Average-Rate Claims With Emphasis On Catastrophe Loss Options

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Abstract

This article studies the valuation of options written on the average level of a Markov process. The general properties of such options are examined. We propose a closed-form characterization in which the option payoff is contingent on cumulative catastrophe losses. In our framework, the loss rate is a mean-reverting Markov process, with no continuous martingale component. The model supposes that high loss levels have lower arrival rates. We analytically derive the cumulative loss process and its characteristic function. The resulting option model is promising.

I Introduction

This article studies the valuation of Asian options contingent on the average level of a Markov process. Option and forward contracts written on the price of crude oil transported by tankers are Asian. The put option is exercised when the average crude oil price over the shipping period is below the strike price. Financial houses routinely offer quotations on European options on the average exchange rate, the average interest rate yield and on swaptions based on average interest rates. Average-rate interest rate derivatives are traded actively on the Paris capital markets, and are now prime instruments for hedging interest rate and exchange rate exposures. Realizing their growth potential, in 1996, the London Metal Exchange began trading average rate options on copper and aluminum. Other prominent examples include the catastrophe insurance option contract traded on the Chicago Board of Trade and commodity-linked bonds on the average bond price.[1]

To appreciate these financial products further, note that average-rate options offer two appealing features. First, when making option markets on thinly traded assets, the averaging feature can lessen the incentives for market manipulation. Maybe, for this reason, many corporate takeovers and mergers are average stock price contingent.[2] Second, when building long-term earning projections and constructing budget policies, the corporate treasurer is often concerned with the average commodity price (or, the exchange rate). The average-rate options are easier to hedge, which explains their usage in risk management.

Despite the wide applicability of contingent claims on the average price, relatively little attention has been devoted to this topic in the academic community.[3] The purpose of this

[1]To give some real world applications, AEGON, a Dutch insurance company, bought Asian options on the Dutch guilder/U.S. dollar with the purpose of averaging fluctuations in the dollar. Besides offering a perfect hedge, these options worked on the same basis as the insurance company's accounting books (Source: Derivatives Week (December 7, 1992)). More recently, Banque Paribas and other investment houses wrote average-rate options on FTSE embedded in long-term notes. Merrill Lynch marketed structured notes linked to emerging market stocks. The deal is a five year zero-coupon note with an embedded Asian option. See also the analysis of contingent value rights and other Asian option related applications in Chen, Chen, and Laiss (1995) and Dammon and Spatt (1992).

[2]See, for example, the merger of Dow Chemical with Marion Laboratories, or that of Rhone-Poulenc with the Rhorer group.

[3]For recent average-rate option related research, see Bakshi and Madan (2000), Boyle (1993), Chacko and Das (1997), Curran (1994), Fu, Madan, and Wang (1996), Geman and Yor (1993), Hull and White (1993), Ju (1997, 2000), Kemna and Vorst (1990), Levy (1992), Milevsky and Posner (1997), and Turnbull and Wakeman (1991). To our knowledge, with the exception of Bakshi and Madan (2000) and Geman and Yor (1993), none of these papers have exact analytical results on the pricing of average-rate options. Chacko and Das (1997) propose a solution to the Digital option on the average interest rate, and Ju (1997)

paper is to fill this void by offering a framework for the valuation of Asian-style contingent claims. Following Bergman, Grundy, and Wiener (1996), Cox and Ross (1976) and Merton (1973), we first provide a characterization of the general properties of Asian-style claims. As is now known, the price path dependence of the option payoff renders the average-rate option valuation problem difficult even for the geometric Brownian motion case. In their contribution, Geman and Yor (1993), offer the Laplace transform of the call price, which is only an intermediate step for determining the Asian call price (and the complete problem is still unsolved). The difficulty lies in inverting Laplace tranforms for some cases. The derived comparative properties are then a way to understand and predict how the Asian contingent claim price may respond to changes in various underlying economic entities. Our analysis sheds light on how the Asian and European option prices compare.

When the laws of motion for the asset price are a one-dimensional Markov diffusion, it is shown that the Asian call price is non-decreasing and convex in the asset price. Moreover, the upper bound attained by the delta is strictly below unity. One implication of this result is the lower variability of the hedged profit and loss accounts. We also show that the Asian call price is non-decreasing and convex in the average-to-date price, and rises with the volatility of the spot price process. Additionally, higher dividend yields lower Asian call option prices. Finally, it is possible for the call price to decrease when the interest rate rises.

We consider the closed-form analysis of options written on catastrophic losses. According to Litzenberger, Beaglehole and Reynolds (1996), the recent hurricane and earthquake related property losses have contributed to their growing popularity among property reinsurers. In particular, as catastrophe related losses are uncorrelated with stock and bond markets, catastrophe insurance options are now regarded as alternative mechanisms for tactical asset diversification and for securitizing catastrophic risk. In our approach, losses are modeled as a mean-reverting process with one-sided jumps. The use of one-sided jumps is realistic and ensures that cumulative losses are increasing. The mean-reversion captures the half-life of processing losses through the system. Specifically, total losses can be decomposed into a mean-reverting component and an exponentially-damped compound Poisson process. Our model incorporates the intuitive property that high level of losses are associated with

suggests a methodology to compute the density of the arithmetic average. Finally, Ju (2000) has developed an accurate approximation to value Asian and basket options when asset prices follow geometric Brownian motion. This paper examines average-rate claims from many different perspectives.

low arrival rates (and vice-versa). In contrast to Cummins and Geman (1995), we analytically derive, from Bakshi and Madan (2000), the characteristic function of the cumulative losses, and option prices on the cumulative losses. Providing a closed-form solution for the catastrophe insurance option contract is, in our view, an important contribution of this paper.

The article is organized as follows. Section II characterizes the properties of average-rate options. Section III considers the valuation of catastrophe insurance options in some detail. The last section IV provides concluding remarks. The Appendix contains a proof of each result.

II Properties of Options on the Arithmetic Average

Average-rate contingent claims are written on the average level of a process. To fix the convention for the remainder of the paper, characterize the average payoff on the contingent claim asset as follows. First, denote the time-t spot price of a non-dividend paying asset by $P(t)$, and the constant interest rate by r.[4] When the time-$t+\tau$ payoff is dependent on the arithmetic price average $\frac{1}{t+\tau}\int_0^{t+\tau} P(u)\,du$, the claim will be referred to as an average-rate claim or simply an Asian claim. Notice that the claim payoff depends on the entire asset price history, where the initial date for the averaging is set to zero (the reference date).[5] Unless otherwise stated, the payoff is contingent on the price path from time 0 to $t+\tau$.

Consider now a European option with strike price K and τ-periods remaining to expiration. Let $C(t,\tau)$ and $H(t,\tau)$ denote the time-t price of the arithmetic average call and put. At expiration, the respective payoff must be:

$$C(t+\tau,0) \;=\; \max\left(\frac{1}{t+\tau}\int_0^{t+\tau} P(u)\,du \,-\, K, 0\right), \tag{1}$$

[4] While studying the general properties, the interest rate can be allowed to be stochastic or deterministic [as in Bakshi, Cao, and Chen (1997) and Bergman, Grundy, and Wiener (1996)]. The assumption about constant interest rates is driven by simplicity.

[5] The option claims we examine belong to a broader family of averaging rules of the type: $\left(\frac{1}{t+\tau}\int_0^{t+\tau} P^{\psi}(u)\,du\right)^{\frac{1}{\psi}}$. For example, the *arithmetic* average is obtained when $\psi=1$; the *geometric* average when $\psi=0$; *quadratic* average when $\psi=2$; and *harmonic* average when $\psi=-1$. In principle, the option claim can be contingent on any of these averages. For practical purposes, our main interest lies in characterizing the arithmetic average class.

and $H(t+\tau,0) = \max\left(K - \frac{1}{t+\tau}\int_0^{t+\tau} P(u)\,du\,,0\right)$. The time-$t$ price of the average-rate call is thus

$$C(t,\tau) = E_t^Q\left\{e^{-r\tau}\max\left(\frac{1}{t+\tau}\int_0^{t+\tau} P(u)\,du \;-\; K, 0\right)\right\}, \tag{2}$$

where $E_t^Q\{.\}$ is the expectation operator under an equivalent martingale measure. Let $A(t) \equiv \int_0^t P(u)\,du$ be the integral of the price path to date. Expressing the expected discounted value of the final average price in terms of the current price and the integral-to-date by the relationship $E_t^Q\left\{\frac{e^{-r\tau}}{t+\tau}\int_0^{t+\tau} P(u)\,du\right\} = P(t)\left\{\frac{1-e^{-r\tau}}{r(t+\tau)}\right\} + \frac{e^{-r\tau}}{t+\tau}A(t)$, one may express the average-rate call option price as:

$$C(t,\tau) = P(t)\left\{\frac{1-e^{-r\tau}}{(t+\tau)r}\right\}\Pi_1(t,\tau) - e^{-r\tau}\left\{K - \frac{A(t)}{t+\tau}\right\}\Pi_2(t,\tau), \tag{3}$$

where $\Pi_1(t,\tau)$ and $\Pi_2(t,\tau)$ are risk-neutralized probabilities under different probability measures

$$\Pi_1(t,\tau) \;\equiv\; \mathrm{Prob}^{\tilde{Q}}\left(\int_t^{t+\tau} P(u)\,du > [t+\tau]K - A(t)\right), \tag{4}$$

$$\Pi_2(t,\tau) \;\equiv\; \mathrm{Prob}^{Q}\left(\int_t^{t+\tau} P(u)\,du > [t+\tau]K - A(t)\right). \tag{5}$$

The measure $\tilde{Q}$ in equation (4) is defined by the Radon-Nikodym derivative, $\frac{d\tilde{Q}}{dQ} = \frac{e^{-r\tau}\int_t^{t+\tau} P(u)\,du}{E_t^Q\left\{e^{-r\tau}\int_t^{t+\tau} P(u)\,du\right\}}$.

