Markowitz Models of Portfolio Selection: The Inverse Problem

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Abstract

Predictions about investor portfolio holdings can provide powerful tests of asset pricing theories. In the context of Markowitz portfolio selection problem, this paper develops an algorithm which determines the structural parameters in both the investor's return-generating process and the utility function based upon the actual portfolio choices made by each investor. We refer to this problem as the inverse of portfolio selection. Furthermore, through the introduction of a set of investor-specific characteristics, the methodology accommodates either homogeneous or heterogeneous anticipated rates of return–a contribution over the existing returns–generating models. Generalization of the algorithm to Black (1972) and Tobin (1965) models where the efficiency frontier is known in closed form is direct and immediate. The methodology is useful for understanding the investor risk-return tradeoffs and, in particular, can be considered as the micro-level counterpart of the determination of the "taste parameters" in Hansen-Singleton (1982) and the simultaneous determination of the parameters driving the forcing process and the "taste parameters" in Duffie-Singleton (1993).

I Introduction

The portfolio selection model of Markowitz (1952, 1959) consists of two interrelated modules:

- a nonlinear programming problem where risk-averse investors solve a utility maximization problem involving the risk and the expected rate of return of any portfolio, subject to the constraint of an efficiency frontier. The latter is defined *pointwise*, as a sequence of solutions to a quadratic programming problem which minimizes the risk associated with each possible portfolio's expected rate of return subject to the constraint that the elements of the portfolio be non-negative and sum to unity, and

- a parametric stochastic returns-generating process by which, in each period, the investors determine the requisite vector of expectations and the variance-covariance matrix of the investors anticipated rates of return on all risky assets.

Thus, given the inputs from above, that is, expected rates of return and their associated covariance inputs, along with knowledge of the parameters in the investor's specific mean-variance utility function, the investor's optimal portfolio can be computed, but only *numerically*, as a solution to the nonlinear programming problem. We refer to these two steps as the *forward solution* to the portfolio selection problem in which all the above parameters are determined, *ex ante.*

Our present interest, however, is to offer an algorithm for the *inverse* of the portfolio selection problem which confronts an external investigator. That is, given

- a time-series of actual portfolios for a sample of investors;

- a time-series of preceding observed prices and dividends of the relevant set of risky assets;

and

- a set of socio-economic characteristics for each investor in the sample;

in conjunction with a well-behaved specification of the parametric form of the investor's utility function and returns generating process; one can infer or otherwise determine the parameter values in each investor's utility function and, where also of interest, the associated parameters in the returns generating process based on their reflection in the *actual* portfolio choices of investors?

Once the parameters in the complete system have been determined, forward solutions of the model may be employed to compute the optimal portfolio for particular investors in subsequent time periods.

For several reasons, the inverse of the portfolio selection has proved to be a difficult task when approached by conventional econometric methods. First, while the portfolio selection problem is a specialized form of consumer demand model couched in budget− share form, the fact that the optimal portfolio, relative to any feasible configuration of parameter values, can only be computed numerically, and is not representable as a closed-form system of Seemingly Unrelated Regression equations, has been problematic. This creates fundamental problems of model specification for the investigator under standard econometric practices.

Second, and more importantly, the fact that certain assets are *not* selected in the investor's optimal portfolio causes serious difficulties.[1] The traditional econometric approach to accommodating the presence of non-negativity constraints is to formulate an underlying *stochastic* Limited Dependent Variable model, in which there is an accumulation of probability mass at all boundary values−each requiring the evaluation of multiple numerical integrals [Maddala (1983)]. In the present generic case, not only are such models extremely difficult to formulate in a logically consistent manner; but, even if possible, suffer from three remaining fundamental flaws:

1. the functional form of each of the asset equations is *ad hoc*, in the sense that, while the domain restrictions on the dependent variable, in principle, may be accommodated, the *functional form* of the asset demand relationships are not "derived" from the posited underlying constrained optimization problem;

2. the mechanism by which infeasible values of the "latent dependent variables" are mapped back onto the boundary of the feasible decision space is *ad hoc*; and

3. even if the issues in the above can be resolved satisfactorily, present computer limitations [see Hausman and Wise (1978) or Quandt (1983)] restrict the analysis to no more than four multiple numerical integrations. Unfortunately, any realistic analysis of portfolio selection would encompass substantially more than four risky assets.

In order to circumvent the above problems, we have formulated a deterministic version of the portfolio selection problem.[2] Specifically, the use of deterministic Neoclassical Econometric Methods advanced in Hartley (1986, 1988, 1994) permits the calibration of the structural parameters in a specified constrained optimization model in cases where the optimal solution, relative to any feasible parameter vector, can only be computed numerically. This contrasts with the requirement of closed-form analytic solution for the optimal portfolio under traditional econometric practice. The occurrence of "corner solutions," which arise when one or more assets are added to or removed from the prevailing portfolio in the course of the implementation of the proposed algorithm causes the elements in the prevailing Jacobian matrix of partial derivatives of the currently predicted optimal portfolio with respect to the current structural parameters to exhibit "jump discontinuities" in the neighborhood of a corner solution. Such cases frequently arise, and avoiding these problems in the context of our algorithm requires various possible "smoothing" techniques [Hardle (1990)]. Moreover, the fact that the model formulation is deterministic implies that we do not have to concern ourselves with the accumulation of probability mass that would otherwise be associated with all boundary solutions for the portfolio problem, whether corner or not.

Only some special cases of the inverse of the portfolio selection problem have been previously examined in the asset pricing literature. For instance, take as an example the Euler equation tests of the consumption-based asset pricing model [Bakshi and Chen (1996), Hansen-Singleton (1982) and Ferson (1994)]. Using the generalized methods of moments of Hansen (1982), the "taste parameters" in the representative agent's utility function and his subjective discount factor have been inferred using aggregate consumption and asset pricing data. In addition, Mehra and Prescott (1985) and Constantinides (1990), among many others, calibrated the mean and the standard deviation of the aggregate consumption process and observed that aggregate consumption is "too smooth" to fit the aggregate market risk premium and the risk-free real interest rate without resorting to implausibly high values of the risk aversion parameter. In the context of an intertemporal asset pricing model, Merton (1980), Bakshi and Chen (1994), and Harvey (1991) calculate the reward-to-risk ratio to infer the economy-wide risk aversion.

The simulated moments estimators of Duffie and Singleton (1993) and McFadden (1989) are important contributions, and are appealing on the grounds that the methodology jointly determines the unknown parameters driving the time-homogeneous stochastic forcing process as well as the structural parameters in the agents utility function. However, in the case of portfolio selection, a drawback of the simulated moments estimator is that it cannot avoid the "curse of dimensionality". Furthermore, as demonstrated in Section 4, the calibration methods are computationally much simpler and the proposed quasi-Newton algorithm is able to accommodate the corner solutions and the associated discontinuities which are likely to be troublesome in simulated moment estimators.

The rest of the paper proceeds as follows. Section 2 outlines the basic features of the Markowitz portfolio selection problem. Section 3 introduces a simple autoregressive model as the basis for the stochastic returns generating process; and permits the parameters to vary with the set of investor characteristics. Thus, in any period in which investment decision is called for, we may accommodate either homogeneous or heterogeneous expected rates of return across all investors, along with a homogeneous covariance matrix. Section 4 presents the Neoclassical Econometric Methods applicable to the inverse of the portfolio selection problem for given structural parameters in the returns-generating process obtained *ex ante* from historical data. In this case, only the structural parameters in the utility function require determination. This section also illustrates how the unknown efficiency frontier can be determined using numerical methods and offers an analytically tractable procedure to evaluate the Jacobian matrix of partial derivatives that serves as a basic input to the modified Newton algorithm [Hartley (1961)]. Section 5 tackles the more demanding task of calibrating the structural parameters of the investor's utility function and the returns-generating process, as reflected ex post in the actual portfolios of investors. We explain, in Section 6, how the proposed framework extends in a simple way to the Black (1972) and Tobin (1965) portfolio selection models. Concluding remarks are offered in Section 7. Mathematical details relating to the algorithm are sketched in Appendices A, B, C.

II The Markowitz Model of Portfolio Selection

This section is devoted to the Markowitz theory of portfolio selection for a set of risky assets.[3] Markowitz postulated that each investor generates a vector of expected returns, denoted μ, and an associated covariance matrix, Σ, for the set of risky assets under consideration. Given these inputs, he assumed that a risk-averse investor will trade-off between the portfolio expected rate of return and risk as embodied in the utility function. Consequently, the optimal Markowitz portfolio may be computed from an optimization problem in which the mean-variance utility function is maximized, subject to the constraint that all prospective portfolio's lie on an "efficiency frontier". However, the latter is defined *pointwise* from a set of solutions to a series of quadratic programming problems in which, for each specified rate of return, the variance of the portfolio is minimized, subject to the conditions that:

- the proportions invested in each risky asset are proper weights, that is, are non-negative and sum to unity.
- the rate of return on a feasible portfolio equals a specified value, ranging from the smallest to the largest rate of return on all individual assets.

Specifically, let $\pi \equiv \pi_a$, for $a = 1 \cdots A$ denote the vector of portfolio weights for any risky asset portfolio for a given investor. The expected rate on any portfolio, ϕ, is therefore given by

$$\phi = \pi'\mu = \sum_{a=1}^{A} \pi_a \mu_a \tag{1}$$

and the standard deviation of the portfolio is defined as

$$\omega = (\pi'\Sigma\pi)^{\frac{1}{2}} = \left[\sum_{a=1}^{A} \sum_{a\prime=1}^{A} \pi_a.\sigma_{aa'}.\pi_{a'} \right]^{\frac{1}{2}} \tag{2}$$

for any portfolio, π. These have an admissible domain,

$$\Pi = \{\pi : \pi \geq 0 \ \text{ and } \ \imath'\pi = 1\} \tag{3}$$

where $\imath$ denotes the unit vector. Hence, the elements, π_a of π are proper weights.

From mean-variance mathematics referred to, among many others, in Merton (1972) and Roll (1977), it is clear that the image of Π under the transformations, equations (1) and (2), to (ϕ, ω)-space is an "umbrella-shaped" region, which we can write as:

$$F^M = \left\{ (\phi, \omega) \equiv (\pi'\mu, (\pi'\Sigma\pi)^{\frac{1}{2}} \ ) : \pi \in \Pi \right\}.$$

The "northwest" boundary function,

$$b^M(\phi, \omega) \equiv b^M \left(\pi'\mu, (\pi'\Sigma\pi)^{\frac{1}{2}} \right) = 0, \tag{4}$$

of F^M is a *piecewise-continuous*, concave function or concave spline in (ϕ, ω)-space, and defines the set of all efficient portfolios of risky assets for a risk-averse investor, where the "join-points" arise when one or more assets are either introduced or deleted from the prevailing portfolio.

Following Markowitz (1959), one can represent the arbitrary investors utility function in risk-return space or in terms of the portfolio space as follows:[4]

$$u(\phi,\omega;x,\xi) = u\left(\pi'\mu,\ \ (\pi'\Sigma\pi)^{\frac{1}{2}};x,\xi\right), \tag{5}$$

where we have introduced the K-element vector, x, of the investor's socio-economic characteristics to capture differences in risk aversion, age, household income, size of investment fund, sex, wealth, and so on. This allows us to incorporate heterogeneity in an individual or institutional investor population. Accordingly, ξ is a P-element vector of unknown "taste parameters" which is to be determined through our calibration exercises in the next sections.

The portfolio selection can now be stated as a nonlinear programming problem in risk-return or portfolio space:

$$\max_{\pi}\ \ u(\phi,\omega;x,\xi), \quad \text{s.t.}\ \ b^M(\phi,\omega) = 0, \tag{6}$$

where as previously mentioned, $b^M = 0$ denotes the *unknown* efficiency frontier to be determined pointwise and embodying the constraints, $\pi \geq 0$ and $\imath'\pi = 1$.

As is well known, "efficiency" for any portfolio with a specified rate of return, say ϕ^*, is defined as that portfolio which minimizes the variance, subject to the conditions that π represents a set of proper portfolio weights. Thus, an associated portfolio satisfying $b^M = 0$ obtains as the solution to the quadratic programming problem:[5]

$$\min_{\pi}\ \ \frac{1}{2}\pi'\Sigma\pi \quad \text{s.t.}\ \ \imath'\pi = 1,\ \ \mu'\pi = \phi^*, \quad \pi \geq 0. \tag{7}$$

Thus, the solution associated with a specific value of ϕ^* defines an efficient portfolio, $\pi^* = \pi(\phi^*)$, which, in turn, defines the associated mean, $\pi^{*\prime}\mu$, and the standard deviation, $\omega^* = (\pi^{*\prime}\Sigma\pi^*)^{\frac{1}{2}}$ of this efficient portfolio.

Let $\phi_{min}= \min_a\{\mu_a\}$ denote the smallest element of μ and analogously let ϕ_{max} denote the largest element. Then, by spanning the range $[\phi_{min}, \phi_{max}]$, over a set of successively-finer one-dimensional grids of ϕ- values, for any investor, one can calculate the corresponding efficient portfolios and the standard deviations. Consequently, for any investor, the efficiency frontier, $b^M = 0$, is defined pointwise. Accordingly, by evaluating the utility function in (5) at each point along the efficiency frontier, the optimal Markowitz portfolio, π^{*M}, and the associated optimal expected return, ϕ^{*M}, and the standard deviation, ω^{*M}, may be determined by one-dimensional grid search methods which are explained in detail in Section 4.2 and 5.2.

Observe that in traversing the efficiency frontier over the values $\phi \in [\phi_{min}, \phi_{max}]$, the associated efficient portfolio's, π^*, will be one of the three types:

1. an interior solution, or
2. a boundary, non-corner solution, or
3. a boundary, corner solution.

