

Variance Bounds on the Permanent and Transitory Components of Stochastic Discount Factors*

Gurdip Bakshi^{a†} Fousseni Chabi-Yo^{b‡}

^a*Smith School of Business, University of Maryland, College Park, MD 20742, USA*

^b*Fisher College of Business, Ohio State University, Columbus, OH 43210, USA*

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Abstract

When the transitory component of the stochastic discount factors (SDFs) prices the long-term bond, and the permanent component prices other assets, we develop lower bounds on the variance of the permanent component and the transitory component, and on the variance of the ratio of the permanent to the transitory components of SDFs. A salient feature of our bounds is that they incorporate information from average returns and the variance-covariance matrix of returns corresponding to a generic set of assets. Relevant to economic modeling, we examine the tightness of our bounds relative to Alvarez and Jermann (2005, *Econometrica*). Exactly solved eigenfunction problems are then used to study the empirical attributes of asset pricing models that incorporate long-run risk, external habit persistence, and rare disasters. Specific quantitative implications are developed for the variance of the permanent and the transitory components, the return behavior of the long-term (infinite-maturity) bond, and the comovement between the transitory and the permanent components of SDFs.

KEY WORDS: Stochastic discount factors, permanent component, transitory component, variance bounds, asset pricing models, eigenfunction problems.

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[†]Tel.: +1-301-405-2261; fax: +1-301-405-0359. *E-mail address:* gbakshi@rhsmith.umd.edu

[‡]Tel.: +1-614-292-8477; fax: +1-614-292-2359. *E-mail address:* chabi-yo_1@fisher.osu.edu

1. Introduction

In an important contribution, Alvarez and Jermann (2005) lay the foundations for deriving bounds when the stochastic discount factor (hereby, SDF), from an asset pricing model, can be decomposed into a permanent component and a transitory component (see also the contribution of Hansen and Scheinkman (2009) and Hansen, Heaton, and Li (2008)). Our objective in this paper is to propose a bounds framework, in the context of permanent and transitory components, and we show that our bounds are fundamentally different from Alvarez and Jermann (2005). A rationale for developing our bounds is that the bounds postulated in Alvarez and Jermann (2005) hinge on the return properties of the long-term discount bond, the risk-free bond, and a generic equity portfolio. Our approach generalizes to an asset space with a dimension greater than three, and it departs by relying on the variance-measure (as in Hansen and Jagannathan (1991)).

Building on our treatment, we address the key questions of (i) how useful our bounds are in assessing asset pricing models, and (ii) how our bounds compare, in the theoretical and empirical dimension, with the counterparts in Alvarez and Jermann (2005) (wherever applicable). The first question is pertinent to economic modeling, while the second question is pertinent to the incremental value-added and the tightness of our bounds in empirical applications. In particular, the gist of our analysis is that the bound implications for the permanent component of the SDFs in the three-asset case is considerably weaker than those reported in our multiple-asset context.

In our setup, we develop a lower bound on the variance of the permanent component of the SDF and a lower bound on the variance of the transitory component of SDFs, and then a lower bound on the variance of the ratio of the permanent to the transitory components of SDFs. The lower bound on the variance of the permanent/transitory component can be viewed as a generalization of the Alvarez and Jermann (2005) bounds. Our lower bound on the variance of the ratio of the permanent to the transitory components of SDFs allows us to assess whether asset pricing models are capable of describing the additional dimension of joint pricing across markets, and has no analog in Alvarez and Jermann (2005). A salient feature of our bounds is that they incorporate information from average returns as well as the variance-covariance matrix of returns from a generic set of assets. Probing further, our analysis reveals that non-normalities in the data can bring deviations between our measure and the Alvarez and Jermann (2005) counterpart.

An essential link between our variance bounds and asset pricing is the eigenfunction problem of Hansen and Scheinkman (2009), which facilitates an analytical expression for the permanent and transitory components of the SDF for an asset pricing model. The posited bounds, in conjunction with an analytical solution

to the eigenfunction problem, can provide a setting for discerning whether the time-series properties underlying an asset pricing model are consistent with observed data from financial markets.