Note that the call price in (3) inherits the structure of the traditional call pricing formula. Specifically, the Asian call price is the time-t (scaled) forward average of the underlying asset times a probability element, less the present value of the adjusted strike price (i.e., $K - \frac{1}{t+\tau}A(t)$) times another probability element. Whether the Asian call finishes in-the-money is determined by the distribution of the remaining spot price uncertainty $\int_t^{t+\tau} P(u)\,du$, rather than that of the terminal spot price. For all values of r and τ, we have $P(t)\frac{1-e^{-r\tau}}{(t+\tau)r} < P(t)$. This implies that the first term in equation (3) is smaller than the analogous term for the regular call price. Consequently, the working of such factors can make the Asian call price lower or higher than the traditional call price. We revisit this issue by employing a parametric option pricing model.

When the spot price obeys a proportional stochastic process, the traditional option price is homogeneous of degree one in the spot price and the strike price. For Asian options, one may observe that the probabilities Π_1 and Π_2 are homogeneous of degree zero in P and the adjusted strike. It follows that the Asian call price is homogeneous of degree one in the triplet (P, K, A).

Unfortunately, just based on (3)–(5), not much further can be said about the basic properties of Asian options. Inspired by Merton (1973), in the first subsection to follow, we present bounds on Asian option prices. Next, to generate sharper predictions about average-rate claims, we restrict the (risk-neutral) spot price process to be one-dimensional Markov diffusions. Using this family of stochastic processes as the basis, we study the comparative properties of Asian options. In a later section, we analytically characterize options written on catastrophic losses.

A Bounds on Average-Rate Options

Exploiting the traditional relationships between option contracts and forward contracts, and noting, in particular, the value of the underlying forward average, one derives counterparts to the traditional option inequalities. Two basic relationships are options dominate the value of the forward when the forward delivery price equals the strike, and the option is worth less than the forward asset.

To derive the lower bound on the Asian call price, consider two alternative time-t investment policies. Take a long position in an Asian call with strike price K and term-to-expiration τ, and invest $K\, e^{-r\tau}$ in a zero-coupon bond. If the call option is exercised, the time-$t+\tau$ payoff is $\frac{1}{t+\tau}\int_0^{t+\tau} P(u)\, du$, and K otherwise. Alternatively, buy a claim that delivers $\frac{1}{t+\tau}\int_0^{t+\tau} P(u)\, du$ at time–$t+\tau$. By comparing the terminal payoff functions of each strategy, we have

$$(6) \qquad C(t,\tau) \;\geq\; \max\left(0, P(t)\left\{\frac{1-e^{-r\tau}}{(t+\tau)\, r}\right\} + \frac{e^{-r\tau}}{t+\tau}\, A(t) - K e^{-r\tau}\right).$$

The difference between the call price and the intrinsic value

$$(7) \qquad \max\left(0, C(t,\tau) - P(t)\left\{\frac{1-e^{-r\tau}}{(t+\tau)\, r}\right\} - \frac{e^{-r\tau}}{t+\tau}\, A(t) + K e^{-r\tau}\right)$$

represents the time-value of the Asian call.

The upper bound on the call price is

$$0 \leq C(t,\tau) \;\; \leq \;\; P(t)\left\{\frac{1-e^{-r\tau}}{(t+\tau)\,r}\right\} + \frac{e^{-r\tau}}{t+\tau}\,A(t), \tag{8}$$

which is an equivalent way of saying that a long position in the call is always dominated by a claim that delivers $\frac{1}{t+\tau}\int_0^{t+\tau} P(u)\,du$. The classical bounds are now altered, as the averaging for the Asian option induces an eventual decline in the value of the underlying averaged forward asset.

The put-call parity below is also based on the no-arbitrage principle:

$$H(t,\tau;K) + P(t)\left\{\frac{1-e^{-r\tau}}{(t+\tau)\,r}\right\} + \frac{e^{-r\tau}}{t+\tau}\,A(t) \;\; = \;\; C(t,\tau;K) + K\,e^{-r\tau}. \tag{9}$$

That is, go long a call option with strike price K and term-to-expiration τ, and invest $K\,e^{-r\tau}$ in a zero-coupon bond. Simultaneously purchase a put option (with the same strike and maturity as the call) and buy a claim to deliver $\frac{1}{t+\tau}\int_0^{t+\tau} P(u)\,du$ units of the stock at time–$t+\tau$. Equation (9) then follows by equating the payoffs from each strategy. The put option bounds can now be derived by employing equation (9).

B Comparative Statics for the Spot Price Change

An understanding of the general properties of option deltas is critical to risk management via delta hedging. Attention is focused here on the generic properties of Asian option deltas. To establish these economic properties, make the simplifying assumption that the risk–neutral dynamics for the non-dividend paying asset obeys a continuous–time Markov process (Bergman, Grundy, and Wiener (1996), Cox and Ross (1976), and Merton (1973))

$$\frac{dP(t)}{P(t)} = r\,dt + \sigma[P(t),t]\,d\omega(t), \tag{10}$$

where the volatility coefficient $\sigma[P(t),t]$ is at most a function of the spot price $P(t)$ and possibly time, and ω represents a standard Brownian motion. Under this assumption, a simple application of Ito's lemma yields the partial differential equation (hereby PDE) for the Asian call price

$$\frac{1}{2}\,\sigma^2[P,t]\,P^2\,\frac{\partial^2 C}{\partial P^2} + r\,P\,\frac{\partial C}{\partial P} - r\,C - \frac{\partial C}{\partial \tau} = -P\,\frac{\partial C}{\partial A}, \tag{11}$$

with $C(t+\tau,0) = \frac{1}{t+\tau}\max\left(\int_t^{t+\tau} P(u)\,du - \tilde{K}, 0\right)$. Here $\tilde{K} = [t+\tau]K - A(t)$ is the adjusted strike price, and $A(t)$ summarizes prior price history. Now we can state:

Theorem 1 *When the spot price is governed by the Markov process (10), the following statements are true:*

(a) The Asian call price is non–decreasing and convex in the spot asset price with the call delta, $\frac{\partial C(t,\tau)}{\partial P(t)}$*, bounded by*

$$0 \le \frac{\partial C(t,\tau)}{\partial P(t)} \le \frac{1-e^{-r\tau}}{[t+\tau]r}. \tag{12}$$

(b) The time–t value of the average Digital can be obtained as the solution to the following:

$$\frac{\partial C(t,\tau)}{\partial P(t)} = E_t^{Q^*}\left\{\int_t^{t+\tau} \frac{\partial C(u,\tau)}{\partial A(u)}\,du\right\}, \tag{13}$$

where $E_t^{Q^*}$ *denotes an expectation operator under the transformed stochastic processes*

$$\frac{dP(t)}{P(t)} = \left\{r + \sigma^2[t,P(t)] + P\,\sigma[t,P(t)]\frac{\partial\sigma[P,t]}{\partial P(t)}\right\}dt + \sigma[t,P(t)]\,d\omega(t); \tag{14}$$

$$d\,A(t) = P(t)\,dt; \tag{15}$$

and $\frac{\partial C(u,\tau)}{\partial A(u)}$ *is a stand-in for the continuous dividend yield on the Digital claim.*

Proof: By a standard representation, the call price (2) is a solution to

$$C(t,\tau) = \frac{1}{t+\tau}\int_{\tilde{K}}^{\infty} e^{-r\tau}\left(W - \tilde{K}\right)\,\Phi\,[W]\;dW, \tag{16}$$

where $\Phi[W]$ is the density function for $W \equiv \int_t^{t+\tau} P(u)\,du$. Write $W[P(t),r,v]$, or simply $W[P(t),v]$, where v characterizes the remaining uncertainty with density $\bar{\Phi}[v]$. Integrating with respect to the density of v, we may now re-express (16) as[6]

$$C(t,\tau) = \frac{1}{t+\tau}\int_{\tilde{K}}^{\infty} e^{-r\tau}\left(W[P(t),v] - \tilde{K}\right)\bar{\Phi}\,[v]\;dv. \tag{17}$$