In the case of interior solutions, the prevailing portfolio includes all of the A risky assets with $\imath'\pi^* = 1$, and $\pi^* > 0$. With a large number of risky assets, this will rarely occur. Instead, a boundary solution, in which certain risky assets are not held, is more typical. Here, again $\imath'\pi^* = 1$; but, upon reordering the assets as $\pi^* \equiv [\pi_1^{*\prime}, {\pi_2}^{*\prime}]'$, we find that $\pi_1^* > 0$, but $\pi_2^* = 0$-thus defining a boundary solution portfolio. Of these, we may encounter a boundary corner portfolio, in which one or more risky assets have been deleted from or added to the neighboring portfolios associated with a slightly smaller or larger rate of return than ϕ^*.

Obviously, the optimal Markowitz portfolio will also be one of the three types noted above. While this causes no particular difficulty in calculating π^{*M}, the optimal Markowitz portfolio in the forward solution, it poses some difficulties in the inverse problem. In particular, the cases (1) and (2) represent "tangency solutions," π^{*M}, relative to either the universe of risky assets, A, or the smaller dimensional problem involving $A_1 < A$ risky assets in π_1^{*M}. However, the case of a corner solution in (3) gives rise to "jump-discontinuities" in the associated Jacobian matrix of partial derivatives of the prevailing portfolio with respect to the structural parameters, as shown in Section 4, requiring the use of "smoothing techniques" to implement the portfolio selection algorithm.

III Inputs to the Markowitz Model: The Returns-Generating Process

Before discussing the algorithm for the inverse selection problem, this section considers a simple prototype model for the underlying stochastic returns generating process which embodies both homogeneous and heterogenous expected rate of return. However, in both cases, on grounds of parsimony, we retain a homogenous covariance matrix across individuals in any investment period.

A A Homogeneous Returns Generating Process

Let L denote the maximum number of lagged time-periods of actual data considered by each investor in assessing future expected rate of return and the associated covariance matrix. It follows that the observed rate of return for any asset "a", $a = 1, \cdots, A$, during time-t is calculable as:

$$r_{a,t} = \frac{p_{a,t} + d_{a,t} - p_{a,t-1}}{p_{a,t-1}}, \quad \forall a = 1 \cdots A; \quad \forall t = -L+1, \cdots, 1, \cdots T,$$

where d_t and p_t, respectively, denotes the time-t dividends and prices for the risky asset "a".

Consider first a simple constant-coefficient autoregressive model for the one-period ahead anticipated rate of return for each investor:[6]

$$r_{i,a,t} = \alpha_{0,a} + \sum_{l=1}^{L} \alpha_{l,a}.r_{i,a,t-l} + e_{i,a,t}; \quad a = 1, \cdots, A; \quad t = 1, \cdots, T. \tag{8}$$

Note that the "systematic" part on the right-hand-side of equation (8) is independent of the socio-economic characteristics of the individual investor; and thus defines a set of homogenous return-generating process where the covariance matrix associated with $\{e_{i,a,t}\}$, denoted Σ, is a constant over all investors and time periods. Further assume that, $e_{i,t} \equiv (e_{i,a,t})$ which denotes the $A \times 1$ vector of random disturbances follows the probability density function:

$$e_{i,t} \sim NID(0, \Sigma). \tag{9}$$

In this case equation (9) defines a linear Vector Autoregressive (VAR) process for the vector of anticipated rates of return, $r_{i,t} \equiv (r_{i,a,t})$, involving A.(L+1) autoregressive parameters, $\{\alpha_{l,a} : l = 0, 1, \cdots, L; a = 1, \cdots, A\}$.[7] [8] This implies that we may obtain consistent and asymptotically normal (CAN) estimates, $\{\hat{\alpha}_{l,a}\}$, from which we may calculate the homogeneous expected rates of return for each risky asset, as,[9]

$$\hat{\mu}_{i,a,t} = \hat{\alpha}_{0,a} + \sum_{l=1}^{L} \hat{\alpha}_{l,a}.r_{i,a,t-l}; \quad a = 1, \cdots, A, \quad t = 1, \cdots, T. \tag{10}$$

This model can be updated for each investment period-t; and results in the sequence, $\{\hat{\mu}_{i,a,t} : i = 1, \cdots, N; a = 1, \cdots A; \ \ t = 1, \cdots T\}$, of expected rates of return that are identical across each investor. Finally, the $A \times A$ covariance matrix, Σ can be estimated by its sample analog,

$$\hat{\Sigma} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (r_{i,t} - \hat{\mu}_{i,t})(r_{i,t} - \hat{\mu}_{i,t})',$$

provided that $NT > A$.

Observe that the homogeneous inputs imply that, in any period, all investors have identical, ex ante beliefs regarding the expectations and the covariance matrix of future anticipated rates of return on all risky assets. Were this be so, it is difficult to imagine a viable market for *any* risky asset. This requires, at a minimum, different expected rates of returns, in which some investors expect the expected returns on a given risky asset to be lower than currently prevailing, while others-presumably with the same information, expect them to be higher in the future. This motivates our discussion of the heterogeneous returns generating process in the next section.

B A Heterogeneous Returns Generating Process

This section discusses the econometric implementation of a heterogeneous returns generating process which is more characteristic of the diversity in investors beliefs and of portfolio selection in the "real world".[10]

The generic approach is to utilize the vector of observable investor characteristic, $x_{i,t}$, to generate a parsimoniously parameterized, investor-specific, return-generating process for the anticipated rates of return for each investor, $i = 1, \cdots, N$, in each time period. We write this below as:

$$r_{i,a,t} = \alpha_{0,a}(x_{i,t}; \gamma_0) + \sum_{l=1}^{L} \alpha_{l,a}(x_{i,t}; \gamma_i).r_{i,a,t-l} + e_{i,a,t}; \quad a = 1, \cdots, A; \quad t = 1, \cdots, T. \tag{11}$$

where again, $e_{i,t} \equiv (e_{i,a,t})$ is a normal random disturbance with p.d.f given in equation (9). Here, both the intercept and the L autoregressive "slope coefficients" vary with the vector of socio-economic characteristics, $x_{i,t}$. One plausible specification is to define the $\{\alpha_{l,a}\}$ as linear function of the $x_{i,t}$ values, that is,

$$\alpha_{l,a}(x_{i,t}; \gamma_i) = \alpha_{i,0,l} + \sum_{k=1}^{K} \gamma_{i,k,l}.x_{i,k,t}$$

separately for each investor, $i = 1, \cdots, N$, with $l = 0, \cdots, L$ and $a = 1, \cdots, A$. This formulation yields the following investor- specific returns-generating process which will be extensively exploited in our investigation,

$$r_{i,a,t} = \left(\alpha_{i,0,0} + \sum_{k=1}^{K} \gamma_{i,k,0}.x_{i,k,t}\right) + \sum_{l=1}^{L} \left(\gamma_{i,0,l} + \sum_{k=1}^{K} \gamma_{i,k,l}.x_{i,k,t}\right) r_{i,a,t-l} + e_{i,a,t} \tag{12}$$

The above model involves (K+1).(L+1) parameters, and not only captures the investor's "bullish" or "bearish" anticipations of future asset prices relative to the current state of the market, as manifest in the intercept of equation (12); but also permits each investor to place different "weights" $\sum_{l=1}^{L}\left(\gamma_{i,0,l}+\sum_{k=1}^{K}\gamma_{i,k,l}.x_{i,k,t}\right)$, on the preceding rates of return. Thus, if feasible on grounds of dimensionality, we would fit the model in equation (11) jointly to all risky assets. This again involves formulating a linear VAR model for estimating the parameters, $\{\gamma_{i,k,l}: \ a=1,\cdots,A; \ k=0,1,\cdots,K; \ l=1,\cdots,L\}$, for each investor. The resulting heterogeneous returns-generating process may then be defined as:[11]

$$\hat{\mu}_{i,a,t}=\left(\hat{\alpha}_{i,0,0}+\sum_{k=1}^{K}\hat{\gamma}_{i,k,0}.x_{i,k,t}\right)+\sum_{l=1}^{L}\left(\hat{\gamma}_{i,0,l}+\sum_{k=1}^{K}\hat{\gamma}_{i,k,l}.x_{i,k,t}\right)r_{i,a,t-l} \tag{13}$$

and applies to each investor, i.

There are clearly innumerable alternative model specifications for the returns generating process. In particular, the empirically attractive single-index model of Sharpe (1963) and the multiple index models, especially Ross (1976), which have been considered in the existing literature on portfolio selection can be analyzed separately, and used in the calibration of portfolio choice.

IV Direct Ex Ante Calibration of Portfolio Choice

Having discussed the portfolio selection problem and the specification of the returns generating process, let us focus our attention on the inverse selection problem. Taking parametrically, the time series of actual portfolio's for a sample of investors, the returns generating process for the relevant set of risky assets, and the set of socio-economic characteristics of each investor; one can determine the parameter values in each investor's utility function and the associated parameters in the returns generating process based upon their reflections in the *actual* portfolio choices of investors. Having determined these structural parameters, the optimal portfolios can computed using the techniques described in Section 2.[12] The goal in this section is, therefore, to discuss the econometric methods associated with *ex-ante* calibration of the "taste parameter" within the utility function of a Markowitz model based upon actual portfolio decisions of investors.[13]

A The Quasi-Newton Algorithm

Recall that the heterogeneous autoregressive model for the anticipated rates of return outlined in equation (11) involves estimating the parameters $\{\hat{\gamma}_{i,k,l} : i = 1, \cdots, N;\ k = 1, \cdots, K;\ l = 0, 1, \cdots, L\}$, which can be denoted $(K+1).(L+1).N \times 1$ element vector, $\hat{\gamma}$. Given $\hat{\gamma}$, we can define, for each investor, the expected rate of return vectors, $\{\hat{\mu}_{i,t} : i = 1, \cdots, N\}$, and the estimated covariance matrix, $\hat{\Sigma}$ for each time-period, $\{t = 1, \cdots, T\}$. These serve as inputs to the portfolio selection problem; and, in turn, permit the computation of the optimal Markowitz portfolios, $\{\pi^{*M}_{i,t} : i = 1, \cdots, N,\ t = 1, \cdots, T\}$, via the methods of Section 2. Thus, each investor at time$-t$ will confront a different efficiency frontier; and in the *ex- ante case*, the only remaining problem is to calibrate the vector of "taste parameters", ξ in the investors utility function, $u(\pi'\mu,\ (\pi'\Sigma\pi)^{\frac{1}{2}}; x, \xi)$.

For this purpose, consider the following formulation for the actual portfolio held by investor i at time-t,

$$\pi_{i,t} = \pi^{*M}_{i,t} + v_{i,t} \equiv \pi^{*M}_{i,t}\left(x_{i,t}, \hat{\mu}_{i,t}, \hat{\Sigma}\right) + v_{i,t}; \tag{14}$$

where $\pi_{i,t}$ is an $A \times 1$ vector denoting the actual portfolio held by investor in period-t; and $\pi^{*M}_{i,t}$ denotes the $A \times 1$ vector defining the corresponding "optimum" Markowitz portfolio. The latter is, however, defined relative to the prevailing set of taste parameters, ξ, with given values for the inputs, $\hat{\mu}_{i,t}, \hat{\Sigma}$, and $x_{i,t}$. Here, since $\pi \geq 0$, $\pi^{*M}_{i,t}$ can only be determined numerically, the functional form the right-hand-side will not be known to the investigator, though the variables which affect the numerical value of $\pi_{i,t}^{*M}$ can be specified. Finally, $v_{i,t}$ is defined as an $A-$element vector of discrepancies between the actual value of the i^{th} investor's portfolio, $\pi_{i,t}$, and the predicted value, $\pi_{i,t}^{*M}$, relative to a given value of parameter vector, ξ.

In the present context, we must distinguish between the calibration of the structural parameters, ξ, where the $\{v_{i,t}\}$ are treated as *constant discrepancies*, and the estimation of the "true" parameters, in which $\{v_{i,t}\}$ are regarded as *random disturbances*. Here, the presence of "A" inequality domain restrictions, $\pi \geq 0$, gives rise to A^{th} order multiple integral in implementing a *stochastic* Limited Dependent Variable approach. However, since no more than four multiple numerical integrals can at present be accommodated [Hausman and Wise (1978), Quandt (1983)], we opt for the specified deterministic approach outlined in Hartley (1984, 1986, 1994). This implies that we must forego the customary rituals of statistical inference and hypothesis testing, but in turn, as we show below, may determine the parameter values for a much larger number of assets, "A".

Having defined *constant discrepancies* as the difference between actual investor portfolio's and the prevailing Markowitz portfolio, define the quadratic loss function as:

$$Q^1(\xi) = \frac{1}{2}|\ \Omega^1\ |^{-\frac{1}{TN}} \ \sum_{t=1}^{T}\sum_{i=1}^{N}\left(\pi_{i,t} - \pi_{i,t}^{*M}\right)' . \left[\Omega^1\right]^+ \left(\pi_{i,t} - \pi_{i,t}^{*M}\right), \tag{15}$$

which has to be minimized with respect to, ξ, where $[\Omega^1]^+$ is the $A \times A$ weighting matrix,[14]

$$\Omega^1 = \frac{1}{TN}\sum_{t=1}^{T}\sum_{i=1}^{N}\left(\pi_{i,t} - \pi_{i,t}^{*M}\right) . \left(\pi_{i,t} - \pi_{i,t}^{*M}\right)', \tag{16}$$

representing the Mahalanobis distance criterion which may be obtained by solving $\frac{\partial Q^1}{\partial \Omega^1} = 0$, for Ω^1.