To provide wider foundations for our empirical examination, we focus on the eigenfunction problem for a broad class of asset pricing models, namely, (i) long-run risk (Bansal and Yaron (2004) as also parameterized in Kelly (2009)), (ii) external habit persistence (Campbell and Cochrane (1999) as also parameterized in Bekaert and Engstrom (2010)), and (iii) rare disasters (as parameterized in Backus, Chernov, and Martin (2011)). Our derived solution for the eigenfunction problem for the five models illustrate that a range of distributional properties for the permanent and the transitory components of SDFs can be accommodated within our applications. Then we are led to ask a question of economic interest: What can be learned about asset pricing models that consistently price the long-term discount bond, the risk-free bond, the equity market, and a multitude of other assets.¹ The importance of studying long-run risk, external habit persistence, and rare disasters under a common platform is also recognized by Hansen (2009).

A number of empirical insights can be garnered about the relative performance of asset pricing models in the context of equity and bond data. One implication of our findings is that the variance of the permanent component of the SDF in models is of an order lower than the corresponding bound reflected in returns of bonds, equity market, and portfolios sorted by size and book-to-market. Reliability of this conclusion is affirmed through p -values constructed from a simulation procedure. The model with rare disasters exhibits the highest variance of the permanent component, a feature linked to occasional consumption crashes.

Next, we observe that the transitory component of the SDF in models fails to meet the lower variance bound restriction. This bound is tied to the Sharpe ratio of the long-term bond. We further characterize the expected return (variance) of a long-term bond, and find that while each model quantitatively depicts the real return of a risk-free bond, they fall short in reproducing the long-term bond properties. This metric of inconsistency matters since prospective models typically offer reconciliation with the equity premium, while often ignoring the return of long-term bonds. Our inquiry uncovers that the misspecified transitory component is a source of the limitation in describing the return behavior of long-term bonds.

With regard to joint dynamics of the permanent and the transitory components of SDFs, we make two key observations. First, we show that it is possible to recover the comovement between the permanent and the transitory components from historical data without making any distributional assumptions. While

¹In addition, our framework is amenable to investigating the suitability of SDFs to address several asset pricing puzzles together, for instance, by combining elements of the value premium, the equity premium, the risk-free return, and the bond risk premium. In this regard, our work is related to, among others, Ait-Sahalia, Parker, and Yogo (2004), Beeler and Campbell (2009), Cochrane and Piazzesi (2005), Koijen, Lustig, and Van Nieuwerburgh (2010), Lettau and Wachter (2007), Routledge and Zin (2010), Santos and Veronesi (2010), and Zhang (2005).

the data tells us that the two components should move in the same direction, our analysis reveals that this feature is not easily imitated by the models. For instance, the models exhibit a negative (or near-zero) association between the permanent and the transitory components. Second, we show that the model-based variance of the ratio of the permanent to the transitory components of the SDFs is insufficiently high, conveying the need to describe more plausibly the joint dynamics of the returns of bonds and other assets.

Pivotal to modeling, we also establish the tightness of our variance bound on the permanent component relative to Alvarez and Jermann (2005). Specifically, it is the sharpness of the bound, in combination with an expanded set of empirically relevant test assets, that can improve the ability of the bounds approach to differentiate between models. The eigenfunction problem offers further guidance for asset price modeling.

One may additionally ask: What is the advantage of employing our approach to assess asset pricing models versus matching some moments of the returns data? The heart of the bounds approach is that bounds are derived under the assumption that the permanent and the transitory components of SDFs correctly price a set of assets. On the other hand, the approach of matching some sample moments of the returns data may fail to internalize broader aspects of asset return dynamics. Our variance bounds could be adopted as a complementary device to examine the validity of a model, apart from matching sample moments, slope coefficients from predictive regressions, and correlations. In this sense, our approach retains the flavor of the Hansen and Jagannathan (1991) bounds, and yet offers the tractability of comparing the variance of the permanent and the transitory components of SDFs to those implied by the data.

Taken all together, our non-parametric approach highlights the dimensions of difficulty in reconciling observed asset returns under some adopted parameterizations of asset pricing models. Our work belongs to a list of studies that searches for the least misspecified model, and explores the implications of asset pricing models, as outlined, for example, in Beeler and Campbell (2009), Ferson, Nallareddy, and Xie (2010), Hansen, Heaton, and Li (2008), Wachter (2011), and Yang (2010, 2011).

