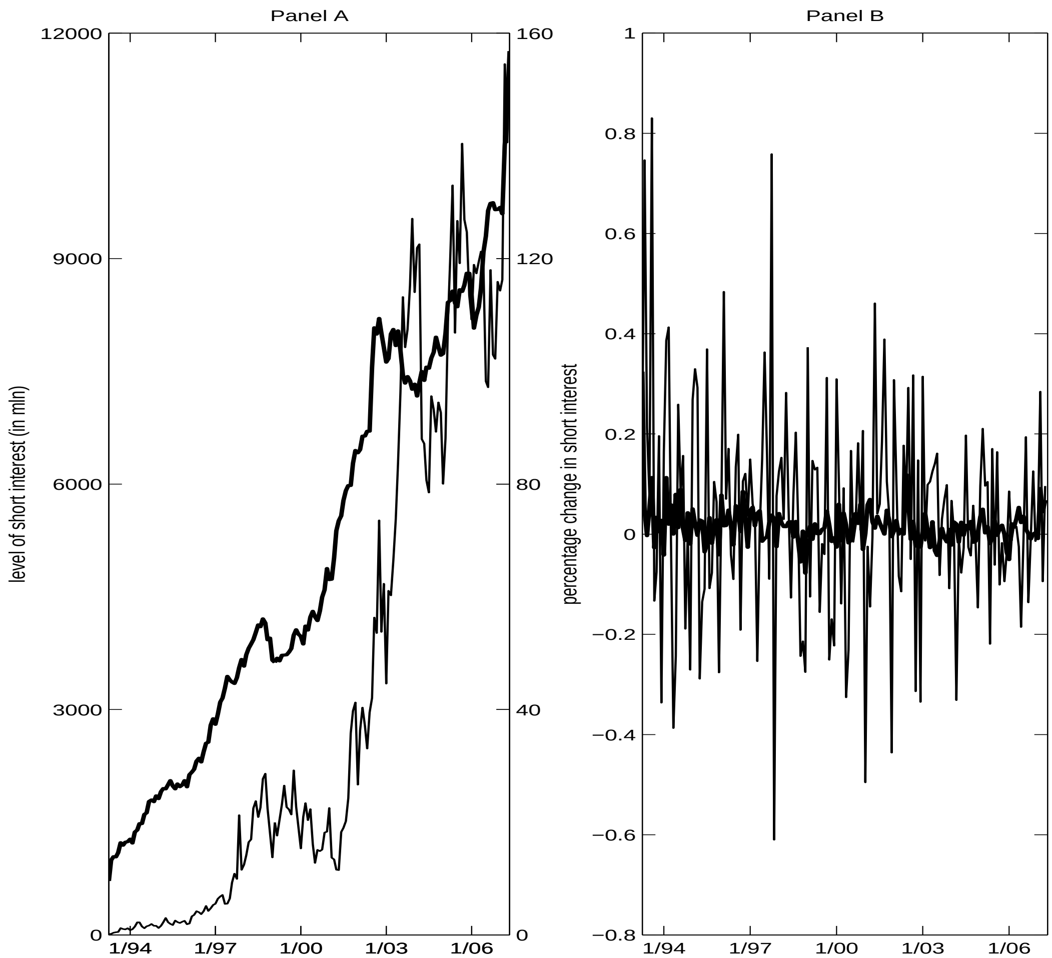

Fig. 3. **Time series behavior of short interest on NYSE and Spiders**

Panel A plots the level of short interest at the NYSE (left axis) and on the Spiders ETF (right axis) over time, while Panel B plots the percentage change in short interest. The sample period is 04/1993 to 05/2007 and the data source is the NYSE Members Reports, Barron's and Bloomberg. The thick line represents short interest on the NYSE, and the thin line represents short interest on the Spiders.