Note from equation (14) that $\imath' \pi_{i,t} = 1 = \imath' \pi^{*M}_{i,t}$, so that $\imath' \Omega^1 \imath = 0$ and the $A \times A$ matrix, Ω, is of rank, $A-1$. Consequently, we must delete one of the risky assets from the model. Without loss of generality, let the deleted risky asset be the A^{th}.[15] According, using the symbol, $\bar{}$, to denote an $(A-1)$- dimensional vector or matrix extracted from an A-dimensional one, we may implicitly define the following $(A-1)$-dimensional expressions:

$$\pi_{i,t} \equiv \left[\bar{\pi}'_{i,t} \mid \pi_{i,t,A}\right]'; \;\; \pi^{*M}_{i,t} \equiv \left[(\bar{\pi}^{*M}_{i,t})' \mid \pi^{*M}_{i,t,A}\right]'; \;\; v_{i,t} \equiv \left[\bar{v}'_{i,t} \mid v_{i,t,A}\right]'; \hat{\mu}_{i,t} \equiv \left[\hat{\bar{\mu}}'_{i,t} \mid \mu_{i,t,A}\right]';$$

Other variables are similarly defined. In this situation, we must definite the basic budget-share model as one involving only the first $A-1$ risky assets as in:

$$\bar{\pi}_{i,t} = \bar{\pi}^{*M}_{i,t} + \bar{v}_{i,t} \equiv \bar{\pi}^{*M}_{i,t}\left(x_{i,t}, \hat{\mu}_{i,t}, \hat{\Sigma}\right) + \bar{v}_{i,t}; \tag{17}$$

and, in the next step, the loss function as:

$$\bar{Q}^1(\xi) = \frac{1}{2} \mid \bar{\Omega}^1 \mid^{-\frac{1}{TN}} \sum_{t=1}^{T} \sum_{i=1}^{N} \left(\bar{\pi}_{i,t} - \bar{\pi}^{*M}_{i,t}\right)' . \left[\bar{\Omega}^1\right]^+ . \left(\bar{\pi}_{i,t} - \bar{\pi}^{*M}_{i,t}\right). \tag{18}$$

Relative to the loss function in equation (18), and holding $\bar{\Omega}^1$, the first order conditions are defined by

$$\frac{\partial \bar{Q}^1}{\partial \xi} = -\sum_{t=1}^{T} \sum_{i=1}^{N} \{\bar{J}^1_{i,t}\}' . \left[\bar{\Omega}^1\right]^+ . \left(\bar{\pi}_{i,t} - \bar{\pi}^{*M}_{i,t}\right) = 0, \tag{19}$$

where $\bar{J}^1_{i,t} \equiv \frac{\partial \bar{\pi}^{*M}}{\partial \xi}$ is an $(A-1) \times P$ Jacobian matrix of partial derivatives with respect to the parameter vector, ξ. Since equation (19) is highly nonlinear in ξ, we must resort to iterative methods to obtain a solution, $\hat{\xi}$, for ξ.

Let $n = 0, 1, \ldots$ denote an iteration index, and consider the standard quasilinearization [Bellman and Roth (1983)] of the optimal portfolio vector, $\bar{\pi}_{i,t}^{*M}$, around the current iterate, $\left[\bar{\pi}_{i,t}^{*M}\right]^n$, that is,

$$\bar{\pi}_{i,t}^{*M} = \left[\bar{\pi}_{i,t}^{*M}\right]^n + \bar{J}_{i,t}^1 \left(\xi - [\xi]^n\right). \tag{20}$$

Then, substituting, equation (20) into (19) with both $\bar{J}_{i,t}^1$ and $\bar{\Omega}^1$ evaluated at $[\xi]^n$, and solution for $\xi = [\xi]^{n+1}$, leads to the *modified* quasi-Newton algorithm [Hartley (1961)],

$$[\xi]^{n+1} = [\xi]^n - [\psi]^n . \left[\bar{H}^1\right]^n . \left[\bar{g}^1\right]^n \tag{21}$$

Here,

$$[\bar{H}^1]^n = \sum_{t=1}^{T} \sum_{i=1}^{N} \left[\{\bar{J}_{i,t}^1\}'\right]^n . \left[\bar{\Omega}^{1n}\right]^+ . \left[\bar{J}_{i,t}^1\right]^n \tag{22}$$

will be referred as the $P \times P$ Hessian matrix under the method of quasilinearization;

$$[\bar{g}^1]^n = -\sum_{t=1}^{T} \sum_{i=1}^{N} \left[\{\bar{J}_{i,t}^1\}'\right]^n . \left[\bar{\Omega}^{1n}\right]^+ . \left(\bar{\pi}_{i,t} - [\bar{\pi}_{i,t}^{*M}]^n\right) \tag{23}$$

denotes the $P \times 1$ gradient vector; and $0 \leq [\psi]^n \leq 1$ denotes a suitable step-size parameter [see Dennis and Schnabel (1983)] chosen to ensure that in each iteration:

1. ξ^{n+1} is feasible.

2. $[\bar{Q}^1]^{n+1} \leq [\bar{Q}^1]^n$ which ensures monotonicity.

Given $[\xi]^n, [H^1]^n$ and $[g^1]^n$, along with an initial value of $[\psi]^n = 1$, equation (21) determines the "full-step" parameter value of $[\xi]^{n+1}$. In particular, a line-search algorithm is then employed-first to establish feasibility; and then to establish monotonicity within any iteration.[16] Finally, assuming that we can determine the elements in the Jacobian matrix, $[\bar{J}^1_{i,t}]^n$, in each iteration, n, the quasi-Newton algorithm, described in equation (21), will converge to the limit point,

$$\hat{\xi} = \lim_{n \to \infty} [\xi]^n. \tag{24}$$

To implement the above algorithm, we determine the optimal portfolio, $\bar{\pi}^{*M}$, and seek an expression for the elements of the $(A-1) \times P$ Jacobian sub$-$matrix, $\bar{J}^1_{i,t} \equiv \frac{\partial \bar{\pi}^{*M}}{\partial \xi}$. We discuss these in turn in the next sections. These steps are critical, since in each iteration, $[\bar{J}^1_{i,t}]$, determines the Hessian matrix, $\bar{H}^1$, and the gradient vector, $\bar{g}^1$.

B Efficiency Frontier Approximations

Our discussion in Section 2 emphasized that the efficiency frontier in either portfolio or risk-return space is only defined *pointwise* when $\pi_{i,t} \geq 0$. In particular, such points $\pi^{*M}_{i,t}$ in portfolio space or $(\phi^*_{i,t}, \omega^*_{i,t})$ in risk-return space, lie on the unknown efficient boundary function

$$b^M_{i,t}(\phi^*_{i,t}, \omega^*_{i,t}) \equiv b^M_{i,t}\left(\pi^{*\prime}_{i,t}\hat{\mu}_{i,t},\ \ (\pi^{*\prime}_{i,t}\hat{\Sigma}\pi^*_{i,t})^{\frac{1}{2}}\right) = 0.$$

These are obtained by solving the quadratic programming problem, in equation (7), over a succession of finer grids, each being (say) one tenth of the grid size, G, of the former. More specifically, we begin by evaluating the utility function,

$$u_{i,t}^{*} \equiv u\left(\phi_{i,t}^{*}, \omega_{i,t}^{*}\right) \equiv u\left(\pi_{i,t}^{*}{}'\hat{\mu}_{i,t},\ \ (\pi_{i,t}^{*}{}'\hat{\Sigma}\pi_{i,t}^{*})^{\frac{1}{2}}; x_{i,t}, \xi\right), \tag{25}$$

relative to the prevailing value of ξ, over the initial grid $[\phi_{min}, \phi_{min}+\delta_0, \phi_{min}+2.\delta_0, \cdots, \phi_{max}-\delta_0, \phi_{max}]$, involving (say) eleven ϕ^* values, where, in the initial grid, $\delta_0 = \frac{\phi_{max}-\phi_{min}}{10}$. This defines an initial "optimal" utility function value, $u_{i,t}^{*1}$, at $\phi_{i,t}^{*1}$, with associated values for $\omega_{i,t}^{*1}$ and $\pi_{i,t}^{*1}$.

The values of $\phi_{i,t}^{*1}$ and δ_0 define our initial grid conditions. Let $m = 1, \cdots, M$ define the index number of each successive grid search and let $\delta_m = \frac{\delta_{m-1}}{G}$ define the density of each such grid. Then we may define the points within arbitrary m^{th} grid as follows:

$$\left[\phi_{i,t}^{*m} - \delta_{m-1} + \delta_m, \phi_{i,t}^{*m} - \delta_{m-1} + 2.\delta_m, \ldots, \phi_{i,t}^{*m}, \ldots, \delta_{m-1} - 2.\delta_m, \phi_{i,t}^{*m} - \delta_{m-1} - \delta_m\right].$$

Thus, for example, the choice of $G = 10$ would result in each successive grid size being one-tenth of preceding one and bounded by the points, $\left[\phi_{i,t}^{*m} - \delta_{m-1}, \phi_{i,t}^{*m} + \delta_{m-1}\right]$, which have already been evaluated. The successive evaluation of these grids stops when $\phi_{i,t}^{M}$ has been determined with pre-specified accuracy, and, in turn, permits the evaluation of the associated values of $u_{i,t}^{*M}$, $\omega_{i,t}^{*M}$ and $\pi_{i,t}^{*M}$, the Markowitz portfolio.

For subsequent purposes, the succession of all grid-values then should be ranked ordered from, ϕ_{min} to ϕ_{max}. Thus, for each value of the expected rate of return, $\phi_{i,t}^{*}$, employed in the preceding succession of grid searches, these may be associated tabularly with the corresponding values of the standard deviation, $\omega_{i,t}^{*}$, the value of the maximum utility level, $u_{i,t}^{*}$, and the "optimal" portfolio vector, $\pi_{i,t}^{*}$. We note that the design of the grid search we have advocated "bunches" many evaluation points around the $\phi_{i,t}^{*M}$ of interest. Two criteria are relevant to the choice of M and G in the succession of grids:

1. The grid-size, G, must be small enough to determine $\pi_{i,t}^{*M}$ to a pre-specified level of accuracy,

2. the succession of grids, M, must be sufficiently dense around $\pi_{i,t}^{*M}$ that is possible to fit a polynomial of suitable order [cases (i) and (ii) below] or a "polynomial spline" [case (iii) below] to the expected rates of return and standard deviations, $(\phi_{i,t}^{*}, \omega_{i,t}^{*})$ in the neighborhood of the point, $(\phi_{i,t}^{*M}, \omega_{i,t}^{*M})$, which determines the optimal Markowitz portfolio, $\pi_{i,t}^{*M}$.

As we show next and in the Appendixes A and B, these points will prove helpful in evaluating the elements of the Jacobian matrix. Our procedure for the optimal portfolio selection can be contrasted with the algorithm in Lewis (1988) which maximizes a concave utility function but only with linear constraints. Alexander (1976, 1977) and Elton-Gruber (1991) also discuss quadratic programming algorithms applicable to the Markowitz (1959) and related models.[17]

Suppose, for instance, we have completed the aforementioned grid search to determine the optimal Markowitz portfolio. Then, a simple inspection of the optimal Markowitz portfolio would immediately reveal to the investigator, whether or not we have an interior optimum, i.e., $\pi_{i,t}^{*M} > 0$, or, alternatively whether a boundary portfolio, $\pi_{1i,t}^{*M} > 0$ and $\pi_{2i,t}^{*M} = 0$, has been obtained; but not immediately inform the investigator whether or not the prevailing optimal boundary portfolio is a non- corner or a corner solution. This, however, requires that we inspect the composition of the closest adjacent portfolios associated with the values, $\phi^{*M} - \delta_M$ and $\phi^{*M} + \delta_M$, to reveal whether one or more risky assets have been introduced or deleted relative to the optimal portfolio, $\pi_{i,t}^{*M}$. Now we investigate each of the possible cases for the prevailing portfolio and the associated boundary function approximations in turn.[18]

i. Interior Solutions: Suppose we have an interior solution for $\pi_{i,t}^{*M}$, involving all the risky assets. In the next step, inspect all of the adjacent portfolios, $\pi_{i,t}^{*} > 0$, in the neighborhood of $\pi_{i,t}^{*M}$ to identify which of them are also interior points. This will include portfolios from each of the current (m=M) and preceding ($m = 1, \cdots, M-1$) grids that have already been rank ordered by their expected rate of return, $\phi_{i,t}^{*}$. Let $\phi_{i,t}^{min}$, and $\phi_{i,t}^{max}$, respectively, denote the smallest and largest expected rate of return associated with an interior portfolio. Suppose that this includes a total of $S_1 + 1$ interior points-one of which is $\phi_{i,t}^{*M}$. Then over the range, $\left[\phi_{i,t}^{min}, \phi_{i,t}^{max}\right]$, we have the set of efficient boundary points, $\{(\phi_{i,t}^{s}, \omega_{i,t}^{s}) : s = 0, 1, \cdots, S_1\}$ in risk-return space, and we can approximate the unknown efficient boundary frontier by an interpolation polynomial of degree S_1. This algorithm permits us to replace the unknown efficient frontier, $b^M(\phi_{i,t}^{*}, \omega_{i,t}^{*}) = 0$, by a known polynomial approximation that passes through each of the $S_1 + 1$ grid points and is twice continuously differentiable. For further details refer to Appendix A.

ii. Boundary Non-Corner Solution: Consider, next, the case of boundary non-corner solution, $\phi_{i,t}^{*M}$, resulting from a succession of grid searches. Inspection of the composition of $\phi_{i,t}^{*M}$, upon reordering the elements, reveals that the A_1-element sub-vector $(A_1 < A)$ $\phi_{1,i,t}^{*M} > 0$, whereas the A_2-element sub-vector, $\phi_{2,i,t}^{*M} = 0$, with $A_1 + A_2 = A$. We may therefore proceed exactly as before and approximate the unknown implicit efficient boundary frontier by an interpolation polynomial of degree S_2. Again, for further details, refer to Appendix A.

iii. Boundary Corner Solution: It remains to consider the case where $\phi_{i,t}^{*M}$ is a boundary *corner* solution. In this case, the two portfolios associated with $\phi_{i,t}^{*M} - \delta_M$ and $\phi_{i,t}^{*M} + \delta_M$ will be of different types, reflecting the fact that one or more assets have been deleted from and/or added to the portfolio $\phi_{i,t}^{*M}$, associated with the expected rate of return, $\phi_{i,t}^{*M}$.[19] It follows that a *different* polynomial approximation function to the efficient boundary function, $b_{i,t}^{M}(\phi_{i,t}^{*}, \omega_{i,t}^{*}) = 0$, must be fit to the common portfolio types that have expected rates of return that are greater or equal to $\phi_{i,t}^{*M}$, as opposed to those with expected returns that are less than or equal to $\phi_{i,t}^{*M}$. Moreover, the two approximation functions must be coincident at the common "joint point," $(\phi_{i,t}^{*M}, \omega_{i,t}^{*M})$. Thus, in the case of the corner solution, we assume that there are S_{31} points with smaller ϕ-values than $\phi_{i,t}^{*M}$ and S_{32} relevant grid points greater than $\phi_{i,t}^{*M}$- plus the inclusion of $\phi_{i,t}^{*M}$, itself, in each.