The paper is organized as follows. Section 2 develops the theoretical framework for our bounds, and Section 3 investigates the relevance of our bounds in the context of Alvarez and Jermann (2005). Section 4 derives the permanent and transitory components of the SDF for models in the class of long-run risk, external habit, and rare disasters. Section 5 is devoted to empirically analyzing aspects of asset pricing models. Conclusions and possible extensions are offered in Section 6.

2. Bounds on the permanent and transitory components

This section presents theoretical bounds related to the unconditional variance of permanent and transitory components of SDFs, and the ratio of the permanent to the transitory components of SDFs.

Since asset pricing models often face a hurdle of explaining asset market data based on unconditional bounds, we develop our results in terms of unconditional bounds, instead of the sharper conditional bounds. Appendix B presents the conditional variance bounds for completeness.

2.1. Motivation for developing bounds based on the properties of a generic set of asset returns

We adopt notations similar to that in Alvarez and Jermann (2005), and let $\{M_t\}$ be the process of strictly positive pricing kernels. As in Duffie (1996) and Hansen and Richards (1987), we use the absence of arbitrage opportunities to specify the current price of an asset that pays D_{t+k} at time $t+k$ as

$$V_t [D_{t+k}] = E_t \left(\frac{M_{t+k}}{M_t} D_{t+k} \right), \quad (1)$$

where $E_t(\cdot)$ represents the conditional expectation operator. The SDF from t to $t+1$ is represented by $\frac{M_{t+1}}{M_t}$.

To differentiate returns offered by different types of assets, we first define $R_{t+1,k}$ as the gross return from holding, from time t to $t+1$, a claim to one unit of the numeraire to be delivered at time $t+k$. Then, the return from holding a discount bond with maturity k from time t to $t+1$, and the long-term discount bond is, respectively,

$$R_{t+1,k} = \frac{V_{t+1} [1_{t+k}]}{V_t [1_{t+k}]} \quad \text{and} \quad R_{t+1,\infty} \equiv \lim_{k \rightarrow \infty} R_{t+1,k}. \quad (2)$$

The case of $k=1$ in equation (2) corresponds to the gross return of a risk-free bond.

Next we denote by R_{t+1} , the gross return of a broad equity portfolio or the equity market. Such an asset captures the aggregate equity risk premium, and plays a key role in the formulations of Alvarez and Jermann (2005).

To build on the asset space, it is of interest to define $\mathbf{R}_{t+1} = (R_{t+1,1}, R_{t+1}, \mathbf{R}_{t+1,a})$, which constitutes an $n+2$ -dimensional vector of gross returns. The n -dimensional vector $\mathbf{R}_{t+1,a}$ contains a finite number of risky assets that excludes the equity market and the long-term discount bond.

Consider the set of SDFs that consistently price $n+3$ assets, that is, the long-term discount bond, the

risk-free bond, the equity market, and additionally n other risky assets,

$$\mathbb{S} \equiv \left\{ \frac{M_{t+1}}{M_t} : E \left(\frac{M_{t+1}}{M_t} R_{t+1,\infty} \right) = 1 \text{ and } E \left(\frac{M_{t+1}}{M_t} \mathbf{R}_{t+1} \right) = \mathbf{1} \right\}, \quad (3)$$

where $\mathbf{1}$ is a vector of ones conformable with \mathbf{R}_{t+1} , and $E(\cdot)$ represents the *unconditional* expectation operator. The mean of the SDF is given by $\mu_m \equiv E \left(\frac{M_{t+1}}{M_t} 1_{t+1} \right)$.

In the discussions to follow, we refer to $Var[u] = E(u^2) - (E(u))^2$ as the variance-measure for some random variable u , and use it to quantify the importance of transitory and permanent components of SDFs.

The L -measure-based bounds framework in Alvarez and Jermann (2005) is intended specifically for a long-term discount bond, a risk-free bond, and a single equity portfolio, whereas equation (3) allows one to expand the asset space to a dimension beyond three. Our motivation for studying variance-measure-based bounds, as opposed to L -measure-based bounds, will be articulated in Section 3.