With the aid of Leibnitz's differentiation rule

$$\frac{\partial C(t,\tau)}{\partial P(t)} = \frac{1}{t+\tau}\int_{\tilde{K}}^{\infty} e^{-r\tau}\frac{\partial W[P(t),v]}{\partial P(t)}\bar{\Phi}\,[v]\;dv. \tag{18}$$

Next, by the requirement that the spot price be a martingale

$$\int_{0}^{\infty} e^{-r\tau}\,W[P(t),v]\,\bar{\Phi}\,[v]\;dv = P(t)\left\{\frac{1-e^{-r\tau}}{r}\right\}, \tag{19}$$

one arrives at the unconditional expectation

$$\int_{0}^{\infty} e^{-r\tau}\frac{\partial W[P(t),v]}{\partial P(t)}\bar{\Phi}\,[v]\;dv = \frac{1-e^{-r\tau}}{r}. \tag{20}$$

By exploiting the non-crossing property (Bergman, Grundy, and Wiener (1996)) and using equation (20), we have $\frac{\partial W[P(t),v]}{\partial P(t)} > 0$. In summary, one may now rewrite (18) as:

$$\frac{\partial C(t,\tau)}{\partial P(t)} = \frac{1-e^{-r\tau}}{(t+\tau)r}\times\left\{\frac{\int_{\tilde{K}}^{\infty} e^{-r\tau}\frac{\partial W[P(t),v]}{\partial P(t)}\bar{\Phi}\,[v]\;dv}{\int_{0}^{\infty} e^{-r\tau}\frac{\partial W[P(t),v]}{\partial P(t)}\bar{\Phi}\,[v]\;dv}\right\}.$$

Since the expression in $\{.\}$ is a probability function, the desired bound on the delta (12) follows. The proof of (13) and the convexity of the Asian call is adapted from Bergman, Grundy, and Wiener (1996), and provided in (56) and (57) of Appendix A. □

The bound on the delta of the Asian call in (12) is informative. When the spot price changes, the response of the Asian call price is positive, but never more than $\frac{1-e^{-r\tau}}{(t+\tau)r}$. The upper bound is related to the interest rate, the term-to-expiration of the option contract and the length of the payout averaging interval $t+\tau$. Ceteris paribus, a higher spot interest rate or a longer term-to-expiration induces a larger option delta, but still less than

[6]By the Markov property and the stochastic differential equation (10), we can write $P(T) = \Lambda[P(t),\omega(u),t \leq u < T]$ for some functional $\Lambda[.]$. Since the probability measure on the path of the standard Brownian motion is independent of $P(t)$, the assertion that $\bar{\Phi}(v)$ is independent of $P(t)$ is immediate.

unity. Conversely, as $\tau \to 0$, the delta is zero regardless of the spot price change. The dependence on the average price effectively renders the expiration period delta immune to the terminal spot price. Interestingly, in contrast to standard vanilla options, the Asian call delta approaches zero at both ends of the maturity spectrum.

Equation (13) of Theorem 1 provides the intuition that the time-t delta claim price can be viewed as the conditional expectation of the undiscounted value of the entire future dividend stream: $\frac{\partial C(u,\tau)}{\partial A(u)} \geq 0$ (it is an integral in the time dimension). It is established in Theorem 2 (for ease of presentation) that $\frac{\partial C(u,\tau)}{\partial A(u)} = \frac{e^{-r(t+\tau-u)}}{t+\tau}\Pi_2(u,\tau)$, which is a scaled probability that the Asian option expires in-the-money. By substituting this expression into (13), we can observe that a higher time-t price has the effect of diminishing the entire sequence of future critical strike prices: $(t+\tau)K - \int_0^\ell P(\ell)\, d\ell$, for $t+\tau \geq \ell \geq t$. Then, according to (13), the delta claim holder is entitled to the accumulated sum of each scaled probability, where the expectation, and hence the probabilities, must be computed under the transformed processes (14)-(15). This transformed price process (14) shares the same diffusion coefficient with its counterpart (10), but the drift coefficient can be higher or lower (depending on the relationship between the diffusion coefficient and the spot price).

Theorem 1 is also useful to us for two different reasons. First, as $A(t)$ is locally deterministic, the call can be dynamically replicated by taking a position in the underlying asset and cash

$$C\left(t,\tau; P(t), A(t)\right) = b_0 + P(t)\,\Delta_P(t,\tau), \tag{21}$$

where b_0 represents the (residual) cash position, with $\Delta_P(t,\tau) \equiv \frac{\partial C(t,\tau)}{\partial P(t)}$ determined via (13). The convexity of the call and the bound on the delta restricts the stock and the non–stock position in (21). By the convexity of the call and the condition $C(t,\tau;0,A(t)) = 0$, note that the option elasticity $\Omega(t,\tau) \equiv \frac{\Delta_P(t,\tau)}{\frac{C}{P}}$ is greater than one. Clearly then $b_0 = -[P(t)\,\Delta_P(t,\tau) - C(t,\tau)] \leq 0$. It is shown in Appendix A that the non–stock position in the replicating portfolio is bounded by

$$-\frac{1}{t+\tau}\left[\frac{P(t)}{r}\left(1 - r - e^{-r\tau}\right) + Ke^{-r\tau}\right] \leq \;\; b_0 \;\; \leq 0. \tag{22}$$

By rearranging the bounds on the non–cash position in (22), one obtains the desired bounds on the option elasticity reported below:

$$1 \leq \quad \Omega(t,\tau) \quad \leq 1 + \frac{1}{(t+\tau)\, C(t,\tau)} \left\{ \frac{P(t)}{r} \left(1 - r - e^{-r\tau}\right) + K e^{-r\tau} \right\}.$$

Second, the results in Theorem 1 can be employed to establish cross-partial derivatives related to the change in the spot price. By Leibnitz's rule and (18)

$$\frac{\partial^2 C(t,\tau)}{\partial P(t)\, \partial \tilde{K}} = - \frac{e^{-r\tau}}{t+\tau} \frac{\partial W[P(t), \tilde{K}]}{\partial P(t)} \bar{\Phi}\left(\tilde{K}\right) \leq 0. \tag{23}$$

From the chain rule, $\frac{\partial^2 C(t,\tau)}{\partial P(t) \partial A} \geq 0$ and $\frac{\partial^2 C(t,\tau)}{\partial P(t) \partial K} \leq 0$. Thus, holding other variables constant, the option delta is non-decreasing in $A(t)$ and non-increasing in $\tilde{K}$.

One crucial question remains unanswered: Will the average-rate option price be higher or lower than the traditional counterpart? Will it command a lower Arrow-Debreu security price Π_2? Answering this requires knowledge of the risk–neutral densities for $\int_t^{t+\tau} P(u)\, du$ and $P(t+\tau)$. With unknown $\sigma[P(t),t]$ in (10), the problem is generally intractable. For this purpose, specialize, for now, $\sigma\, P(t)^{-1/2} \equiv \sigma[P(t),t]$. This parameterization is of special interest to our investigation, as an analytical solution is available for both average-rate option price and the traditional option price (see Cox and Ross (1976) for the latter). That is, substituting the price dynamics into (2), we arrive at the call formula (3), where $\Pi_j(t,\tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{e^{-i\phi \tilde{K}} F_j(t,\tau;\phi)}{i\phi} d\phi$. From Bakshi and Madan (2000), the characteristic function $F_1(t,\tau;\phi)$ is (details are in (59)-(60) of Appendix A):

$$\begin{aligned} F_1(t,\tau;\phi) \quad = \quad & \frac{2\, r\, F_2(t,\tau;\phi)}{1 - e^{r\tau}} \left\{ \frac{2\phi}{\sqrt{\zeta}} \left(\frac{\frac{1}{2} + (1 - e^{-\zeta\tau})^{-1} - \tau\zeta e^{-\zeta\tau}(1 - e^{-\zeta\tau})^{-2}}{\left(\zeta(1 - e^{-\zeta\tau})^{-1} - \zeta - r\right)^2} \right) \right. \\ & \left. - \frac{1}{2\zeta(1 - e^{-\zeta\tau})^{-1} - \zeta - r} \right\}, \end{aligned}$$

where $\zeta(\phi) \equiv \sqrt{r^2 - 2i\phi\sigma^2}$ and $F_2(t,\tau;\phi) = \exp\left(\frac{2i\phi P(t)}{2\zeta(1-e^{-\zeta\tau})^{-1} - \zeta - r}\right)$.