In short, at this juncture, the approximation function is a concave spline with a joint-point at $(\phi_{i,t}^{*M}, \omega_{i,t}^{*M})$, where the "left-hand" derivative of the approximation function associated with smaller expected rates of return evaluated at the value, $\phi_{i,t}^{*M}$, will be larger than the "right-hand" derivative of the corresponding portfolio with larger expected rates of return-see Figure 1. This results in "jump-continuities" in the derivative of the original function and the approximating polynomial function to the left and the right of the value, $\phi_{i,t}^{*M}$, at which point the derivative is not defined. Our approach to this dilemma is to replace the approximating polynomials in the immediate vicinity of $(\phi_{i,t}^{*M}, \omega_{i,t}^{*M})$ by a circle of sufficiently small radius in $(\phi_{i,t}, \omega_{i,t})$-space, and to evaluate the requisite derivative relative to $\phi_{i,t}^{*M}$ as the slope of the "circular approximant"-see Appendix A for details.

C Evaluation of the Jacobian Matrix

The burden of the preceding subsection has been to argue that in the neighborhood of the optimal portfolio, $\phi_{i,t}^{*M}$, the unknown efficient frontier, $b^{M}(\phi_{i,t}^{*}, \omega_{i,t}^{*}) = 0$, may be approximated by an analytic, twice-continuously-differentiable function, $b^{*M}(\phi^*_{i,t}, \omega^*_{i,t}) = 0$, regardless of which type of portfolio is encountered. Thus at this junction, we may exploit the fact that both the utility function, $u_{i,t}$, and the boundary function, b^{*M}, are known differentiable functions.

Since the optimal Markowitz portfolio, $\pi^{*M}_{i,t}$, can only be computed by numerical methods, it may be tempting to conclude that its partial derivatives, $J^1_{i,t} \equiv \frac{\partial \pi^{*M}_{i,t}}{\partial \xi}$, must also be computed by numerical approximation. This would involve perturbing each element of ξ in turn by a small $\epsilon > 0$; and using finite-difference approximations, relative to the prevailing value of ξ-see, for example, Quandt (1983)-to evaluate each of the partial derivatives required in $J^1_{i,t}$. This would entail repeating the same steps required to calculate $\pi^{*M}_{i,t}$ a further $p = 1, \cdots, P$ times, as each parameter in ξ is successively perturbed by ϵ and the optimal portfolio is recomputed. In the context of our algorithm, therefore, it is of considerable importance to note that, even though $\pi^{*M}_{i,t}$ must be computed by numerical methods, once $\pi^{*M}_{i,t}$ is known, its Jacobian matrix may be calculated directly using a closed-form expression. Since this must be done for each (i,t) in the sample over the iterations, $n = 0, 1, \cdots$, this saves vast amounts of computer time.

Given our twice continuously differentiable approximation, $b^{*M}\left(\phi^*_{i,t}, \omega^*_{i,t}\right) = 0$, to the efficient boundary function in the neighborhood of $\pi^{*M}_{i,t}$, to derive this expression, recast the utility maximization problem as:

$$\max_{\pi} \ u\left(\pi'_{i,t}\hat{\mu}_{i,t}, (\pi'_{i,t}\hat{\Sigma}_t\pi_{i,t})^{\frac{1}{2}}; x_{i,t}, \xi\right), \quad \text{s.t.} \quad b^{*M}_{i,t}\left(\pi'_{i,t}\hat{\mu}_{i,t}, (\pi'_{i,t}\hat{\Sigma}_t\pi_{i,t})^{\frac{1}{2}}\right) = 0, \qquad (26)$$

Formulate a Lagrange function and denote the Lagrange multiplier as $\lambda_{i,t}$. Elementary total differentiation of the two first order conditions with respect to $\pi_{i,t}$, $\lambda_{i,t}$, ξ, with all the partials evaluated at ξ^n, leads to the following set of equations,

$$\left[J_{i,t}^{1*}\right]^n .dz_{i,t} \equiv \begin{bmatrix} S_{i,t}^{1n} & s_{i,t}^{1n} \\ s_{i,t}^{1n\prime} & 0 \end{bmatrix} . \begin{bmatrix} d\pi_{i,t} \\ d\lambda_{i,t} \end{bmatrix} = \begin{bmatrix} -R_{i,t}^{1n}.d\xi \\ 0 \end{bmatrix}, \tag{27}$$

where $z_{i,t} \equiv \left[\pi'_{i,t} \mid \lambda_{i,t}\right]'$; and the expression of $S_{i,t}^1$, $s_{i,t}^1$ and $R_{i,t}^1$ are quite complicated, and relegated to Appendix B. It follows that using the formula for partitioned inverses [see, for example, Rao (1973, p. 73)], we may solve equation (27) for a closed-form expression for the full Jacobian matrix,

$$[J_{i,t}^1]^n \equiv \frac{\partial[\pi_{i,t}^{*M}]^n}{\partial\xi} = -[R_{i,t}^1]^n \left[\{S_{i,t}^{1n}\}^{-1} - \frac{\{S_{i,t}^{1n}\}^{-1}.s_{i,t}^{1n}.s_{i,t}^{1n\prime}.\{S_{i,t}^{1n}\}^{-1}}{s_{i,t}^{1n\prime}.\{S_{i,t}^{1n}\}^{-1}.s_{i,t}^{1n}}\right] \tag{28}$$

Thus the desired sub-matrix $\bar{J}_{i,t}^{1n}$ can be extracted from $J_{i,t}^{1n}$, and inserted into the algorithm in equation (21).

V Indirect Ex Post Calibration of Portfolio Choice

Tests based on predictions about investor portfolio holdings can provide powerful tests of asset pricing theories. The previous section described one such econometric method by which parameters in the investor's utility function can be determined on the basis of their reflections in the actual portfolio decisions of investors. It is, however, arguable as to whether investors' expectations of future rates of returns are best modelled as manifestations of the preceding, historical behavior of actual prices; or whether such investor expectations are better reflected in the actual portfolio decisions that investors make. Accordingly in this section, we generalize the econometric problem to the case in which *both* the parameters in the investors utility function *and* those in the model adopted for the expected rates of return are simultaneously determined, *ex post*, from the actual portfolios selected by investors.

Specifically, in this section, we generalize our calibration methods to include not only the $P \times 1$ vector of taste parameters, ξ in the investors utility function, but also include the $(K+1).(L+1).N$ parameters in the variable-coefficients autoregressive models that surrogates for heterogeneity in the investor population. However, it is important to emphasize that the proposed calibration methodology is general enough to accommodate any reasonably parameterized heterogeneous forcing process for asset returns. We proceed as before in three steps; describing first the quasi-Newton algorithm, and then characterizing the optimal Markowitz portfolio relative to the prevailing parameter values and the associated boundary function approximation using numerical methods, and finally evaluating the elements of the Jacobian matrix.

A The Quasi-Newton Algorithm

Let the $\hat{\mu}_{i,t}(\gamma_i) \equiv (\hat{\mu}_{i,a,t}(\gamma_i))$ denote the A-element vector of expected rates of return for investor i in period-t. It has previously been noted that the parameters, $\{\gamma_{i,k,l} : k = 0, 1, \cdots, K;\ l = 0, 1, \cdots, L\}$, are independent of the particular asset "a" and the time period-t. Consequently, let

$$\gamma_i = [\gamma_{i,0,0}, \gamma_{i,1,0}, \cdots, \gamma_{i,K,0} \mid \gamma_{i,0,1}, \gamma_{i,1,1}, \cdots, \gamma_{i,K,1} \mid \cdots \mid \gamma_{i,0,L}, \gamma_{i,1,L}, \cdots, \gamma_{i,K,L}]', \qquad (29)$$

denote the $(K+1).(L+1)$ autoregressive parameters applicable to the i^{th} investor. Furthermore let $\theta \equiv [\xi' \mid \gamma_1', \cdots, \gamma_i', \cdots, \gamma_N']'$ denote the $B \equiv P + (K+1).(L+1).N$ parameters in the complete model, and note that for any investor i and time-period t, the portfolio model contains only the parameters, ξ and γ_i-the former common to each observation in the sample. Thus, we have the complete model which can be written, as before as:

$$\pi_{i,t} = \pi_{i,t}^{*M} + v_{i,t} \equiv \pi_{i,t}^{*M}\left(x_{i,t}, \hat{\mu}_{i,t}(\gamma_i), \hat{\Sigma}; \xi\right) + v_{i,t}, \tag{30}$$

and define the quadratic loss function as:

$$Q^2(\theta) = \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}\left(\pi_{i,t} - \pi_{i,t}^{*M}\right)' . \left[\Omega^2\right]^{+}\left(\pi_{i,t} - \pi_{i,t}^{*M}\right), \tag{31}$$

which now has to be minimized with respect to, θ, where Ω^2 is the $A \times A$ weighting matrix,

$$\Omega^2 = \frac{1}{TN}\sum_{t=1}^{T}\sum_{i=1}^{N}\left(\pi_{i,t} - \pi_{i,t}^{*M}\right) . \left(\pi_{i,t} - \pi_{i,t}^{*M}\right)'. \tag{32}$$

Again, using the same arguments as before, Ω^2 must be of rank $A-1$, so one of the risky assets must be deleted-say the A^{th}. Accordingly, this results in the modified model,

$$\bar{\pi}_{i,t} = \bar{\pi}_{i,t}^{*M} + \bar{v}_{i,t} \equiv \bar{\pi}_{i,t}^{*M}\left(x_{i,t}, \hat{\mu}_{i,t}(\gamma_i), \hat{\Sigma}; \xi\right) + \bar{v}_{i,t}, \tag{33}$$

Form a quadratic loss function, $\bar{Q}^2$ analogous to $\bar{Q}^1$, and consider the first-order conditions with respect to the parameter vector θ,

$$\frac{\partial \bar{Q}^2}{\partial \theta} = -\sum_{t=1}^{T}\sum_{i=1}^{N}\{\bar{J}_{i,t}^2\}' . \left[\bar{\Omega}^2\right]^{+} . \left(\bar{\pi}_{i,t} - \bar{\pi}_{i,t}^{*M}\right) = 0, \tag{34}$$

where now $\bar{J}_{i,t}^2 \equiv \frac{\partial \bar{\pi}^{*M}}{\partial \theta}$ is an $(A-1) \times B$ Jacobian matrix of partial derivatives. Following the same quasilinearization procedure in Section 4.1, we are lead to the *modified* quasi-Newton algorithm,

$$[\theta]^{n+1} = [\theta]^n - [\psi]^n . \left[\bar{H}^2\right]^n . \left[\bar{g}^2\right]^n , \tag{35}$$

where

$$[\bar{H}^2]^n = \sum_{t=1}^{T} \sum_{i=1}^{N} \left[\{\bar{J}_{i,t}^2\}'\right]^n . \left[\bar{\Omega}^{2n}\right]^+ . \left[\bar{J}_{i,t}^1\right]^n \tag{36}$$

denotes the $B \times B$ Hessian matrix under the method of quasilinearization;

$$[\bar{g}^2]^n = -\sum_{t=1}^{T} \sum_{i=1}^{N} \left[\{\bar{J}_{i,t}^2\}'\right]^n . \left[\bar{\Omega}^{2n}\right]^+ . \left(\bar{\pi}_{i,t} - [\bar{\pi}_{i,t}^{*M}]^n\right) \tag{37}$$

denotes the $B \times 1$ gradient vector; and $[\psi]^n$ denotes a suitable step-size parameter defined over the unit interval, [0,1], and chosen to ensure the feasibility and the monotonicity of θ^{n+1}. Then by similar arguments presented to obtain equation (24), the sequence of the parameter values, $\{\theta^n\}$, will converge to a limit point,

$$\hat{\theta} = \lim_{n \to \infty} [\theta]^n . \tag{38}$$

The next two section discuss the derivation and evaluation of the Jacobian matrix used above in the context of optimal portfolio choice.