There are reasons to expand the set of assets in our theoretical and empirical analysis. Note that the crux of the bounds approach is that the bounds are derived under the assumption that (i) the transitory component prices the long-term bond, (ii) the permanent component prices other assets, and (iii) the SDF correctly prices the entire set of assets, while accounting for the relation between the transitory and the permanent components. In this context, we show that the bounds implication for the three-asset case of Alvarez and Jermann (2005) is considerably weaker than those reported in our empirical illustrations involving many risky assets.

Equally important, our variance bounds treatment is intended for all sorts of assets, which is in the vein of, among others, Shanken (1987), Hansen and Jagannathan (1991), Snow (1991), Cecchetti, Lam, and Mark (1994), Kan and Zhou (2006), Luttmer (1996), Balduzzi and Kallal (1997), and Bekaert and Liu (2004). In this sense, the viability of asset pricing models can now be judged by their ability to satisfactorily accommodate the risk premium on a spectrum of traded assets, and not just the equity premium.

2.2. *Bounds on the permanent component of SDFs*

Alvarez and Jermann (2005, Proposition 1), and Hansen and Scheinkman (2009, Corollary 6.1) show that any SDF can be decomposed into a transitory component and a permanent component. Inspired by their analyses, we presume that there exists a decomposition of the pricing kernel M_t into a transitory and

a permanent component of the type:

$$M_t = M_t^T M_t^P, \quad \text{with} \quad E_t(M_{t+1}^P) = M_t^P. \quad (4)$$

The permanent component M_t^P is a martingale, while the transitory component M_t^T is a scaled long-term interest rate. In particular,

$$R_{t+1,\infty} = \left(\frac{M_{t+1}^T}{M_t^T} \right)^{-1}, \quad (5)$$

which follows from Alvarez and Jermann (2005, Assumptions 1 and 2, and their proof of Proposition 2). The transitory component prices the long-term bond with $E\left(\frac{M_{t+1}^T}{M_t^T} R_{t+1,\infty}\right) = 1$. Completing the description of the decomposition (4), the transitory and permanent components of the SDF can be correlated.

We assume that the variance-covariance matrix of \mathbf{R}_{t+1} , $\mathbf{R}_{t+1}/R_{t+1,\infty}$, $\mathbf{R}_{t+1}/R_{t+1,\infty}^2$ are each nonsingular. Our proof follows.

Proposition 1 *Suppose the relations in equations (4) and (5) hold. Then the lower bound on the unconditional variance of the permanent component of SDFs $\frac{M_{t+1}^P}{M_t^P} \in \mathbb{S}$ is*

$$\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] \geq \sigma_{pc}^2 \equiv \left(\mathbf{1} - E \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right) \right)' \Omega^{-1} \left(\mathbf{1} - E \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right) \right), \quad (6)$$

where $\Omega \equiv \text{Var} \left[\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right]$.

Proof: See Appendix A. ■

We note that our variance bound is general and not specific to the Alvarez and Jermann (2005) or Hansen and Scheinkman (2009) decomposition. The bound applies to any SDF that can be decomposed into a permanent and a transitory component.

The σ_{pc}^2 in inequality (6) is computable, given the return time-series of long-term bond, risk-free bond, equity market, and other assets (for instance, equity portfolios sorted by size and book-to-market, and bonds of various maturities). Our development facilitates a variance bound on the permanent component of the SDF that can accommodate the return properties of the desired number of assets contained in \mathbf{R}_{t+1} .

Inequality (6) bounds the variance of the permanent component of the SDF, which can be a useful object for understanding what time-series assumptions are necessary to achieve consistent risk pricing across a multitude of asset markets. In addition to using the information content of returns across different asset classes, the bound is essentially model-free and can be employed to evaluate the empirical relevance of the

permanent component of any SDF, regardless of its distribution. The permanent component of the SDF from any asset pricing model should respect the bound in (6).

The quadratic form for the lower bound in (6) departs fundamentally from the corresponding L -measure-based bound for the three-asset case in Alvarez and Jermann (2005, equation (4)):

$$\text{Lower bound on the permanent component in Alvarez and Jermann is } E(\log(R_{t+1}/R_{t+1,\infty})). \quad (7)$$

Under their approach, it is the expected return spread that unpins the bound. On the other hand, the stipulated bound in (6) combines information from the vector of average returns (scaled by the long-term bond return) and the variance-covariance matrix of returns.