Each for the average-rate call and the traditional call, we set $t = 0$, $\tau = 1$, $r = 0.10$, $P = 1$, and $\sigma = 0.20$. Keeping $A(0) = 0$, we generate, for each candidate model, the call price and the Arrow-Debreu security price, $\Pi_2(t,\tau)$ and display the differences in Figures 1 and 2. One can observe that the average-rate call price is generally lower than the traditional counterpart. The gap between the model prices is positively-sloped initially as the option moneyness shifts from deep-in-the money to at-the-money. Later, the slope becomes negative with the moneyness turning point being $\frac{P}{K} = 1$. For a moneyness level of 0.80, for example, the traditional call is priced at \$0.281 versus \$0.228 for the average-rate call, which amounts to a price divergence of 5.30% (as a fraction of spot price). When the moneyness level is exactly 1, the traditional (average-rate) call price is \$0.1327 (\$0.0805).

Fluctuating in an S-shaped pattern (see Figure 2), the disparity across the Π_2's is negative for deep-in-the money options (the difference in the probability Π_1 mimics the same pattern). Subsequently, as the moneyness increases, the price of the average-rate delta security becomes higher for out-of-the-money calls. Thus, in summary, the price of the average-rate Arrow-Debreu security can be higher or lower than the traditional Digital call, but the average-rate call option can be expected to be cheaper than the traditional call option contract.

C Comparative Statics for the Change in Average-to-Date Price, the Strike Price, and Asset Riskiness

Theorem 2 *When the spot price is governed by the Markov process (10), the following characterizations can be obtained:*

(a) The Asian call price is non–decreasing and convex in $A(t)$

$$\frac{\partial C(t,\tau)}{\partial A(t)} = \frac{e^{-r\tau}}{t+\tau} Prob\left[\int_t^{t+\tau} P(u)\,du > (t+\tau)\,K - A(t)\right]$$

$$(24) \qquad = \frac{e^{-r\tau}}{t+\tau}\Pi_2(t,\tau) \geq 0,$$

with $0 \leq \frac{\partial C(t,\tau)}{\partial A(t)} \leq \frac{e^{-r\tau}}{t+\tau}$*; and*

$$(25) \qquad \frac{\partial^2 C(t,\tau)}{\partial A^2(t)} = \frac{e^{-r\tau}}{t+\tau}\,\Phi\left[(t+\tau)\,K - A(t)\right] \geq 0,$$

where $\Phi[.]$ *represents the density of* $\int_t^{t+\tau} P(u)\,du$.

(b) The price of the Asian call is non-increasing and convex in the strike price K (and the adjusted strike price $\tilde{K}$)

$$\frac{\partial C(t,\tau)}{\partial K} = -e^{-r\tau}\,\Pi_2(t,\tau) \le 0, \tag{26}$$

with $-e^{-r\tau} \le \frac{\partial C(t,\tau)}{\partial K} \le 0$*; and*

$$\frac{\partial^2 C(t,\tau)}{\partial K^2} \;=\; (t+\tau)\,e^{-r\tau}\,\Phi\left[(t+\tau)\,K - A(t)\right] \ge 0. \tag{27}$$

(c) The Asian call price is positively related to the riskiness of the spot asset. That is, if $\bar{\sigma}[P(t),t] \ge \sigma[P(t),t]$*, then* $C(t,\tau;\bar{\sigma}) \ge C(t,\tau;\sigma)$ *for all* t*.*

Proof: From equation (16) and Leibniz's differentiation rule

$$\begin{aligned} \frac{\partial C(t,\tau)}{\partial \tilde{K}} &= -\frac{e^{-r\tau}}{t+\tau}\int_{\tilde{K}}^{\infty} \Phi\left[W\right]\, dW \\ &= -\frac{e^{-r\tau}}{t+\tau}\,\text{Prob}\left[\int_t^{t+\tau} P(u)\,du > K(t+\tau) - A(t)\right]. \end{aligned} \tag{28}$$

The rest of the proof relies on the relationships: $\frac{\partial C(t,\tau)}{\partial A(t)} = \frac{\partial C(t,\tau)}{\partial \tilde{K}} \times \frac{\partial \tilde{K}}{\partial A(t)} = \frac{e^{-r\tau}}{t+\tau}\,\Pi_2(t,\tau)$ and $\frac{\partial C(t,\tau)}{\partial K} = \frac{\partial C(t,\tau)}{\partial \tilde{K}} \times \frac{\partial \tilde{K}}{\partial K} = e^{-r\tau}\,\Pi_2(t,\tau)$. The second-order partial derivatives (25) and (27) are due to Leibnitz's rule. Proof of Part (c) of the Theorem is a variant of the corresponding one in Bergman, Grundy and Wiener (1996). Details can be found in Appendix A. □

Theorems 1 and 2 equivalently state that the Asian call price is non–decreasing in the term–to-expiration of the option contract (i.e., decreasing in the passage of time). In the valuation PDE of the call price

$$\frac{\partial C}{\partial \tau} = \frac{1}{2}\,\sigma^2[P,t]\,P^2\,\frac{\partial^2 C}{\partial P^2} + P\,\frac{\partial C}{\partial A} + r\left\{P\,\frac{\partial C}{\partial P} - C\right\}, \tag{29}$$

all the terms on the right hand side of (29) are non-negative. Consequently, the Asian call price has positive time-decay.

D Comparative Statics for Spot Interest Rate Change

The interest rate exposure of Asian option is a more involved calculation, as the time averaging brings into play complete yield curve considerations in comparison with the traditional option. Here, we focus on the first-order level effect by developing the response in a constant interest rate environment.

Theorem 3 *When the spot price is governed by the one-dimensional Markov process (10), the response of the Asian call price to a change in the spot interest rate is*

$$\frac{\partial C(t,\tau)}{\partial r} = -\tau C(t,\tau) + \frac{P(t)\,(r\tau - 1 + e^{-r\tau})}{(t+\tau)\,r^2} \times \left\{ \frac{\int_{\tilde{K}}^{\infty} e^{-r\tau} \frac{\partial W(P(t),r,v)}{\partial r}\,\bar{\Phi}\,[v]\;dv}{\int_{0}^{\infty} e^{-r\tau} \frac{\partial W(P(t),r,v)}{\partial r}\,\bar{\Phi}\,[v]\;dv} \right\}, \tag{30}$$

with $-\frac{P}{(t+\tau)\,r^2}\,(1-(1+r\tau)\,e^{-r\tau}) \leq \frac{\partial C(t,\tau)}{\partial r} \leq \frac{P}{(t+\tau)\,r^2}\,(e^{-r\tau} - 1 + r\tau)$.

Proof: The proof of this result can be constructed in two steps. First, we may write

$$\frac{\partial C(t,\tau)}{\partial r} = -\tau\, C(t,\tau) + \frac{1}{t+\tau} \int_{\tilde{K}}^{\infty} e^{-r\tau}\, \frac{\partial W[P(t),r,v]}{\partial r}\,\bar{\Phi}\,[v]\;dv. \tag{31}$$

Moreover, by the unconditional expectation (19) and algebraic manipulation

$$\int_{0}^{\infty} e^{-r\tau}\, \frac{\partial W[P(t),r,v]}{\partial r}\,\bar{\Phi}\,[v]\;dv = \frac{\tau P}{r} - \frac{P}{r^2}(1-e^{-r\tau}). \tag{32}$$

By combining (31) and (32) and recognizing from the no-crossing principle that $\frac{\partial W(P(t),r,v)}{\partial r} > 0$, confirms equation (30). To substantiate the rest of the assertion, note that when $\tilde{K} \to \infty$, both the terms in (31) are zero, and hence $\frac{\partial C(t,\tau)}{\partial r} = 0$. Now let $\tilde{K} = 0$, which makes the final term $\{.\}$ in (30) unity. Using equations (3) and (32) and relying on $\Pi_1(t,\tau) = 1$ yields

$$\frac{\partial C(t,\tau)}{\partial r} = -\frac{P}{(t+\tau)\,r^2}\left(1-(1+r\tau)\,e^{-r\tau}\right) \leq 0.$$

Now the lower bound on the average-rate call price is

$$\begin{aligned}
\frac{\partial C(t,\tau)}{\partial r} &= -\frac{\tau}{t+\tau}\int_{\tilde{K}}^{\infty} e^{-r\tau}\left(W[P(t),v] - \tilde{K}\right)\,\bar{\Phi}\,[v]\;dv + \frac{1}{t+\tau}\int_{\tilde{K}}^{\infty} e^{-r\tau}\frac{\partial W[P(t),r,v]}{\partial r}\,\bar{\Phi}\,[v]\;dv \\
&\leq -\tau C(t,\tau) + \frac{1}{t+\tau}\int_{0}^{\infty} e^{-r\tau}\frac{\partial W[P(t),r,v]}{\partial r}\,\bar{\Phi}\,[v]\;dv \\
&\leq \frac{P}{(t+\tau)\,r^2}\left(e^{-r\tau} - 1 + r\tau\right)
\end{aligned}$$

from (32). For large values of $\tilde{K}$, the partial derivative $\frac{\partial C(t,\tau)}{\partial r}$ can assume a positive sign. □