B Efficiency Frontier Approximations

Consider the arbitrary investor i in period-t, and seek an expression for the $(A-1) \times B$ sub-matrix, $\bar{J}_{i,t}^2 \equiv \frac{\partial \bar{\pi}^{*M}}{\partial \theta}$, of the complete Jacobian matrix $J_{i,t}^2$. As before, the unknown efficiency frontier which is only defined *pointwise*, where, for any specified $\phi^*_{i,t}$, with associated $\omega^*_{i,t}$ and $\pi^*_{i,t}$, we have

$$b^M_{i,t}(\phi^*_{i,t}, \omega^*_{i,t}) \equiv b^M_{i,t}\left(\pi^{*\prime}_{i,t}\hat{\mu}_{i,t}(\gamma_{i,t}),\ \ (\pi^{*\prime}_{i,t}\hat{\Sigma}\pi^*_{i,t})^{\frac{1}{2}}\right) = 0. \tag{39}$$

We proceed in analogous fashion to Section 4.2 and define a sequence of grids at which $\omega^*_{i,t}$, $\pi^*_{i,t}$, $u^*_{i,t}$ are evaluated. These, in turn, lead to the determination of the optimal portfolio, $\pi^{*M}_{i,t}$ relative to the prevailing parameter values, ξ, and γ_i. Moreover, by inspection of the composition of $\pi^{*M}_{i,t}$ and its two adjacent portfolios $\{\phi^{*M} - \delta_M\}$ and $\{\phi^{*M} + \delta_M\}$, we may again determine whether $\pi^{*M}_{i,t}$ is an interior solution (case (i)), a boundary non-corner solution (case ii) or a boundary corner solution (case iii). Consequently, for the boundary function approximations, a similar procedure to those advanced in Section 4.2 applies to each of the possible types of portfolio for $\pi^{*M}_{i,t}$

i. Interior Solutions: Using the points, $\{(\phi^s_{i,t}, \omega^s_{i,t}) : s = 0, 1, \cdots, S_1\}$ obtained from tabular presentation of the ordered $\phi^{*s}_{i,t}$-values, Appendix A provides the methods by which the unknown efficient boundary function, $b^M(\phi^*_{i,t}, \omega^*_{i,t}) = 0$, of equation (39) can be replaced by a twice continuously differentiable approximation function, $b^{*M}(\phi^*_{i,t}, \omega^*_{i,t}) = 0$, in the neighborhood of $\pi^{*M}_{i,t}$.

ii. Boundary Non-Corner Solution: Apart from the fact that now we are dealing with $A_1 < A$ assets, the same procedures as discussed in Section 4.2 apply to fitting an interpolation polynomial approximation to the efficient boundary function to a boundary non-corner solution involving the points, $\{(\phi^s_{i,t}, \omega^s_{i,t}) : s = 0, 1, \cdots, S_2\}$-see Appendix A.

iii. Boundary Corner Solution: In the case of a boundary corner solution, as before, we must fit a spline which is piecewise twice continuously differentiable to the "left" and the "right" of the "join point" value, $\phi^{*M}_{i,t}$. This involves $S_{31}+1$ points, $(\phi^{*}_{i,t}, \omega^{*}_{i,t})$, for expected rates of return with values less than or equal to $\phi^{*M}_{i,t}$, and $S_{32}+1$ points with greater than or equal expected rates of return than $\phi^{*M}_{i,t}$. For details, the reader can refer to Appendix A and C.

C Evaluation of the Jacobian Matrix

With the approximation to the efficient boundary function in hand, we may reformulate the investors utility maximization problem as follows:

$$\max_{\pi_{it}} \ u\left(\pi'_{i,t}\hat{\mu}_{i,t}(\gamma_{i,t}), (\pi'_{i,t}\hat{\Sigma}_t\pi_{i,t})^{\frac{1}{2}}; x_{i,t}, \xi\right), \quad s.t. \ \ b^{*M}_{i,t}\left(\pi'_{i,t}\hat{\mu}_{i,t}(\gamma_{i,t}), (\pi'_{i,t}\hat{\Sigma}_t\pi_{i,t})^{\frac{1}{2}}\right) = 0,$$

which differs from equation (26) in that dependence of $\hat{\mu}_{i,t}$ on γ_i is now explicit. Denote the Lagrange multiplier for this maximization problem as $\lambda_{i,t}$. Total differentiation of the first order conditions with respect to $\pi_{i,t}$, $\lambda_{i,t}$, and θ can be written instructively as:

$$\left[J^{2*}_{i,t}\right]^n .dz_{i,t} \equiv \begin{bmatrix} S^{2n}_{i,t} & s^{2n}_{i,t} \\ s^{2n\prime}_{i,t} & 0 \end{bmatrix} . \begin{bmatrix} d\pi_{i,t} \\ d\lambda_{i,t} \end{bmatrix} = \begin{bmatrix} -[R^2_{i,t}]^n.d\theta \\ 0 \end{bmatrix}, \tag{40}$$

where the $A \times A$ matrix, $S^2_{i,t}$, the $A \times 1$ vector, $s^2_{i,t}$, and the $A \times P + (K+1).(L+1)$ matrix, $R^2_{i,t}$ are defined in Appendix C, and are evaluated at the prevailing parameter value, θ^n. Here, the last of these, $R^2_{i,t}$, exhibits a patterned structure due to the fact that only γ_i appears in the $(i,t)^{th}$ term. It follows from the use of partitioned inversion of the leading matrix, $[J^{2*}_{i,t}]^n$, in equation (40) that a closed-form expression for the full Jacobian matrix can be derived as:

$$J^{2n}_{i,t} \equiv \frac{\partial \pi^{*Mn}_{i,t}}{\partial \xi} = -R^{2n}_{i,t}\left[\{S^{2n}_{i,t}\}^{-1} - \frac{\{S^{2n}_{i,t}\}^{-1}.s^{2n}_{i,t}.s^{2n\prime}_{i,t}.\{S^{2n}_{i,t}\}^{-1}}{s^{2n\prime}_{i,t}\{S^{2n}_{i,t}\}^{-1}.s^{2n}_{i,t}}\right] \tag{41}$$

Thus the sub-matrix $\bar{J}^{2n}_{i,t}$, of interest in the algorithm may be simply be inserted into equation (35).

VI Extensions and Related Issues

The proposed calibration methods are potentially useful in several other asset pricing applications, especially in situations where it is important to introduce institutional restriction. Consider first the portfolio selection model due to Tobin (1958, 1965), who explored the consequence of introducing a risk-free asset into an otherwise risky portfolio. As is well known, in this case of riskless borrowing and lending, the Sharpe portfolio or the market portfolio, denoted "s", can be defined as the point on the efficiency frontier at which a line emanating from the point, $(r^f, 0)$, in risk-return space, has maximum slope. This reflects the fact that any convex combination of the risk-free asset and a portfolio of risky assets is feasible. Apart from the increase in dimensionality of the portfolio selection problem, the econometric accommodation of the Tobin model poses no new substantive issues. For instance, the algorithm discussed in Sections 4.2, 5.2 and Appendix A must now simply be modified to locate that point $(\phi^s_{i,t}, \omega^s_{i,t})$, on the efficiency boundary that maximizes the slope, $\frac{\phi^s_{i,t} r^f_t}{\omega^s_{i,t}}$, of the line,

$$\phi_{i,t} = r^f_t + \left[\frac{\phi^s_{i,t} - r^f_t}{\omega^s_{i,t}} \right] \omega_{i,t} \tag{42}$$

Under homogeneous returns, this search determines the Sharpe (market) portfolio; and then it is straightforward to determine the optimal utility maximizing portfolio, $[w \mid (1-w).s']'$, where w is the proportion of the risk-free asset held and $(1-w)$ is the proportion held in the form of the Sharpe risky asset portfolio.

The institutional distinction that borrowing rates (r^{fB}) must be greater than the lending rates (r^{fL}), i.e., $r^{fB} > r^{fL}$, as emphasized in Brennan (1971), can be similarly handled. Let s^L and s^B, respectively, denote the Sharpe portfolios relative to the lending and borrowing rates. The Brennan efficient boundary function, $b^{BR}(\phi, \omega) = 0$ consists of three pieces: (1) a straight line connecting the point, $(r^{fL}, 0)$, to the point, (ϕ^{sL}, ω^{sL}), where ($\phi^{sL} = s^{L'}\mu$, and $\omega^{sL} = \left(s^{L'}\Sigma s^L\right)^{\frac{1}{2}}$; (2) the Markowitz efficient boundary function, $b^M(\phi, \omega) = 0$, for (ϕ, ω) values between (ϕ^{sL}, ω^{sL}) and (ϕ^{sB}, ω^{sB}); and (3) a straight line emanating from the point, (ϕ^{sB}, ω^{sB}), with slope $\frac{\phi^{sB} - r^{fB}}{\omega^{sB}}$. With this set-up, the calibration procedures are essentially the same: the forward solution, the Brennan efficient boundary function is searched for the value which maximizes utility, subject to any institutional restrictions on lending and borrowing which may also restrict the domain of w. In the inverse problem, however, problem arise with possible corner solutions at either of the two join points, which as discussed before, can be handled by circular approximants.

Suppose now, the non-negativity restriction on the portfolio's is removed as in Black (1972). This implies that the Black efficiency frontier, $b^B(\phi, \omega)$, is defined as a solution to the following quadratic programming problem:

$$\min_{\pi} \quad \frac{1}{2}\pi'\Sigma\pi \quad \text{s.t.} \quad \imath'\pi = 1, \;\; \mu'\pi = \phi^*. \tag{43}$$

This leads to a closed-form hyperbolic expression for the investor's efficiency frontier [Ingersoll (1987), Huang and Litzenberger (1988), Alexander-Francis (1986) and Elton-Gruber (1991)],

$$b^B(\phi,\omega) = \omega - \left\{ [\phi \;\; \imath] . D^{-1} . \begin{bmatrix} \phi \\ \imath \end{bmatrix} \right\}^{\frac{1}{2}} = 0; \quad D = \begin{bmatrix} \mu' \\ \imath' \end{bmatrix} . \Sigma . [\mu \;\; \imath]; \tag{44}$$

and, by standard solution of the first order conditions or by numerical grid-search methods, the utility function may be maximized. In the *mixed* Markowitz-Black model, short sales without margin requirements are permitted on only a subset, π_1, of the A risky assets, with no forward markets for the remainder, π_2. Thus, the optimization problem confronting the investor is identical to equation (6), except that the number of non−negativity constraints in $\pi \geq 0$ is replaced by $\pi_1 \geq 0$. Thus, the Black (1972) model and the mixed Markowitz-Black models permits calibration of the structural parameters as discussed above, as would be expected from a special case.

In situations where short selling of risky assets is permitted but typical margin required are imposed, the analysis of Dyl (1975), Ross (1977), and Sharpe (1991) leads to the following relationship between realized long (r^{lo}), and short rates (r^{sh})

$$r^{sh}_{a,t} = -\frac{r^{lo}_{a,t}}{c_{a,t}}$$

where $c_{a,t}$ is proportion of the short-sale proceeds deposited with the broker. This mixed Markowitz-Black case is straightforward application of our econometric methodology and involves an increase in the dimensionality of the problem to $2.A$ risky assets.

Observe that the market−based Capital Asset Pricing Model (CAPM) and its multi-beta interpretations in Merton (1973) only exploit the homogeneous input assumptions. This assumption implies that all the optimal portfolio's will lie on the same linear function in the risk-return space and the proportion of the risky assets contained in the Sharpe or "market portfolio," s, will be identical across all investors, so that the optimal portfolio's will only differ in the proportions, $w_{i,t}$, of risk-free versus risky assets that are held. But surely, the essence of a stock market is that different investors arrive at different expectations as to the future course of prices and dividends, and that is why there are both buyers and sellers to "make a market" and, consequently, motivated the econometric implementation of portfolio choice under heterogeneous expectations.

VII Concluding Remarks

In this paper, we have developed an econometric methodology associated with the inverse of the portfolio selection problem. In particular, given a time series of actual observed portfolio of risky assets for a sample of investors, a set of socio-economic characteristics for each investor in the sample, and a time series of preceding rates of returns for the set of risky assets, the algorithm determines the parameter values in each investor's utility function and the associated parameters in the returns generating process. It also determines the optimal current portfolios at the same time for all sample members. The proposed econometric framework can, therefore, accommodate either homogeneous or heterogeneous expected returns and covariance matrix. Specifically, in the context of market-based CAPM, note that heterogenous expectations causes investors to face different efficiency frontiers and destroys the linearity of the *security market line* and this forces the analysis into the domain of searching for the *structural* parameters which underlies the economic system, rather than just the "α" and "β" parameters.

The paper makes two important methodological contributions in terms of the algorithm for portfolio selection and its inverse.[20] First, since the efficient boundary function can only be determined pointwise, we have employed Lagrange interpolation polynomials fitted to the points on an increasingly-finer, one dimensional grids. This permits the approximating polynomials to pass through each of the grid points, and preserves the concavity of the efficient boundary function. However, in cases where the calibration algorithm encounters a "corner solution" which results in "jump- discontinuities" in the partial derivatives efficiency frontier, our approach is to use "circular approximants." This allows us to approximate the efficiency frontier by embedding a small circle into the immediate vicinity of the corner point. Second, whereas only *numerical* solutions can be computed for the optimal Markowitz portfolio, we demonstrate that given this optimal portfolio, the Jacobian matrix of partial derivatives required to implement the quasi-Newton procedure can be derived as a closed−form analytic expression. This avoids the need for the use of numerical approximations to the derivatives in each iteration and, therefore, saves vast amounts of computer time.

Clearly, the existing asset pricing theories such as the market- based CAPM and the consumption-based CAPM have the distinct advantage that they can be fitted to aggregate-time series data. However, predictions about individual or panels of portfolio holdings can be provide powerful tests of asset pricing theories. Consequently, a natural question, in the present framework, is how to go from a micro-theory of the individual or institutional investor's asset-portfolio mix to the aggregate behavior of the market for risky and risk-free assets. If the set of panel data represents a stratified sample of individual/institutional investors, then by applying suitable sampling weights to the numerical solution values in the sample, one may develop an internally-consistent micro-macro model with no further restrictions on the functional forms of the underlying functional relationships-the investor's returns-generating process and the utility function-than the customary stipulations of "well– behavedness".