Although the Hansen and Jagannathan (1991) bound was not developed to differentiate between the permanent and the transitory components of the SDF, the variance bound on the permanent component σ_{pc}^2 is receptive to an interpretation, as in the Hansen and Jagannathan (1991) bound (see also Cochrane and Hansen (1992)). To appreciate this feature, note that $\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \simeq \mathbf{1} + \log(\mathbf{R}_{t+1}) - \log(R_{t+1,\infty}) \mathbf{1}$, and suppose that \mathbf{R}_{t+1} consists of a risk-free bond, equity market, and equity portfolios sorted by size and book-to-market. Then following Hansen and Jagannathan (1991) and Cochrane (2005, Sections 5.5–5.6), the variance bound on the permanent component of SDFs can be interpreted as the maximum Sharpe ratio when the investment opportunity set is composed of the risk-free bond with excess return relative to the long-term bond, the equity market with excess return relative to the long-term bond, and the equity portfolios with excess return relative to the long-term bond.

In a manner akin to the lower bound on the volatility of the SDF in Hansen and Jagannathan (1991), we establish, via equation (22), that σ_{pc}^2 corresponds to the volatility of the permanent component of SDFs exhibiting the lowest variance. Thus, there is a key difference between the lower bound on the variance of SDFs and the lower bound on the variance of the permanent component of SDFs.

Koijen, Lustig, and Van Nieuwerburgh (2010) highlight the economic role of $\text{Var}\left[\frac{M_{t+1}^P}{M_t^P}\right] / \text{Var}\left[\frac{M_{t+1}}{M_t}\right]$ in affine models. When both the permanent and transitory components of the SDF are lognormally distributed, they specifically show that $\text{Var}\left[\frac{M_{t+1}^P}{M_t^P}\right] / \text{Var}\left[\frac{M_{t+1}}{M_t}\right]$ implied within their model is almost perfectly correlated with the Cochrane and Piazzesi (2005) factor. Under our formulation, the analysis of Appendix A leads to a bound on the ratio:

$$\frac{\text{Var}\left[\frac{M_{t+1}^P}{M_t^P}\right]}{\text{Var}\left[\frac{M_{t+1}}{M_t}\right]} \geq \frac{\sigma_{pc}^2}{\sigma_{pc}^2 + 1 - \mu_m^2}. \quad (8)$$

The upshot is that the lower bound on the *size* of the permanent component of the SDFs in equation (8) is also analytically distinct from its *L*-measure and three-asset based counterpart in Alvarez and Jermann (2005, equation (5)).

The relative usefulness of our bounds in empirical applications is the focal point of the exercises in Sections 3.3, 5.1, and 5.6. Our contention in Section 3.3 is also that the bounds appear quantitatively stable to how the return of a long-term bond is proxied.

2.3. Bounds on the transitory component of SDFs

While a central constituent of any SDF is the permanent component, a second constituent is the transitory component, which equals the inverse of the gross return of an infinite-maturity discount bond and governs the behavior of interest rates. Absent a transitory component, the excess returns of discount bonds are zero, which contradicts empirical evidence (e.g., Fama and Bliss (1987), Campbell and Shiller (1991), and Cochrane and Piazzesi (2005)).

To gauge the ability of SDFs to explain aspects of the bond market data, while consistently pricing the remaining set of assets \mathbf{R}_{t+1} , as described in equation (3), we provide a lower bound on the variance of the transitory component of SDFs.

Proposition 2 *Suppose the relations in equations (4) and (5) hold. Then the lower bound on the unconditional variance of the transitory component of SDFs $\frac{M_{t+1}}{M_t} \in \mathbb{S}$ is*

$$\text{Var} \left[\frac{M_{t+1}^T}{M_t^T} \right] \geq \sigma_{tc}^2 \equiv \frac{\left(1 - E \left(\frac{M_{t+1}^T}{M_t^T} \right) E(R_{t+1,\infty}) \right)^2}{\text{Var}[R_{t+1,\infty}]} . \quad (9)$$

Proof: See Appendix A. ■

The variance of the transitory component of any SDF that consistently prices the long-term bond should be higher than the bound depicted in equation (9). There is no analog in Alvarez and Jermann (2005) to our analytical bound on $\text{Var} \left[\frac{M_{t+1}^T}{M_t^T} \right]$.