E Comparative Statics for Dividend Yield Change

So far our analysis assumes a spot asset that pays no dividends over the life of the option. To relax this assumption and to understand the impact of dividend yield on the valuation of Asian options, let us now assume that the spot asset pays continuous dividends at the rate z. Accommodate this dividend yield payout by modifying the spot price dynamics in equation (10) to

$$\frac{dP(t)}{P(t)} = (r - z)\,dt + \sigma[t, P(t)]\,d\omega(t) \qquad t \geq 0. \tag{33}$$

Theorem 4 *When the spot price is governed by the Markov process (33), the Asian call price is non–increasing in the dividend yield z*

$$\frac{\partial C(t,\tau)}{\partial z} = -\frac{P(t)e^{-z\tau}}{(r-z)^2}\left\{e^{-(r-z)\tau} + (r-z)\tau - 1\right\} \times \left\{\frac{\int_{\bar{K}}^{\infty} e^{-r\tau}\frac{\partial W(P(t),r,z,v)}{\partial z}\,\bar{\Phi}\,[v]\,dv}{\int_0^{\infty} e^{-r\tau}\frac{\partial W(P(t),r,z,v)}{\partial z}\,\bar{\Phi}\,[v]\,dv}\right\} \leq 0,$$

with $-\frac{P(t)e^{-z\tau}}{(r-z)^2}\left\{e^{-(r-z)\tau} + (r-z)\tau - 1\right\} \leq \frac{\partial C(t,\tau)}{\partial z} \leq 0.$

Proof: See Appendix A. □

In developing the above rich set of economic implications, our emphasis was on one-dimensional diffusions. When the drift $r(t)$ of the spot price process is changing according to some pre-specified independent diffusion law, the same qualitative characterizations emerge with minor modifications to each of the proofs. For the remainder of the paper, we concentrate on a closed-form illustration and study its promise.

III Catastrophe Insurance Option Contracts

Consider now the pricing of catastrophe insurance option contracts, traded on the Chicago Board of trade. As emphasized in Cummins and Geman (1995), Litzenberger, Beaglehole and Reynolds (1996), and Tilley (1995), this option contract is instrumental in diversifying and securitizing catastrophe risk. It is also found to be an attractive alternative to the traditional property/casualty reinsurance programs.

The option contract is a European instrument written on the Property Claim Services catastrophe loss indices. This index tracks natural disasters (e.g., hurricane, floods, fires, or wind damage). Specifically, the terminal payoff is based on actuarially assessed cumulative losses over the contract period. Let $X(t)$ represent the (dollar) loss index at date t (each index point is equivalent to a loss estimate of 100 million). The payoff on the catastrophe loss (CAT) option contract is equal to:

$$\max\left(\int_0^{t+\tau} X(u)\,du - K, 0\right), \tag{34}$$

for strike price K and time-to-maturity τ. With the exception of a scaling factor, the payoff (34) resembles the average-rate payoff (1), and is contingent on the integral of the level of a process. Currently there are nine loss indices. One each for the five regions in the U.S.; one each for property loss prone states of California, Texas, and Florida; and a composite national index (all 50 states plus Washington D.C.).

A variant of the basic contract, the call option spread, transfers the catastrophic risk exposure to a pre-specified range of strike prices. For example, a 40/60 spread represents a long position in a CAT option with strike of 40 (i.e., 4 billion worth of losses), and a short position in a CAT option with strike of 60. Only the option premiums on the spread are recorded and not separately for each component strike price.

A Modeling the Loss Process

To model losses due to catastrophes, assume a time-evolution of the loss index, $\{X(t), t \geq 0\}$, as shown below:

$$X(t) = \frac{\mu}{\kappa} + \left(X(0) - \frac{\mu}{\kappa}\right) e^{-\kappa t} + \sum_{j=1}^{N(t)} J(t_j)\, e^{-\kappa(t-t_j)} \qquad t \in [0, t+\tau], \tag{35}$$

with $X(0) \geq 0$. The process (35) needs some explanation. First, $\{N(t), t \geq 0\}$ is an integer valued stochastic process and distributed Poisson with intensity parameter $\lambda > 0$. For a later characterization, note that $\text{Prob}[N(t) = n] = \frac{(\lambda t)^n}{n!}\, e^{-\lambda t}$.

Second, $\{J(t_j)\}$ is a sequence of independent identically distributed random variables possessing some common distribution on the *positive* half-line. Denote the characteristic function of the density by $F_J(\phi)$. In essence, $N(t)$ counts the number of loss events, and $J(t_j) > 0$ is the aggregate loss associated with the loss events occurring at time t_j. The claims are paid out at an exponential delay reflecting a processing half-life of $\ln(2)/\kappa$.

Consequently, $X(t)$ is a strictly positive process, and the process for cumulative losses, $\int_t^{t+\tau} X(u)\,du$, is increasing by construction.

Third, we incorporate the feature that high loss levels have low arrival rates. Assume that J is distributed exponential with density $\Phi[J] = \nu\, e^{-\nu J}$. The characteristic function is:

$$F_J(\phi) = \frac{\nu}{\nu - i\,\phi}. \tag{36}$$

The characteristic function (36) will be used to build the characteristic function of the cumulative losses. Assume that $J(t)$ and $N(t)$ are independent.[7]

Fourth, total losses at each date t can be decomposed into two components. One, as seen from (35), there is a deterministic mean-reverting component given by: $\mu/\kappa + (X(0) - \mu/\kappa)\, e^{-\kappa t}$. Two, there is an exponentially-damped compound Poisson process given by: $\sum_{j=1}^{N(t)} J(t_j)\, e^{-\kappa(t-t_j)}$. Using Ito's Lemma for semimartingales (i.e., Theorem 4.57 in Jacod and Shiryaev (1980, page 57)), we arrive at

$$\begin{aligned}
dX(t) &= -\kappa \left(X(0) - \frac{\mu}{\kappa} \right) e^{-\kappa t}\, dt + \sum_{j=1}^{N(t)} (-\kappa)\, J(t_j)\, e^{\kappa(t-t_j)}\, dt + J(t)\, dN(t), && (37)\\
&= (\mu - \kappa\, X(t))\, dt + J(t)\, dN(t), && (38)
\end{aligned}$$

which is a mean-reverting Markov process with loss uncertainty driven by $N(t)$ and $J(t)$.

It is important to observe that the loss process has no continuous martingale component. Moreover, as catastrophe related losses are nature induced phenomenon, they may be uncorrelated with economy-wide risk factors (i.e., the stock and the bond markets). A notable exception is a major economic catastrophe such as the recent earthquake in Japan. Barring such exceptions, investors may not require any risk premium to bear catastrophe risk. With the understanding that catastrophe related contracts are generally zero beta products, we can assume that (37) and (38) hold under both the risk-neutral and the physical measure. Alternatively, if catastrophe losses are priced, we suppose that the risk-neutral dynamics of the loss process are described by equations (37) and (38), with parameters and the distribution of J now altered to its risk-neutral counterpart. Notice that the rate of mean-reversion in (38) is controlled by κ and the long-run loss rate is μ/κ.

[7] The structure of jump arrival rates and the jump amplitudes can be generalized. But, not all candidates meet the requirement that high level of losses have low arrival rates. For this reason, we ruled out gamma, non-central chi-squared distribution and extreme-value distributions.

Finally, with loss assumption (38), the characteristic function of the cumulative losses can be derived analytically. In Appendix B, we show that:

$$F(t,\tau;\phi) \equiv E_t^Q\left\{\exp\left(i\,\phi\int_t^{t+\tau}X(u)\,du\right)\right\}, \tag{39}$$

$$= \exp\left[\,\mathcal{A}(\tau;\phi) + \mathcal{B}(\tau;\phi)\,X(t)\,\right], \tag{40}$$

where defining

$$\mathcal{A}(\tau;\phi) = \frac{i\,\phi\,\mu\,\tau}{\kappa} - \lambda\,\tau - \frac{i\,\phi\,(1-e^{-\kappa\tau})\mu}{\kappa^2} + \frac{\lambda\,\nu}{\nu\kappa - i\phi}\ln\left[\frac{\nu\kappa - i\phi + i\phi e^{-\kappa\tau}}{\nu\kappa e^{-\kappa\tau}}\right], \tag{41}$$

$$\mathcal{B}(\tau;\phi) = \frac{i\,\phi\,(1-e^{-\kappa\tau})}{\kappa}. \tag{42}$$

From equation (40), the mean aggregate loss is

$$\mu_X \equiv E_t^Q\left\{\int_t^{t+\tau}X(u)\,du\right\}$$

$$= \frac{\mu\tau}{\kappa} + \frac{(1-e^{-\kappa\tau})}{\kappa}\left(X(t) - \frac{\mu}{\kappa}\right) + \frac{\lambda}{\kappa}\left(\tau - \frac{(1-e^{-\kappa\tau})}{\kappa}\right)E(J) \tag{43}$$

and the variance of aggregate losses must be

$$\sigma_X^2 \equiv E_t^Q\left\{\left(\int_t^{t+\tau}X(u)\,du - \mu_X\right)^2\right\}$$

$$= \frac{\lambda}{\kappa^2}\left(\tau + \frac{(1-e^{-\kappa\tau})}{2\kappa} - \frac{2(1-e^{-\kappa\tau})}{\kappa}\right)E(J^2) - 2\mu_X^2. \tag{44}$$

The higher the expected value of the jump component (or higher the μ), the higher are the aggregate losses.