References

Alexander, G., 1976, The derivation of the efficient sets. *Journal of Finance and Quantitative Analysis* 11: 817- 830.

Alexander, G., 1977, An algorithm for deriving the capital market line. *Management Science* 23: 1183-1186.

Alexander, G., and J. Francis, 1986, Portfolio Analysis. *Englewood Cliffs*, NJ: Prentice Hall.

Alexander, G., and B. Resnick, 1985, More on estimation risk and simple rules for optimal portfolio selection. *Journal of Finance* 40: 125-133.

Bakshi, G., and Z. Chen, 1994, Baby boom, population aging, and capital markets. *Journal of Business* 67: 165- 202.

Bakshi, G., and Z. Chen, 1996, The spirit of capitalism and stock market prices. *American Economic Review* 86: 133- 157.

Barten, A., 1969, Maximum-Likelihood estimation of a complete system of demand equations. *European Econometric Review* 1: 7-73.

Bellman, A., and R. Roth, 1983, Quasilinearization and the Identification Problem. *Singapore: World Scientific Press.*

Black, F., 1972, Capital market equilibrium with restricted borrowing. *Journal of Business* 45: 444-455.

Brennan, M., 1971, Capital market equilibrium with divergent borrowing and lending rates. *Journal of Financial and Quantitative Analysis* 6: 407-418.

Cheung, S., and C. Kwan, 1988, A note on simple criteria for optimal portfolio selection. *Journal of Finance* 53: 241-245.

Dennis, J., and R. Schnabel, 1983, Numerical Methods for Unconstrained Optimization and Nonlinear Equations. *Englewood Cliffs*, NJ: Prentice-Hall.

Duffie, D., 1988, Security Markets Stochastic Models, *Academic Press, Inc.*, San Diego.

Duffie, D., 1992, Dynamic Asset Pricing Theory, *Princeton University Press*, Princeton, New Jersey.

Duffie, D. and K. Singleton, 1993, Simulated moments estimation of Markov models of asset prices. *Econometrica*, 61, 929-952.

Dumas, B., and E. Luciano, 1991, An exact solution to a dynamic portfolio choice problem under transaction costs. *Journal of Finance* 56: 577-595.

Dyl, E., 1975, Negative betas: the attraction of Selling short. *Journal of Portfolio Management* 1: 74-76.

Dybvig, P., 1984, Short sale restrictions and kinks on the mean variance frontier. *Journal of Finance* 39: 239-244.

Dybvig, P., 1988, Distributional analysis of portfolio choice. *Journal of Business* 61: 369-394.

Dybvig, P., and S. Ross, 1982, Portfolio efficiency sets. *Econometrica* 50: 1525-1546.

Elton, E., M. Gruber, 1991, Modern Portfolio Theory and Investment Analysis. *John Wiley & Sons, Inc.*

Elton, E., M. Gruber, and M. Padberg, 1976, Simple criteria for optimal portfolio selection, *Journal of Finance* 31: 1341-1357.

Ferson, W., 1994, Theory and empirical testing of asset pricing models, Ch. 6. *The Finance Handbook*, ed. R. Jarrow, W. Ziemba, V. Maksimovic, North-Holland.

Green, R., 1986, Positively weighted portfolios on the minimum-variance frontier. *Journal of Finance* 41: 1051-1068.

Hansen, L., 1982, Large sample properties of generalized method of moments estimators. *Econometrica* 45: 1029-179.

Hansen, L. and K. Singleton, 1982, Generalized instrument variable estimation of nonlinear rational expectations models. *Econometrica* 50: 1269-1286.

Hardle, W., 1990, Smoothing-Techniques With Implementation. *Berlin: Springer-Verlag.*

Hartley, H.O., 1961, The modified Gauss-Newton methods for fitting nonlinear regression functions by least squares. *Technometrics* 3: 269-280.

Hartley, M., 1986, Calibration of macroeconomic models with incomplete data. *Computers and Mathematics With Applications: An International Journal Series A, Bellman Memorial Issue* 6, 769-776.

Hartley, M., 1988, Neoclassical econometrics: The kernel. *Journal of Mathematical Analysis and Applications* 2: 313-328.

Hartley, M., 1994, A systems approach to the calibration of deterministic nonlinear simultaneous equation models with incomplete data. *Applied Stochastic Models and Data Analysis*, Forthcoming.

Harvey, C., 1991, The world price of covariance risk. *Journal of Finance* 46: 111-157.

Hausman J., and D. Wise, 1978, A conditional probit model for qualitative choice: discrete decisions recognizing interdependence and heterogeneous preferences. *Econometrica* 46: 403-426.

Huang, C-F, and R. Litzenberger, 1988, Foundations for Financial Economics. *Prentice Hall*, Englewood Cliffs, New Jersey.

Kroll, Y., H. Levy, and H. Markowitz, 1984, Mean-Variance versus direct utility maximization. *Journal of Finance* 39: 47-62.

Isaacson, E., and H. Keller, 1966, Analysis of Numerical Methods. *New York: John Wiley and Sons, Inc.*

Ingersoll, J., 1987, The Theory of Financial Decision Making. *Rowman & Littlefield, Inc.*

Lewis, A., 1988, A simple algorithm for portfolio selection problem. *Journal of Finance* 53: 71-82.

Maddala, G., 1983, Limited dependent and qualitative variables in econometrics. *Cambridge: Cambridge University Press.*

Markowitz, H., 1952, Portfolio selection. *Journal of Finance* 7: 77-91.

Markowitz, H., 1956, The Optimization of a Quadratic Function Subject to Linear Constraints. *Naval Research Logistics Quarterly* 3: 111-133.

Markowitz, H., 1959, Portfolio selection: efficient diversification of investments. *New York: John Wiley & Sons, Inc.*

Markowitz, H., 1987, Mean Variance Analysis in Portfolio Choice and Capital Markets, *Oxford: Basil Blackwell.*

McFadden, D., 1989, A method of simulated moments for estimation of discrete response models without numerical integration *Econometrica* 57: 995-1026.

Merton, R., 1972, An Analytical derivation of the efficient frontier. *Journal of Financial and Quantitative Analysis* 7: 1850-1872.

Merton, R., 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867-888.

Merton, R., 1980, On estimating the expected rate of return on the market: an exploratory investigation. *Journal of Financial Economics* 8: 323-361.

McGuire, T., J. Farley, R. Lucas, and R. Winston, 1968, Estimation and inference for linear models in which subsets of the dependent variables are constrained. *Journal of the American Statistical Association* 63: 1201-1213.

Nielsen, L., 1987, Portfolio selection in the mean- variance world: a note. *Journal of Finance* 47: 1371- 471.

Quandt, R., 1983, Computational problems and methods. *Handbook of Econometrics, Vol. I* , Amsterdam: North-Holland.

Rao, C., 1973, Linear Statistical Inference and Its Applications, *New York: John Wiley & Sons.*

Roll, R., 1977, A critique of asset pricing theory tests–Part 1: on past and potential testability of the theory. *Journal of Financial Economics* 4: 129-176.

Ross, S., 1976, The arbitrage theory of capital asset pricing. *Journal of Economic Theory* 13: 341-360.

Ross, S., 1977, The capital asset pricing model (CAPM), short sale restrictions and related issues. *Journal of Finance* 32, 177-183.

Sharpe, W., 1963, A simplified model for portfolio analysis. *Management Science* 9: 277-293.

Sharpe, W., 1991, Capital asset prices with and without negative holdings. *Journal of Finance* 46: 489- 509.

Tobin, J., 1958, Liquidity preference as behavior towards risk. *Review of Economic Studies* 26: 65-86.

Tobin, J., 1965, The Theory of Portfolio Selection, in *The Theory of Interest Rates*, ed. F. Hahn and F. Brechling, London: MaxMillan and Co., Ltd.

APPENDIX A

Approximations to the Efficient Boundary Function

Interior Solutions and Boundary Non-Corner Solutions

Both the interior and boundary noncorner portfolios require fitting a polynomial to the set of (say) $S+1$ points, $\{(\phi_{i,t}^{*s}, \omega_{i,t}^{*s}) : s = 0, 1, \ldots, S\}$, in risk-return space. It will be convenient for our purposes to employ Lagrange interpolation polynomials [Isaacson and Keller (1966, p.189)], defined as:

$$\omega_{i,t} \equiv f_S(\phi_{i,t}) = \sum_{s=0}^{S} \omega_{i,t}^{*s} \prod_{r=0, r\neq s}^{S} \frac{(\phi_{i,t} - \phi_{i,t}^{*r})}{(\phi_{i,t}^{*s} - \phi_{i,t}^{*r})} \tag{45}$$

Note that at any of the grid points, $\phi_{i,t} = \phi_{i,t}^{*s}$, we have $\omega_{i,t} = \omega_{i,t}^{*s}$ for $s = 0, 1, \ldots, S$, hence the Lagrange interpolation polynomial is of degree S; and it passes through each of the $(S+1)$ grid points, $(\phi_{i,t}^{*s}, \omega_{i,t}^{*s})$. Equivalently, the approximation to the unknown efficiency frontier in the neighborhood of $\phi_{i,t}^{*M}$ can be rewritten for present purposes as the implicit function:

$$b_{i,t}^{*M}\left(\phi_{i,t}, \omega_{i,t}\right) = \omega_{i,t} - \sum_{s=0}^{S} \omega_{i,t}^{*s} \prod_{r=0, r\neq s}^{S} \frac{(\phi_{i,t} - \phi_{i,t}^{*r})}{(\phi_{i,t}^{*s} - \phi_{i,t}^{*r})} = 0 \equiv \omega_{i,t} - f_S(\phi_{i,t}) \tag{46}$$

As we demonstrate in Appendices B and C, to calculate the elements of the Jacobian matrix in either the direct *ex ante* or direct *ex post* cases, we require various expressions for the partial derivatives of $b_{i,t}^{*M}\left(\phi_{i,t}, \omega_{i,t}\right)$, with respect to $\phi_{i,t}$ and $\omega_{i,t}$, which we can write below as:

$$\frac{\partial b_{i,t}^{*M}\left(\phi_{i,t}, \omega_{i,t}\right)}{\partial \phi_{i,t}} = \sum_{s=0}^{S} \omega_{i,t}^{*s} \frac{\sum_{u=0}^{S} \prod_{r=0, r\neq u}^{S} \left(\phi_{i,t} - \phi_{i,t}^{*r}\right)}{\prod_{r=0, r\neq s}^{S} \left(\phi_{i,t}^{*s} - \phi_{i,t}^{*r}\right)} = \frac{\partial f_S\left(\phi_{i,t}, \omega_{i,t}\right)}{\partial \phi_{i,t}}; \quad \frac{\partial b_{i,t}^{*M}\left(\phi_{i,t}, \omega_{i,t}\right)}{\partial \omega_{i,t}} = 1. \tag{47}$$

Similarly, the second partial derivatives are given by:

$$\frac{\partial^2 b_{i,t}^{*M}}{\partial {\phi_{i,t}}^2} = \sum_{s=0}^{S} \omega_{i,t}^{*s} \frac{\sum_{u=0}^{S} \sum_{v=0,v\neq u}^{s} \prod_{r=0,r\neq u,r\neq v}^{S} (\phi_{i,t} - \phi_{i,t}^{*r})}{\prod_{r=0,r\neq s}^{S} (\phi_{i,t}^{*s} - \phi_{i,t}^{*r})}; \quad \frac{\partial^2 b_{i,t}^{*M}}{\partial \omega_{i,t} \partial \phi_{i,t}} = \frac{\partial^2 b_{i,t}^{*M}}{\partial {\omega_{i,t}}^2} = 0 \qquad (48)$$

These expressions, along with the first and second partial derivatives of the utility function specified by the investigator with respect to $\phi_{i,t}$ and $\omega_{i,t}$, are required to define the elements of the Jacobian matrix, as outlined in Appendices B and C.

Boundary Corner Solutions

It remains to approximate the efficient boundary by a twice continuously differentiable function when a corner solution obtains. As noted earlier, such situations occur when one or more assets are introduced into or deleted from the prevailing portfolio. In risk-return space, the presence of a corner solution is revealed to the investigator by the adjacent portfolio with slightly smaller or larger expected rate of return than that associated with $\phi_{i,t}^{*M}$ being of different types.