We can also characterize the *upper bound* on the size of the transitory component (see the proof to

equation (A18) in Appendix A), namely, the counterpart to Alvarez and Jermann (2005, Proposition 3), as:

$$\frac{\text{Var} \left[\frac{M_{t+1}^T}{M_t^T} \right]}{\text{Var} \left[\frac{M_{t+1}}{M_t} \right]} \leq \frac{\text{Var} \left[\frac{1}{R_{t+1,\infty}} \right]}{\left(\mathbf{1} - \mu_m E(\mathbf{R}_{t+1}) \right)' (\text{Var}[\mathbf{R}_{t+1}])^{-1} \left(\mathbf{1} - \mu_m E(\mathbf{R}_{t+1}) \right)}, \quad (10)$$

where the denominator of (10) is the lower bound on the variance of the SDFs, as in Hansen and Jagannathan (1991, equation (12)).

The quantity on the right-hand side of equation (9) is tractable and readily computable from the returns data. A particular observation is that the bound in (9) is a parabola in $\left(E \left(\frac{M_{t+1}^T}{M_t^T} \right), \sigma_{ic}^2 \right)$ space, and σ_{ic}^2 is positively associated with the square of the Sharpe ratio of the long-term bond. We potentially contribute by using our bound (9) to assess the bond market implications of asset pricing models.

2.4. Bounds on the ratio of the permanent to the transitory components of SDFs

A third feature of SDFs is their ability to link the behavior of the bond market to other markets. In this regard, a construct suitable for understanding cross-market relationships is the variance of the ratio $\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T}$, which captures the importance of the permanent component relative to the transitory component.

Proposition 3 *Suppose the relations in equations (4) and (5) hold. Then the lower bound on the unconditional variance of the ratio of the permanent to the transitory components of SDFs is*

$$\text{Var} \left[\frac{\frac{M_{t+1}^P}{M_t^P}}{\frac{M_{t+1}^T}{M_t^T}} \right] \geq \sigma_{pt}^2 \equiv \left(\mathbf{1} - \mu_{pt} E \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2} \right) \right)' \Sigma^{-1} \left(\mathbf{1} - \mu_{pt} E \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2} \right) \right), \quad (11)$$

where $\Sigma \equiv \text{Var} \left[\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2} \right]$ and $\mu_{pt} \equiv E \left(\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right)$.

Proof: See Appendix A. ■

For a given $E \left(\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right)$, Proposition 3 provides the lower bound on $\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right]$, the purpose of which is to assess whether the SDFs can explain joint pricing restrictions across markets. The importance of linking markets has been emphasized by Campbell (1986), Campbell and Ammer (1993), Baele, Bekaert, and Inghelbrecht (2010), Colacito, Engle, and Ghysels (2010), and David and Veronesi (2009). Our bound in Proposition 3 is new, with no counterpart in Alvarez and Jermann (2005).

Further note that $Var \left[\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right]$ can be cast in terms of the mixed moments of the permanent and the transitory components of the SDF. Later we derive an explicit implication regarding a form of dependence between $\frac{M_{t+1}^P}{M_t^P}$ and $\frac{M_{t+1}^T}{M_t^T}$ that can be inputted from the data. In one extreme, a pricing framework devoid of a comovement between the permanent and the transitory components amounts to a flat term structure, a feature falsified by the data.

2.5. Summary and further discussion

Recapping our results so far, the first type of bounds we propose is on the variance of the permanent component of SDFs. Such a bound is beneficial for characterizing the restrictions imposed by time-series assumptions in asset pricing models. Next, we derive a lower bound on the variance of the transitory component of SDFs, which is a potentially useful tool for disentangling which time-series assumptions on the consumption growth process can more aptly capture observed features of the bond market. Finally, we provide a lower bound on the variance of the relative contribution of the permanent to the transitory component of SDFs, which can establish the link between bond pricing and the pricing of other assets.