B The Closed-form Option Characterization

Consistent with the contractual provisions set by the CBOT, the call price is, subject to certain regularity conditions, given by

$$C(t,\tau) = E_t^Q\left\{e^{-r\tau}\max\left(\int_0^{t+\tau}X(u)\,du - K, 0\right)\right\} \tag{45}$$

for constant interest rate r. The price of the call satisfies the integro-differential equation

$$(\mu - \kappa X) \frac{\partial C}{\partial X} - \frac{\partial C}{\partial \tau} - r\,C + X\,\frac{\partial C}{\partial A}$$

$$= -\int_0^{\infty} \lambda\,\{C(t,\tau;X+J) - C(t,\tau;X)\}\,\Phi[J]\,dJ \tag{46}$$

where $\Phi[J]$ is the density for the size of the (loss) jump and $A(t) \equiv \int_0^t X(u)\,du$. Using the approach in Bakshi and Madan (2000), one obtains the following catastrophe insurance option formula:

$$C(t,\tau) \;=\; e^{-r\tau}\,G(t,\tau)\,\Pi_1(t,\tau) - e^{-r\tau}\,(K - A(t))\,\Pi_2(t,\tau), \tag{47}$$

where the risk-neutral probabilities are given by

$$\Pi_j(t,\tau) \;=\; \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\exp[-i\,\phi\,(K - A(t))]\,\frac{F_j(t,\tau;\phi)}{i\phi}\right] d\phi \qquad \text{j=1,2} \tag{48}$$

and defining the security price

$$G(t,\tau) \;=\; \frac{1}{i}\,\frac{\partial F(t,\tau;\phi)}{\partial \phi}\,|_{\phi=0} \tag{49}$$

$$= \frac{\mu\tau}{\kappa} - \frac{(1-e^{-\kappa\tau})\mu}{\kappa^2} + \frac{\lambda\tau}{\nu\kappa} - \frac{\lambda(1-e^{-\kappa\tau})}{\nu\kappa^2} + \frac{(1-e^{-\kappa\tau})}{\kappa}\,X(t), \tag{50}$$

and the first characteristic function

$$F_1(t,\tau;\phi) \;=\; \frac{\frac{\partial F(t,\tau;\phi)}{\partial \phi}}{\frac{\partial F(t,\tau;\phi)}{\partial \phi}\,|_{\phi=0}} \tag{51}$$

$$= \frac{F(t,\tau;\phi)}{G(t,\tau)}\left\{\frac{\mu\tau}{\kappa} - \frac{(1-e^{-\kappa\tau})\mu}{\kappa^2} + \frac{\lambda\nu}{(\nu\kappa - i\phi)^2}\ln\left[\frac{\nu\kappa - i\phi + i\phi e^{-\kappa\tau}}{\nu\kappa e^{-\kappa\tau}}\right] \right.$$

$$\left. -\frac{\lambda\nu}{\nu\kappa - i\phi}\left(\frac{1-e^{-\kappa\tau}}{\nu\kappa - i\phi + i\phi e^{-\kappa\tau}}\right) + \frac{(1-e^{-\kappa\tau})}{\kappa}\,X(t)\right\}, \tag{52}$$

and $F_2(t,\tau;\phi) = F(t,\tau;\phi)$, with the characteristic function, $F(t,\tau;\phi)$ as displayed in (40).

The option formula and the loss process assumption differs fundamentally from the corresponding one in Cummins and Geman (1995). For one, their characterization of the catastrophe option contract is based on the geometric Brownian motion assumption for the instantaneous loss process (perturbed by an orthogonal Poison jump component). The resulting model is analytically intractable and only solvable by simulation methods. Though the jump can be linked to loss events, the relationship of the geometric Brownian motion to such events is unclear. Our approach of directly linking losses to loss events that are processed with exponential delay is in contrast fully tractable.

Moreover, we have a call price that is related analytically to the cumulative losses to date, and the current loss rate; the model is frugal in the structural parameters μ, κ, λ, and ν. Our closed-form representation facilitates the calibration of our model to market CAT option prices.

Given that both the characteristic functions and $G(t, \tau)$ are known analytically and the option deltas are determined internally, our valuation formula (47) can also be employed to construct static as well as dynamic hedging strategies involving portfolio of catastrophic option contracts. Recall that a bull spread catastrophic option contract entails taking a long position in a call with strike price K_1 and a short position with strike price $K_2 > K_1$ (with options having the same maturity). The call spread can then be synthetically computed as: $C(t, \tau; K_1) - C(t, \tau; K_2)$.

Finally, $G(t, \tau)$ can be interpreted as the price of the futures contract on cumulative catastrophe losses. According to CBOT, the time-$t + \tau$ period payoff on the futures contract (for estimated property premium δ) with τ periods remaining to expiration is: $\frac{25000}{\delta} \int_t^{t+\tau} X(u)\, du$. Therefore, the time-$t$ price is $\frac{25000}{\delta} G(t, \tau)$. We have computed this price analytically in (49)-(50).

IV Conclusions and Summary

Stimulated by the interest in the pricing of average-rate options and related claims, we examine their general properties. We show that the upper bound on the average-rate option delta can depart significantly from the traditional one: large in magnitude initially, but then converging to exactly zero at option maturity. As a consequence, they may be difficult to delta hedge early on in its life. By canceling the low prices and high prices and smoothing out volatility fluctuations over the life of the option contract, the average-rate options offer unique tools to hedge price risk.

When the price dynamics follow a one-dimensional Markov diffusion, the Asian call price is non-decreasing and convex in the asset price. We also show that the Asian call price is non-decreasing and convex in the average-to-date price. The Asian call price rises with the volatility of the price process. Our analysis demonstrates that the average call price can decrease when the interest rate rises.

We specifically examine a closed-form example in which the option is contingent on cumulative catastrophe losses over a loss period. In addition, we argue that any theoretical model omitting the pure jump loss feature, or alternatively relying only on diffusion loss dynamics are likely mis-specified. We posit losses as a mean-reverting process with one-sided jumps. In our modeling approach, high level of losses are associated with low arrival rates. The characteristic function for cumulative losses and the catastrophe option price is derived analytically. Our contribution provides the impetus to expand empirical and theoretical research on average-rate claims.

Appendix

Appendix A: Proof of General Properties

Proof of the Convexity of the Average-Rate Call

The valuation PDE of the average-rate call is

$$\frac{1}{2}\,\sigma^2[P,t]\,P^2\,\frac{\partial^2 C}{\partial P^2} + r\,P\,\frac{\partial C}{\partial P} - \frac{\partial C}{\partial \tau} + P\,\frac{\partial C}{\partial A} - r\,C = 0. \tag{53}$$

Now define the variable $\Delta_P(t,\tau) \equiv \frac{\partial C(t,\tau)}{\partial P}$. Differentiate the PDE (53) with respect to the spot price P (as in Bergman, Grundy, and Wiener (1996)), which leads to

$$\begin{aligned}
&\frac{1}{2}\,\sigma^2[P,t]\,P^2\,\frac{\partial^2 \Delta_P}{\partial P^2} + \left\{ r\,P + \sigma^2[P,t]\,P + \sigma[P,t]\,P^2\,\frac{\partial \sigma[P,t]}{\partial P} \right\} \frac{\partial \Delta_P}{\partial P} \\
&-\frac{\partial \Delta_P}{\partial \tau} + P\,\frac{\partial \Delta_P}{\partial A} + \frac{\partial C}{\partial A} = 0.
\end{aligned} \tag{54}$$

Using the Feynman–Kac representation, the solution to the PDE in (54), for $u \geq t$, is equivalently

$$\frac{\partial C(t,\tau)}{\partial P(t)} = E_t^{Q^*} \left\{ \Delta_P(t+\tau, 0) + \int_t^{t+\tau} \frac{\partial C(u,\tau)}{\partial A(u)}\,du \right\}, \tag{55}$$

where the expectation $E_t^{Q^*}$ is taken under the stochastic system

$$\begin{aligned}
\frac{dP(t)}{P(t)} &= \left\{ r + \sigma^2[t,P(t)] + P\,\sigma[t,P(t)]\frac{\partial \sigma[P,t]}{\partial P(t)} \right\} dt + \sigma[t,P(t)]\,d\omega(t), \\
d\,A(t) &= P(t)\,dt.
\end{aligned}$$

The remaining task is to prove the following:

(a) The terminal period delta is zero

$$\Delta_P(t+\tau, 0) = 0, \tag{56}$$

because $C(t+\tau,0) = \max\left(\frac{1}{t+\tau}\,\int_0^{t+\tau} P(u)\,du - K, 0\right)$. Inserting (56) in (55) proves (13) of Theorem 1. Since $\frac{\partial C(u,\tau)}{\partial A(u)} \geq 0$ [see Theorem 2], the non–negativity of the delta is apparent.