Suppose that, in the neighborhood of $\phi_{i,t}^{*M}$, there are $S_1 + 1$ grid points, $\{(\phi_{i,t}^{*s}, \omega_{i,t}^{*s}) : s = 0, 1, \ldots, S_1\}$, associated with the adjacent portfolios with expected rates of return less than or equal to $\phi_{i,t}^{*M} = \phi_{i,t}^{*S_1}$ of (say) type 1; and suppose there are $S_2 + 1$ grid points, $\{(\phi_{i,t}^{*s}, \omega_{i,t}^{*s}) : s = 0, 1, \ldots, S_2\}$, associated with the adjacent portfolios with expected rates of return less than or equal to $\phi_{i,t}^{*M} = \phi_{i,t}^{*0}$ of (say) type 2. By the analysis above, we may fit two separate Lagrange interpolation polynomials, $\omega_{i,t} = f_{S_1}(\pi_{i,t})$ and $\omega_{i,t} = g_{S_2}(\pi_{i,t})$ to the $\{\phi_{i,t}^{*s}, \omega_{i,t}^{*s}\}$-values associated with the portfolio of each type-including the corner portfolio. However, since each Lagrange interpolation polynomial passes through each of the grid points, $\{\phi_{i,t}^{*s}, \omega_{i,t}^{*s}\}$, it follows that the two approximation polynomials have the common "joint point", $\{\phi_{i,t}^{*M}, \omega_{i,t}^{*M}\}$-that is

$\omega_{i,t}^{*M} = f_{S_1}(\pi_{i,t}^{*M}) = g_{S_2}(\pi_{i,t}^{*M})$. However, since $b_{i,t}^{M}(\phi_{i,t}, \omega_{i,t}) = 0$ is strictly concave, then so will be the approximation, $b_{i,t}^{*M}(\phi_{i,t}, \omega_{i,t}) = 0$, in the neighborhood of $\phi_{i,t}^{*M}$. Moreover, since $\phi_{i,t}^{*M}$ is associated with a corner portfolio, $\pi_{i,t}^{*M}$, it follows that the partial derivatives of the "left," $\partial f_{S_1}/\partial \phi_{i,t}$ and "right," $\partial g_{S_2}/\partial \phi_{i,t}$, of $\phi_{i,t}^{*M}$-the joint point-are not equal. This results in a "jump-discontinuity" in the associated elements of the Jacobian matrix, $J_{i,t}$; and we must resort to the use of various possible "smoothing techniques" to avoid the fact that the derivative, $\partial b_{i,t}^{*M}/\partial \phi_{i,t}$, does not exist at $\phi_{i,t}^{*M}$, which we describe below.

Circular Approximants

A comprehensive treatment of the smoothing techniques is given in Hardle (1990). Our development of the circular approximant is dictated by its simplicity when restricted to two-dimensional risk-return space. Essentially, circular approximants involve embedding a simple circle of pre-specified data, ρ, into the immediate vicinity of the corner point $(\phi_{i,t}^{*M}, \omega_{i,t}^{*M})$, which is tangent to both Lagrange interpolation polynomials, f_{S_1} and g_{S_2}; and the requisite partial derivatives at the corner solution value, $\phi_{i,t}^{*M}$, can then be taken relative to the "circular approximant".

The circular approximant is displayed in Figure 1. The two Lagrange interpolation polynomials, f_{S_1} and g_{S_2}, intersect at the corner solution, $(\phi_{i,t}^{*M}, \omega_{i,t}^{*M})$, in risk-return space. The problem in two dimensions is to smooth the approximation function by inserting a circle of suitably small radius so that it is tangent to both f_{S_1} and g_{S_2}, and then to define the approximating function between the two points of the tangency as the circle itself. Let this circular approximant be defined as:

$$(\phi_{i,t} - a)^2 + (\omega_{i,t} - b)^2 = \rho^2 \tag{49}$$

where (a, b) is the unknown center of the circle with specified radius, ρ. Then, the problem is to determine (a, b) from the knowledge of the two functions, f_{S_1} and g_{S_2}, along with a given value for ρ. Despite, the apparent simplicity of this problem, unfortunately there does not appear to be a general closed–form solution.

Let $(\phi^1_{i,t}, \omega^1_{i,t})$ and $(\phi^2_{i,t}, \omega^2_{i,t})$ denote the points at which the circle is both tangent to and coincident with the functions $\omega_{i,t} = f_{S_1}(\phi_{i,t})$ and $\omega_{i,t} = g_{S_2}(\phi_{i,t})$, respectively. Then the equivalence of the two functions implies that:

$$(\phi^1_{i,t} - a)^2 + (f_{S_1}(\phi^1_{i,t}) - b)^2 = \rho^2; \quad (\phi^2_{i,t} - a)^2 + (g_{S_2}(\phi^2_{i,t}) - b)^2 = \rho^2 \tag{50}$$

whereas the equivalence of the slopes at these two points implies that,

$$\frac{(\phi^1_{i,t} - a)}{(f_{S_1}(\phi^1_{i,t}) - b)} = \frac{\partial f_{S_1}(\phi^2_{i,t})}{\partial \phi_{i,t}}; \quad \frac{(\phi^2_{i,t} - a)}{(g_{S_2}(\phi^2_{i,t}) - b)} = \frac{\partial g_{S_2}(\phi^2_{i,t})}{\partial \phi_{i,t}} \tag{51}$$

Here, f_{S_1} and g_{S_2} are Lagrange polynomials of order, $(S_1 + 1)$ and $(S_2 + 1)$; whereas the first partials of with respect to $\phi_{i,t}$ are given in equations (47) and are polynomials of order S_1 and S_2, respectively. Thus, the four-equation system, (50) and (51), must be solved numerically for the four variables: a, b, $\phi^1_{i,t}$ and $\phi^2_{i,t}$.

Thus given a solution to the above problem, the boundary function, $b^{*M}_{i,t}(\phi_{i,t}, \omega_{i,t}) = 0$, is then approximated by the values assumed by the circle over the arc from $(\phi^1_{i,t}, \omega^1_{i,t})$ to $(\phi^2_{i,t}, \omega^2_{i,t})$.

Consequently, the approximation to the boundary function is given by

$$b_{i,t}^{*M}(\phi_{i,t}, \omega_{i,t}) = (\phi_{i,t} - a)^2 + (\omega_{i,t} - b)^2 - \rho^2 = 0 \tag{52}$$

over the specified arc; and the requite partial derivatives are given by:

$$\frac{\partial b_{i,t}^{*M}}{\partial \phi_{i,t}} = 2.(\phi_{i,t}^{*M} - a); \quad \frac{\partial b_{i,t}^{*M}}{\partial \omega_{i,t}} = 2.(\omega_{i,t}^{*M} - b) \tag{53}$$

and,

$$\frac{\partial^2 b_{i,t}^{*M}}{\partial {\phi_{i,t}}^2} = \frac{\partial^2 b_{i,t}^{*M}}{\partial {\omega_{i,t}}^2} = 2; \quad \frac{\partial^2 b_{i,t}^{*M}}{\partial \phi_{i,t} \partial \omega_{i,t}} = 0. \tag{54}$$

Clearly, the method of "circular approximants" can be extended to other conic section-for example, the use of ellipse. □

APPENDIX B

Evaluation of the Jacobian Matrix in the Direct Ex Ante Case

Regardless of the type of the optimal portfolio encountered by the algorithm in the direct *ex ante* case, this appendix demonstrates that given the restriction that the utility function be twice continuously differentiable, and given our twice continuously differentiable approximation to the efficient boundary function shown in Appendix A, the elements of the Jacobian matrix will be known in closed-form.

To prove this assertion, consider the constrained maximization problem from Section 4.3, which we can formulate as:

$$\max_{\pi_{i,t}} \ \Lambda_{i,t}(\pi_{i,t}, \lambda_{i,t}) = u\left(\pi'_{i,t}\hat{\mu}_{i,t}, (\pi'_{i,t}\hat{\Sigma}_t\pi_{i,t})^{\frac{1}{2}}; x_{i,t}, \xi\right) + \lambda_{i,t}.b^{*M}_{i,t}\left(\pi'_{i,t}\hat{\mu}_{i,t}, (\pi'_{i,t}\hat{\Sigma}_t\pi_{i,t})^{\frac{1}{2}}\right), \quad (55)$$

The first order conditions for this maximization are:

$$\frac{\partial\Lambda_{i,t}}{\partial\pi_{i,t}} = \left(\frac{\partial u_{i,t}}{\partial\phi_{i,t}} + \lambda_{i,t}.\frac{\partial b^{*M}_{i,t}}{\partial\phi_{i,t}}\right)\hat{\mu}_{i,t} + \left(\frac{\partial u_{i,t}}{\partial\omega_{i,t}} + \lambda_{i,t}.\frac{\partial b^{*M}_{i,t}}{\partial\omega_{i,t}}\right)\left(\pi'_{i,t}\hat{\Sigma}_t.\pi_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\pi_{i,t} = 0, \quad (56)$$

and,

$$\frac{\partial\Lambda_{i,t}}{\partial\lambda_{i,t}} = b^{*M}_{i,t}(\phi_{i,t}, \omega_{i,t}) = 0 \quad (57)$$

Consider total differentiation of the first− order conditions in equations (56) and (57), with respect to $\pi_{i,t}$, $\lambda_{i,t}$, and ξ, which implicitly defines the expressions for the $A \times A$ matrix, $S^1_{i,t}$; the $A \times P$ matrix, $R^1_{i,t}$; and the $A \times 1$ vector, $s^1_{i,t}$:

$$S^1_{i,t}.d\ \pi_{i,t} + s^1_{i,t}.d\ \lambda_{i,t} + R^1_{i,t}.d\ \xi = 0$$

where,

$$s^1_{i,t} = \frac{\partial b^{*M}_{i,t}}{\partial\phi_{i,t}}.\hat{\mu}_{i,t} + \frac{\partial b^{*M}_{i,t}}{\partial\omega_{i,t}}.\left(\pi'_{i,t}\hat{\Sigma}_t.\pi_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\pi_{i,t},$$

$$R^1_{i,t} = \frac{\partial^2 u_{i,t}}{\partial\phi_{i,t}\partial\xi'_{i,t}}.\hat{\mu}_{i,t} + \left(\hat{\pi}'_{i,t}\hat{\Sigma}_t.\hat{\pi}_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\hat{\pi}_{i,t}.\frac{\partial^2 u_{i,t}}{\partial\omega_{i,t}\partial\xi'_{i,t}},$$

and,

$$\begin{aligned}
S^1_{i,t} \;&= \left[\frac{\partial^2 u_{i,t}}{\partial\phi_{i,t}\partial\phi_{i,t}} + \lambda_{i,t}.\frac{\partial^2 b^{*M}_{i,t}}{\partial\phi_{i,t}\partial\phi_{i,t}}\right].\hat{\mu}_{i,t}.\hat{\mu}'_{i,t} + \\
&\left[\frac{\partial^2 u_{i,t}}{\partial\phi_{i,t}\partial\omega_{i,t}} + \lambda_{i,t}.\frac{\partial^2 b^{*M}_{i,t}}{\partial\phi_{i,t}\partial\omega_{i,t}}\right].\left(\hat{\pi}'_{i,t}\hat{\Sigma}_t.\hat{\pi}_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\pi_{i,t}\hat{\mu}_{i,t} + \\
&\left[\frac{\partial^2 u_{i,t}}{\partial\omega_{i,t}\partial\phi_{i,t}} + \lambda_{i,t}.\frac{\partial^2 b^{*M}_{i,t}}{\partial\omega_{i,t}\partial\phi_{i,t}}\right].\left(\pi'_{i,t}\hat{\Sigma}_t.\pi_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\pi_{i,t}\hat{\mu}_{i,t} + \\
&\left[\frac{\partial^2 u_{i,t}}{\partial\omega_{i,t}\partial\omega_{i,t}} + \lambda_{i,t}.\frac{\partial^2 b^{*M}_{i,t}}{\partial\omega_{i,t}\partial\omega_{i,t}}\right].\left(\pi'_{i,t}\hat{\Sigma}_t.\pi_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\pi_{i,t}.\pi'_{i,t}.\hat{\Sigma}_t + \\
&\left\{\frac{\partial u_{i,t}}{\partial\omega_{i,t}} + \lambda_{i,t}.\frac{\partial b^{*M}_{i,t}}{\partial\omega_{i,t}}\right\}.\left\{\left(\pi'_{i,t}\hat{\Sigma}_t.\pi_{i,t}\right)^{-1/2}.\hat{\Sigma}_t - \left(\pi'_{i,t}\hat{\Sigma}_t.\pi_{i,t}\right)^{-3/2}.\hat{\Sigma}_t.\pi_{i,t}.\pi'_{i,t}.\hat{\Sigma}_t\right\},
\end{aligned}$$

and each of these are to be evaluated at the prevailing parameter vector, $[\xi]^n$.

It is important to note that, given the twice continuously differentiable approximation, $b^{*M}_{i,t}(\phi_{i,t},\omega) = 0$, to the efficient boundary function in the neighborhood of the expected rate of return, $\phi^{*M}_{i,t}$, evaluation of the terms above require expressions for the first and second partial derivatives of the utility function specified by the investigator, as well the use of corresponding expressions with respect to the Lagrange interpolation polynomials for $b^{*M}_{i,t}$.