Our bounds, thus, provide a set of dimensions along which one could appraise asset pricing models.

When a diagnostic test, for instance, the Hansen and Jagannathan (1991) variance bound, rejects an asset pricing model, it often fails to ascribe model failure specifically to the inadequacy of the permanent component of SDFs, the transitory component of SDFs, or to a combination of both. Thus, our framework offers the pertinent measures to investigate which dimension of the SDF can be modified, when the goal is to capture return variation either in a single market or across markets. We elaborate on this issue by analytically solving an eigenfunction problem for asset pricing models in the class of long-run risk, external habit persistence, and rare disasters, and then invoking our Propositions 1, 2, and 3.²

Each of the bounds derived in Propositions 1, 2, and 3 are unconditional bounds. In Appendix B, we scale the returns by conditioning variables and propose bounds that incorporate conditioning information.

²In some models, asset returns are driven by both long- and short-run risk and, hence, the permanent and transitory components of SDFs can impact valuations (e.g., Hansen (2009) and Hansen and Scheinkman (2009)). For example, the SDF implied by the Bansal and Yaron (2004) model, and its extensions, suggest restrictions on the returns of value and growth stocks (e.g., Ai and Kiku (2010)), the cross-section of bond returns, and contingent claims written on market variance, in addition to the return of a broad equity portfolio. A comparable set of restrictions can be deduced from the external habit persistence model (e.g., Campbell and Cochrane (1999) and Santos and Veronesi (2010)), and the rare disasters model.

3. Comparison with Alvarez and Jermann (2005) bounds

Germane to the bounds developed in Section 2 are two central questions: In what way are these bounds distinct from the corresponding bounds in Propositions 2 and 3 in Alvarez and Jermann (2005)? How useful are our proposed bounds, and how do they compare, say, along the empirical dimension, with Alvarez and Jermann (2005)?

To address these questions, we first note that Alvarez and Jermann define the L -measure (entropy) of a random variable u as

$$L[u] \equiv f[E(u)] - E(f[u]), \quad \text{with} \quad f[u] = \log(u). \quad (12)$$

Using $L[u]$ as a measure of volatility, Alvarez and Jermann develop their bounds in terms of the L -measure. Under their characterizations, a one-to-one correspondence exists between the L -measure and the variance-measure of $\log(u)$, when u is distributed lognormally, as in $L[u] = \frac{1}{2}Var[\log(u)]$.

Still, discrepancies between the two dispersion measures can get magnified under departures from lognormality, for example Kelly (2009), Bekaert and Engstrom (2010), and Backus, Chernov, and Martin (2011), as we show, where neither the SDF nor the permanent component are lognormally distributed, as captured by $|L[u] - \frac{1}{2}Var[\log(u)]| > 0$.

The distinction between our treatment and in that of Alvarez and Jermann (2005) is hereby studied from three perspectives. First, we illustrate some differences in the permanent and transitory components of the SDFs across the L -measure and the variance-measure in example economies. Second, we highlight some conceptual differences in the characterization of bounds, focusing on the three-asset setting of Alvarez and Jermann, i.e., we rely on the return properties of the long-term bond, the risk-free bond, and the equity market (which amounts to specializing equation (3)). Third, we provide a comparison of bounds in the data dimension while maintaining the three-asset setting.

3.1. L -measure versus the variance-measure in example economies

Example 1 (Alvarez and Jermann (2005, page 1981)). Suppose the SDF is $\frac{M_{t+1}}{M_t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$, for time preference parameter β and coefficient of relative risk aversion γ , and consumption growth is independently

and identically distributed. In this economy, interest rates are some constant $r_0 > 0$, hence,

$$R_{t+1,1} = R_{t+1,\infty} = 1 + r_0, \quad \text{which implies} \quad \frac{M_{t+1}}{M_t} = \frac{1}{(1 + r_0)} \frac{M_{t+1}^P}{M_t^P}. \quad (13)$$