(b) The option delta is non-decreasing in the spot price

$$\frac{\partial^2 C(t,\tau)}{\partial P(t)^2} = E_t^{Q^*}\left\{\int_t^{t+\tau} \frac{\partial^2 C(u,\tau)}{\partial A(u)\partial P(u)} \times \frac{\partial P(u)}{\partial P(t)}\, du\right\} \geq 0, \tag{57}$$

To prove convexity, it is sufficient to demonstrate $\frac{\partial^2 C(u,\tau)}{\partial A(u)\,\partial P(t)} \geq 0$. Observe from no-crossing (for $u \geq t$) that $\frac{\partial P(u)}{\partial P(t)} \geq 0$, and by equation (23), $\frac{\partial^2 C(u,\tau)}{\partial A(u)\,\partial P(u)} \geq 0\ \forall u$. This is the final step of the proof. □

Proof of the Bound on the Non-Stock Position in (22)

From the replicating portfolio of the call, and by the convexity of the average call price in the spot price

$$\begin{aligned}
b_0 &= -(P(t)\,\Delta_P(t,\tau) - C(t,\tau)) \\
&= -\frac{1}{t+\tau}\int_{\tilde{K}}^{\infty} e^{-r\tau}\left\{P(t)\,\frac{\partial W(P(t),r,v,t)}{\partial P(t)} - W(P(t),r,t)+K\right\}\bar{\Phi}\,[v]\; dv \\
&= -\frac{1}{t+\tau}\left[\int_0^{\infty} e^{-r\tau}\left\{P(t)\,\frac{\partial W(P(t),v)}{\partial P(t)} - W(P(t),r,v,t)+K\right\}\bar{\Phi}\,[v]\; dv\right] \times \text{Prob}[.] \\
&= -\frac{1}{t+\tau}\left[\frac{P(t)}{r}\left(1-r-e^{-r\tau}\right) + K e^{-r\tau}\right] \times \text{Prob}[.] \qquad (58)
\end{aligned}$$

where, for convenience, the probability function Prob[.] is displayed below:

$$\text{Prob}[.] = \frac{\int_{\tilde{K}}^{\infty} e^{-r\tau}\left\{P(t)\,\frac{\partial W(P(t),r,v,t)}{\partial P(t)} - W(P(t),r,t)+K\right\}\bar{\Phi}\,[v]\; dv}{\int_0^{\infty} e^{-r\tau}\left\{P(t)\,\frac{\partial W(P(t),r,v,t)}{\partial P(t)} - W(P(t),r,t)+K\right\}\bar{\Phi}\,[v]\; dv}.$$

Equation (58) leads to the bounds posited in equation (22) of the text. □

Proof of the Average-Rate Call for the Price Process with $\sigma[P,t] = \sigma\, P^{-1/2}$

According to Theorem 1 in Bakshi and Madan (2000), we need the characteristic function of the risk-neutral density. The characteristic function, $F_2(t,\tau;\phi) \equiv E_t^Q\left\{\exp(i\,\phi\,\int_t^{t+\tau} P(u)\,du)\right\}$, satisfies the PDE

$$\frac{1}{2}\,\sigma^2\,P\,\frac{\partial F_2^2}{\partial P^2} + r\,P\,\frac{\partial F_2}{\partial P} - \frac{\partial F_2}{\partial \tau} = -i\,\phi\,P\,F_2 \tag{59}$$

subject to $F_2(t+\tau,0;\phi)=1$. Let $\zeta(\phi) \equiv \sqrt{r^2 - 2\,i\,\phi\,\sigma^2}$. It can be verified by substitution that $F_2(t,\tau;\phi) = \exp\left[\frac{2\,i\,\phi\,P(t)}{2\zeta(1-e^{-\zeta\tau})^{-1}-\zeta-r}\right]$ satisfies (59). From (4), the characteristic function, $F_1(t,\tau;\phi)$, can be derived as

$$F_1(t,\tau;\phi) \equiv \frac{E_t^Q\left\{\left(\int_t^{t+\tau} P(u)\,du\right) \times e^{i\phi\int_t^{t+\tau} P(u)\,du}\right\}}{E_t^Q\left\{\int_t^{t+\tau} P(u)\,du\right\}}.$$

Since this is an option on the level of the uncertainty, we can differentiate $F_2(t,\tau)$ with respect to ϕ, and deduce $f_1(t,\tau)$ as (refer to Theorem 1 of Bakshi and Madan (2000)):

$$F_1(t,\tau;\phi) = \frac{\frac{\partial F_2(t,\tau;\phi)}{\partial\phi}}{\frac{\partial F_2(t,\tau;\phi)}{\partial\phi}\,|_{\phi=0}}. \tag{60}$$

Simplifying the resulting expression proves the characteristic function, $F_1(t,\tau;\phi)$. □

Proof that the Average-Rate Call is Increasing in the Riskiness of the Underlying Asset

We want to demonstrate that $\bar{\sigma}[P(t),t] \geq \sigma[P(t),t]$ implies $C(t,\tau;\bar{\sigma}) \geq C(t,\tau;\sigma)$. Following Bergman, Grundy, and Wiener (1996), make the following transformations: (i) $p(t) = e^{-r\tau}P(t)$; (ii) $a(t) = e^{r\tau}A(t)$; (iii) $c(t,\tau;p(t)) = e^{r\tau}C(t,\tau;\sigma[P(t),t])$ and $\bar{c}(t,\tau;p(t)) = e^{r\tau}C(t,\tau;\bar{\sigma}[P(t),t])$; and (iv) $Y(t,\tau) = \bar{c}(t,\tau;p(t)) - c(t,\tau;p(t))$. Reformulate the PDE of the Asian call price as

$$\frac{1}{2}\,\sigma^2[p,t]\,p^2\,\frac{\partial^2 c}{\partial p^2} - \frac{\partial c}{\partial\tau} + p\,\frac{\partial c}{\partial a} = 0. \tag{61}$$

Relying on the counterpart PDE for $\bar{c}(t,\tau)$, the PDE for $Y(t,\tau)$ becomes

$$\frac{1}{2}\,\sigma^2[p,t]\,p^2\,\frac{\partial^2 Y}{\partial p^2} - \frac{\partial Y}{\partial\tau} + p\,\frac{\partial Y}{\partial a} + \frac{1}{2}\left\{\bar{\sigma}^2[p,t] - \sigma^2[p,t]\right\}\,p^2\,\frac{\partial^2 c}{\partial p^2} = 0. \tag{62}$$

Using the correspondence between (62) and the Feynman-Kac Theorem, it implies that

$$Y(t,\tau) = E^{Q**}\left\{\int_t^{t+\tau}\frac{1}{2}\left\{\bar{\sigma}^2[p(u),t] - \sigma^2[p(u),t]\right\}\,p(u)^2\,\frac{\partial^2 c(u,\tau)}{\partial p^2(u)}\,du\right\} \geq 0$$

where the expectation now is to be taken according to the system

$$\frac{dp(t)}{p(t)} = \sigma[p(t),t]\,d\omega(t),$$
$$d\,a(t) = p(t)\,dt.$$

The above inequality then holds because $\frac{\partial^2 c(u,\tau)}{\partial p^2(u)} \geq 0$ and $\bar{\sigma}^2[p(u),t] - \sigma^2[p(u),t] \geq 0$ (by definition). □

Proof of Theorem 4

From the price dynamics (33)

$$\frac{\partial C(t,\tau)}{\partial z} = \frac{1}{t+\tau}\int_{\tilde{K}}^{\infty} e^{-r\tau}\frac{\partial W(P(t),r,z,v)}{\partial z}\,\bar{\Phi}\,[v]\;dv.$$

Dividing the numerator and the denominator by $\int_0^\infty e^{-r\tau}\frac{\partial W(P(t),r,z,v)}{\partial z}\,\bar{\Phi}\,[v]\;dv$, and computing the unconditional expectation verifies Theorem 4. □