Note that in the case of the interior or boundary non−corner solutions for the optimal portfolio, the use of equations (47) and (48) simplifies the above expressions; whereas, in the case of corner solutions, simplifications obtains from applying equations (53) and (54). □

APPENDIX C

Evaluation of the Jacobian Matrix in the Indirect Ex Post Case

This appendix shows that the elements of the Jacobian matrix will be known in closed-form. For this purpose, consider the constrained maximization problem from Section 5.3, which we can rewrite as:

$$\max_{\pi_{i,t}} \ \Lambda_{i,t}\left(\pi_{i,t},\lambda_{i,t}\right) = u\left(\pi'_{i,t}\hat{\mu}_{i,t}(\gamma_i),(\pi'_{i,t}\hat{\Sigma}_t\pi_{i,t})^{\frac{1}{2}};x_{i,t},\xi\right)+\lambda_{i,t}.b^{*M}_{i,t}\left(\pi'_{i,t}\hat{\mu}_{i,t}(\gamma_i),(\pi'_{i,t}\hat{\Sigma}_t\pi_{i,t})^{\frac{1}{2}}\right), \tag{58}$$

The first order conditions for this maximization are:

$$\frac{\partial\Lambda_{i,t}}{\partial\pi_{i,t}} = \left(\frac{\partial u_{i,t}}{\partial\phi_{i,t}}+\lambda_{i,t}.\frac{\partial b^{*M}_{i,t}}{\partial\phi_{i,t}}\right)\hat{\mu}_{i,t}(\gamma_i)+\left(\frac{\partial u_{i,t}}{\partial\omega_{i,t}}+\lambda_{i,t}.\frac{\partial b^{*M}_{i,t}}{\partial\omega_{i,t}}\right)\left(\pi'_{i,t}\hat{\Sigma}_t.\pi_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\pi_{i,t}=0, \tag{59}$$

and,

$$\frac{\partial\Lambda_{i,t}}{\partial\lambda_{i,t}} = b^{*M}_{i,t}\left(\phi_{i,t},\omega_{i,t}\right)=0 \tag{60}$$

Consider total differentiation of the first$-$ order conditions in equations (59) and (60), with respect to $\pi_{i,t}$, $\lambda_{i,t}$, and $\theta\equiv[\xi'\mid\gamma'_1,\cdots,\gamma'_i,\cdots,\gamma'_N]'$, which results in:

$$S^2_{i,t}.d\ \pi_{i,t}+s^2_{i,t}.d\ \lambda_{i,t}+P^2_{i,t}.d\ \xi+h^2_{i,t}.G^2_{i,t}.d\ \gamma_i=0;$$

where the following expressions defines the $A\times A$ matrix, $S^2_{i,t}$; the $A\times P$ matrix, $P^2_{i,t}$; the $A\times 1$ vector, $s^2_{i,t}$; and the scaler, $h^2_{i,t}$:

$$s^2_{i,t}=\frac{\partial b^{*M}_{i,t}}{\partial\phi_{i,t}}.\hat{\mu}_{i,t}(\gamma_i)+\frac{\partial b^{*M}_{i,t}}{\partial\omega_{i,t}}.\left(\pi'_{i,t}\hat{\Sigma}_t.\pi_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\pi_{i,t},$$

$$P^2_{i,t}=\frac{\partial^2 u_{i,t}}{\partial\phi_{i,t}\partial\xi'_{i,t}}.\hat{\mu}_{i,t}(\gamma_i)+\left(\hat{\pi}'_{i,t}\hat{\Sigma}_t.\hat{\pi}_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\pi_{i,t}.\frac{\partial^2 u_{i,t}}{\partial\omega_{i,t}\partial\xi'_{i,t}},$$

and,

$$\begin{aligned} S_{i,t}^2 &= \left[\frac{\partial^2 u_{i,t}}{\partial\phi_{i,t}\partial\phi_{i,t}} + \lambda_{i,t}.\frac{\partial^2 b_{i,t}^{*M}}{\partial\phi_{i,t}\partial\phi_{i,t}}\right].\hat{\mu}_{i,t}(\gamma_i).\hat{\mu}_{i,t}(\gamma_i)' + \\ &\left[\frac{\partial^2 u_{i,t}}{\partial\phi_{i,t}\partial\omega_{i,t}} + \lambda_{i,t}.\frac{\partial^2 b_{i,t}^{*M}}{\partial\phi_{i,t}\partial\omega_{i,t}}\right].\left(\pi_{i,t}'\hat{\Sigma}_t.\pi_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\pi_{i,t}\hat{\mu}_{i,t}(\gamma_i) + \\ &\left[\frac{\partial^2 u_{i,t}}{\partial\omega_{i,t}\partial\phi_{i,t}} + \lambda_{i,t}.\frac{\partial^2 b_{i,t}^{*M}}{\partial\omega_{i,t}\partial\phi_{i,t}}\right].\left(\pi_{i,t}'\hat{\Sigma}_t.\pi_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\pi_{i,t}\hat{\mu}_{i,t}(\gamma_i) + \\ &\left[\frac{\partial^2 u_{i,t}}{\partial\omega_{i,t}\partial\omega_{i,t}} + \lambda_{i,t}.\frac{\partial^2 b_{i,t}^{*M}}{\partial\omega_{i,t}\partial\omega_{i,t}}\right].\left(\pi_{i,t}'\hat{\Sigma}_t.\pi_{i,t}\right)^{-1/2}.\hat{\Sigma}_t.\pi_{i,t}\pi_{i,t}'.\hat{\Sigma}_t + \\ &\left\{\frac{\partial u_{i,t}}{\partial\omega_{i,t}} + \lambda_{i,t}.\frac{\partial b_{i,t}^{*M}}{\partial\omega_{i,t}}\right\}.\left\{\left(\pi_{i,t}'\hat{\Sigma}_t.\pi_{i,t}\right)^{-1/2}.\hat{\Sigma}_t - \left(\pi_{i,t}'\hat{\Sigma}_t.\pi_{i,t}\right)^{-3/2}.\hat{\Sigma}_t.\pi_{i,t}.\pi_{i,t}'.\hat{\Sigma}_t\right\}, \end{aligned}$$

and,

$$\begin{aligned} h_{i,t}^2 &= \left[\frac{\partial^2 u_{i,t}}{\partial\phi_{i,t}\partial\phi_{i,t}} + \lambda_{i,t}.\frac{\partial^2 b_{i,t}^{*M}}{\partial\phi_{i,t}\partial\phi_{i,t}}\right].\pi_{i,t}.\hat{\mu}_{i,t}'(\gamma) + \\ &\left[\frac{\partial^2 u_{i,t}}{\partial\phi_{i,t}\partial\omega_{i,t}} + \lambda_{i,t}.\frac{\partial^2 b_{i,t}^{*M}}{\partial\phi_{i,t}\partial\omega_{i,t}}\right].\left(\hat{\pi}_{i,t}'\hat{\Sigma}_t.\hat{\pi}_{i,t}\right)^{1/2} + \\ &\left\{\frac{\partial u_{i,t}}{\partial\phi_{i,t}} + \lambda_{i,t}.\frac{\partial b_{i,t}^{*M}}{\partial\phi_{i,t}}\right\} \end{aligned}$$

Depending upon the parametric form of the returns-generating process adopted, the following partial derivative of the utility function can be calculated:

$$\frac{\partial \hat{\mu}_{i,t}}{\partial \gamma_i} = G^2_{i,t} = \begin{bmatrix} 1 & x_{i,1,t} \ldots x_{i,K,t} & r_{i,1,t-1} & x_{i,1,t}.r_{i,1,t-1} \ldots x_{i,1,t}.r_{i,1.t-1} \\ 1 & x_{i,1,t} \ldots x_{i,K,t} & r_{i,2,t-1} & x_{i,1,t}.r_{i,2,t-1} \ldots x_{i,K,t}.r_{i,2,t-1} \\ . & . & \ldots & \ldots \\ 1 & x_{i,1,t} \ldots x_{i,K,t} & r_{i,A,t-1} & x_{i,1,t}.r_{i,A,t-1} \ldots x_{i,K,t}.r_{i,A,t-1} \\ . & . & \ldots & \ldots \\ . & \ldots & r_{i,A,t-L} & x_{i,1,t}.r_{i,A,t-L} \ldots x_{i,K,t-L}.r_{i,A,t-L} \end{bmatrix}$$

This in turn, permits us to define the requisite $A \times P + (K+1).(L+1).N$ matrix, $R^2_{i,t}=$ $\left[P^2_{i,t} 0 \ldots h^2_{i,t}.G^2_{i,t} \ldots 0\right]$, where the $h^2_{i,t}.G^2_{i,t}$ consists the i^{th} position after $P^2_{i,t}$, and each of the zero matrices, 0, are of the same order. Needless to say, the basic algorithm remains the same, if an alternate returns-generating process were adopted by the investigator, then the structure of the matrix $G^2_{i,t}$, would differ.

Again, notice here that in the case of interior or boundary non−corner solutions, one may employ the simplifying results in equations (47) and (48); whereas, with boundary corner solutions using the circular approximants, the results (53) and (54) may be employed. However, as compared to our analysis in Appendix B, the main difference is that, rather than treating $\hat{\mu}_{i,t}$ as a given constant, we now regard it as a parametric function of γ_i. $\square$

NOTES

1. Nielsen (1987) has stressed that in the mean– variance portfolio-selection model, the induced preferences for asset holdings are not necessarily monotone. That is more of an portfolio is not necessarily better even if the portfolio has positive expected return. In particular, for diversification purposes, the portfolio selection problem predicts that an investor typically wants only a limited number of shares of an asset, and beyond that the increase in mean return from acquired additional shares of the asset is not sufficient to compensate for the increased risk.

2. This is the case even though a stochastic returns generating process has been advanced, since the inputs to the portfolio selection problem require the expectation of the anticipated rates of return and the associated covariance matrix– both of which are deterministic.

3. For a further technical discussion, see Ingersoll (1987), Huang and Litzenberger (1988), and Duffie (1988, 1992).

4. Kroll, Levy and Markowitz (1984) have demonstrated that mean-variance analysis is often a sufficient normal approximation to the more exact "direct utility maximization". We further assume that the utility function is well behaved and concave. In particular, the utility be strictly increasing in expected rate of return of the portfolio (ϕ), and nonincreasing in its variance and the indifference curves in risk-return space are a convex set. This will ensure that the first order conditions for interior maximum will be necessary and sufficient.

5. Dybvig and Ross (1982) have shown that under certain conditions the efficient frontier will not be a convex set and Dybvig (1984) has demonstrated that the efficient frontier can have kinks. Throughout this paper assume that the efficient frontier is a convex set although, as shown in Section 4 and Appendix A, the Lagrange interpolation polynomial approximations are

general enough to handle all discontinuities and the efficient frontier will always be concave.

6. In order to conform to the notation employed for the heterogeneous returns-generating process, the subscript i is retained here even though the process is the same for each investor.

7. In certain applications, there may be dimensional restrictions which preclude the joint estimation of the A.(L+1) parameters. In such cases, for each asset, the (L+1) parameters, $\{\alpha_{l,a}\}$, associated with (8) may be estimated consistently using the OLS residuals. As is well known, such a procedure only sacrifices the efficiency of the VAR estimators.

8. The number of parameters that have to be estimated in this homogeneous autoregressive case can be contrasted to the single index model of Sharpe (1963) where 3A+2 parameters need to be estimated.

9. We adopt the customary terminology of "expected" rates of return when referring to the asymptotically normal, asymptotically unbiased and consistent estimates, $\{\hat{\alpha}_{l,a}\}$, of the $\{\alpha_{l,a}\}$ obtained under a VAR specification.

10. The availability of a set of investor characteristics is a *sine qua non* for developing a heterogeneous return-generating process. In cases where such data are not available, the only option may be to use the homogeneous returns-generating process. However, in such cases, one may still determine the structural parameters of the investors' utility function, as well those in the homogeneous returns- generating process, *ex post*.

11. Here, it is appropriate to inquire about the problems of dimensionality. In particular, in the heterogeneous case, there are "N" investors, "A" assets, "K" investor characteristic, and "L" lagged period histories of asset returns to be considered. However, in the present context-particularly since the period of observation on asset returns may be daily, and computerized

records on individual/institutional investors are typically kept over a significant number of periods, T is likely to be large in much of empirical work. Thus, the principal dimensionality issue arises from the fact that the number of potential risky assets actually under consideration may also be large. However, if in reality, investors focus upon the behavior of a limited number of stocks, etc., which may vary from one investor to another-as is quite likely to be the case, then the "curse of dimensionality" may be avoided.

12. It is customary in empirical applications of the Markowitz model to first formulate a returns-generating process for asset returns, and then to estimate its parameters on the basis of the preceding historical time-series data. The remaining parameters in the investors' utility function must be specified, *ex ante* by the investor to compute the optimal portfolio. The goal in this section is, therefore, to discuss the econometric methods associated with *ex-ante* calibration of the "taste parameter" within the utility function of a Markowitz model based upon actual portfolio decisions of investors.

13 The next section considers the more demanding task of jointly calibrating both the "taste parameters" and the parameters of the returns-generating process.

14. Let Φ be an arbitrary square matrix of order $A \times A$. We employ the symbol, Φ^{+}, to denote a suitable inverse of Φ which can calculated using either Cholesky Decomposition, Crout Decomposition or Moore-Penrose pseudo inverse.

15. McGuire, Farley, Lucas, and Winston (1968), Powell (1969) and Barten (1969) have demonstrated that the GLS/ML estimates are invariant with respect to which assets are deleted from s standard budget-share or demand system.

16. In the case of "backtracking," line-search procedures using a quasi-Newton algorithm, Dennis and Schnabel (1983, p.129) recommend imposition of a minimum step-size as part of the algorithm's convergence test. As they note: "This criterion prevents the line-search from looping forever if $(-[H]^{n+}.[g]^{n})$ is not a descent direction. [This sometimes occurs at the final iteration $(n+1)$ of minimizing algorithms, owing to finite-precision errors, especially if the gradient is calculated by finite differences.] (emphasis and notation added)." In such cases, provided the remaining conditions are also satisfied, the algorithm is said to converge in the preceding iteration, n.

17. The work of Elton-Gruber-Padberg (1976), Kwan (1984), Alexander and Resnick (1985), and Cheun and Kwan (1988) provide other relevant criteria for optimal portfolio selection.

18. Here, it is important to stress that it is *only* necessary to approximate the unknown efficient boundary function, $b_{i,t}^{M}(\phi_{i,t}, \omega_{i,t}) = 0$, in the neighborhood of the expected rate of return, $\phi_{i,t}^{*M}$, associated with the prevailing portfolio, $\pi_{i,t}^{*M}$.

19. A portfolio is of the same *type* as another if both contain the same assets in positive amounts, though the non$-zero$ proportions in each may differ.

20. A recent example for an algorithm for optimal portfolio selection is in Lewis (1988). This paper uses Markowitz's (1956) critical line approach to determine optimal portfolio weights when the utility function is dependent upon the mean and the variance, but can only be used when the constraints are linear. Our methodology, however, is general enough to accommodate any institutional restriction, linear or nonlinear. Furthermore, the algorithm simultaneously determines the parameter values which best fit the portfolio selection model to *actual* investor portfolios and the *optimal* values of the investor portfolios.

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