Appendix B: Proof of the Catastrophe Insurance Option Formula

Our goal is to derive the characteristic function of cumulative losses. Consider the characteristic function

$$F(0,t;\phi) \equiv E_0^Q\left\{\exp\left(i\,\phi\int_0^t X(u)\,du\right)\right\} \tag{63}$$

$$= \exp\left[i\,\phi\,\frac{\mu}{\kappa}\,t + i\,\phi\,\frac{(1-e^{-\kappa t})\,(X(0)-\mu/\kappa)}{\kappa}\right] \times f^*(0,t;\phi) \tag{64}$$

where $f^*(0,t;\phi) \equiv E_0^Q\left\{\exp\left(i\,\phi\int_{t_j}^t \sum_{j=1}^{N(t)} J(t_j)\,e^{\kappa(u-t_j)}\,du\right)\right\}$. We then have

$$f^*(0,t;\phi) \equiv E_0^Q\left\{\exp\left(i\,\phi\sum_{j=1}^{N(t)}\int_{t_j}^t J(t_j)\,e^{\kappa(u-t_j)}\,du\right)\right\}$$

$$= E_0^Q\left\{\exp\left(i\,\phi\sum_{j=1}^{N(t)} J(t_j)\,\frac{(1-e^{\kappa(t-t_j)})}{\kappa}\right)\right\} \tag{65}$$

$$= \sum_{n=0}^{\infty} E\left\{\exp\left(\sum_{j=1}^{N(t)} J(t_j)\,\frac{i\,\phi\,(1-e^{\kappa(t-t_j)})}{\kappa}\right) \mid N(t)=n\right\} \times \frac{(\lambda t)^n e^{-\lambda t}}{n!} \tag{66}$$

$$= \sum_{n=0}^{\infty} \prod_{j=1}^{n} E\left\{\exp\left(J(s)\,\frac{i\,\phi\,(1-e^{\kappa(t-s)})}{\kappa}\right)\right\} \times \frac{(\lambda t)^n e^{-\lambda t}}{n!} \tag{67}$$

$$= \sum_{n=0}^{\infty} \left\{\frac{1}{t}\int_0^t F_J\left(\frac{\phi\,(1-e^{\kappa(t-s)})}{\kappa}\right) ds\right\}^n \times \frac{(\lambda t)^n e^{-\lambda t}}{n!} \tag{68}$$

$$= \exp\left(\lambda\, t\left\{\frac{1}{t}\int_0^t F_J\left(\frac{\phi\,(1-e^{\kappa(t-s)})}{\kappa}\right) ds - 1\right\}\right) \tag{69}$$

and exploiting the fact that $\int_0^\tau F_J\left(\frac{\phi\,(1-e^{-\kappa u})}{\kappa}\right)\, du = \frac{\nu}{\nu\kappa - i\phi}\, \ln\left[(\nu\kappa - i\phi + i\phi e^{-\kappa\tau})/(\nu\kappa e^{-\kappa\tau})\right]$, completes our proof. □

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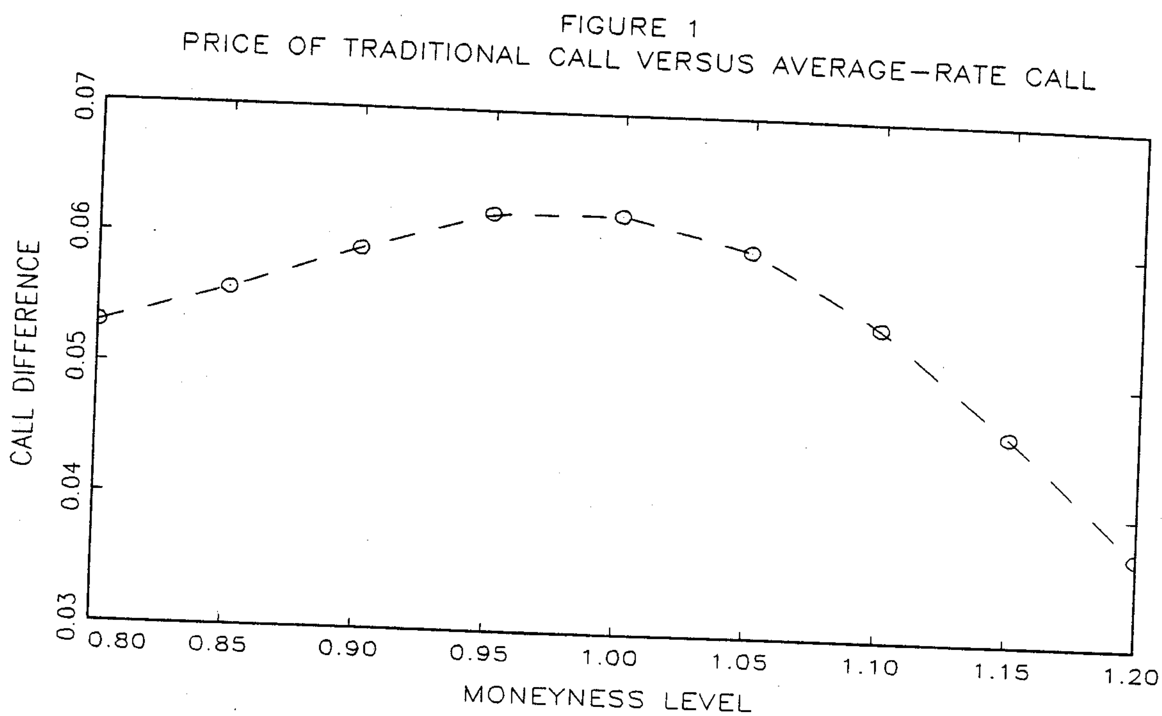
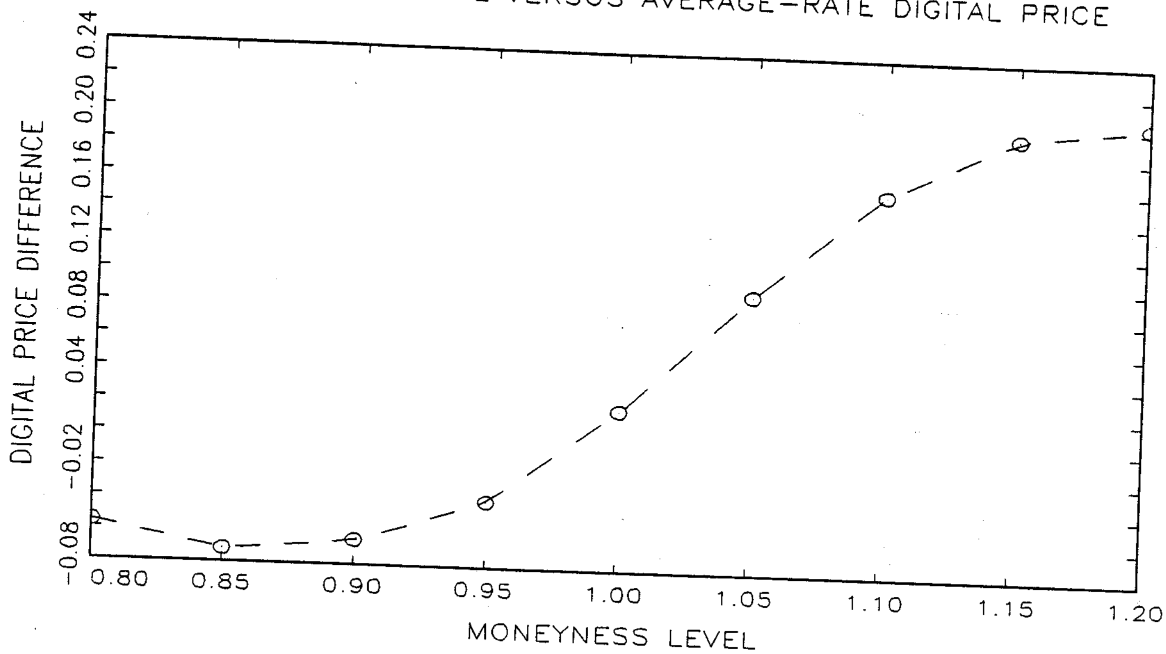


The o-curve illustrates the difference between the fair value of the traditional call and the average-rate call, when the option moneyness varies between 0.80 and 1.20. In deriving each call option price, the spot price dynamics is assumed to follow the process: $\frac{dP(t)}{P(t)} = r\,dt + \sigma P(t)^{-1/2}\,d\omega(t)$. To construct the call price difference curve, we set $r = 10\%$, $\sigma = 20\%$, $P(t) = 1$, $\tau = 1$, and t=0 (the initial date) and $A(0) = 0$.

FIGURE 2

TRADITIONAL DIGITAL VERSUS AVERAGE-RATE DIGITAL PRICE



The o-curve illustrates the difference between the traditional Digital call (i.e., $E_t^Q\{P(t+\tau) \geq K\}$ and the average-rate Digital call (i.e., $E_t^Q\left\{\frac{1}{t+\tau}\int_0^{t+\tau} P(u)du \geq K\right\}$). In deriving each Digital price, the spot price dynamics is assumed to follow the process: $\frac{dP(t)}{P(t)} = r\,dt + \sigma P(t)^{-1/2}\,d\omega(t)$. To construct the curve, we set $r = 10\%$, $\sigma = 20\%$, $P(t) = 1$, $\tau = 1$, and t=0 (the initial date) and $A(0) = 0$.