The L -measure of Alvarez and Jermann implies

$$L \left[\frac{M_{t+1}^P}{M_t^P} \right] = L \left[\frac{M_{t+1}}{M_t} \right], \quad \frac{L \left[\frac{M_{t+1}^P}{M_t^P} \right]}{L \left[\frac{M_{t+1}}{M_t} \right]} = 1, \quad \frac{L \left[\frac{M_{t+1}^T}{M_t^T} \right]}{L \left[\frac{M_{t+1}}{M_t} \right]} = 0. \quad (14)$$

On the other hand, the variance-measure implies

$$\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] = (1 + r_0)^2 \text{Var} \left[\frac{M_{t+1}}{M_t} \right], \quad \frac{\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right]}{\text{Var} \left[\frac{M_{t+1}}{M_t} \right]} = (1 + r_0)^2, \quad \frac{\text{Var} \left[\frac{M_{t+1}^T}{M_t^T} \right]}{\text{Var} \left[\frac{M_{t+1}}{M_t} \right]} = 0, \quad (15)$$

which follows from (13) and invokes $R_{t+1,\infty}^{-1} = M_{t+1}^T / M_t^T$. ♣

Example 1 illustrates that the size of the permanent component is unity under the L -measure, while it is $(1 + r_0)^2$ under the variance-measure. As the SDF only has shocks to the permanent component, the upper bound on the size of the transitory component is zero under both treatments.

Example 2 (Alvarez and Jermann (2005, page 1997)). Suppose the log of the pricing kernel evolves according to an AR(1) process,

$$\log(M_{t+1}) = \log(\beta) + \zeta \log(M_t) + \varepsilon_{t+1}, \quad \text{where} \quad \varepsilon_{t+1} \sim \mathcal{N}(0, \sigma_\varepsilon^2), \quad (16)$$

with $|\zeta| < 1$. In this setting, the log excess bond return at maturity k is $\log \left(\frac{R_{t+1,k}}{R_{t+1,1}} \right) = \frac{\sigma_\varepsilon^2}{2} (1 - \zeta^{2(k-1)})$. It can be shown that the permanent component of the SDF is

$$\frac{M_{t+1}^P}{M_t^P} = \exp \left(-\zeta^2 \frac{\sigma_\varepsilon^2}{2} + \zeta \varepsilon_{t+1} \right), \quad \text{and, hence, } M_t^P \text{ is a martingale} \quad E \left(\frac{M_{t+1}^P}{M_t^P} \right) = 1. \quad (17)$$

The transitory component is

$$\frac{M_{t+1}^T}{M_t^T} = \exp \left(-\frac{\sigma_\varepsilon^2}{2} \left(2 - \zeta^{2(k-1)} \right) - \log(\beta) - (\zeta - 1) \log(M_{t+1}) \right), \quad \text{for a large } k. \quad (18)$$

Now,

$$L \left[\frac{M_{t+1}^P}{M_t^P} \right] = \zeta^2 \frac{\sigma_\varepsilon^2}{2}, \quad \text{and} \quad L \left[\frac{M_{t+1}^T}{M_t^T} \right] = (1-\zeta)^2 \frac{\sigma_\varepsilon^2}{2} \left(\frac{1-\zeta^{2(t+1)}}{1-\zeta^2} \right). \quad (19)$$

Further, under our treatment, the variance of the permanent component is

$$\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] = \exp(\zeta^2 \sigma_\varepsilon^2) - 1, \quad (20)$$

and the variance of the transitory component is

$$\begin{aligned} \text{Var} \left[\frac{M_{t+1}^T}{M_t^T} \right] &= \left(\exp \left((\zeta-1)^2 \frac{\sigma_\varepsilon^2}{2} \left(\frac{1-\zeta^{2(t+1)}}{1-\zeta^2} \right) \right) - 1 \right) \\ &\quad \times \exp \left(-\sigma_\varepsilon^2 (2-\zeta^{2(k-1)}) - 2\zeta^{(t+1)} \log(\beta) + (1-\zeta)^2 \frac{\sigma_\varepsilon^2}{2} \left(\frac{1-\zeta^{2(t+1)}}{1-\zeta^2} \right) \right), \end{aligned} \quad (21)$$

and they do not coincide with the L -theoretic counterparts in (19). ♣

Figure 1 plots the L -measure and the variance-measure of the permanent and the transitory components of the SDF by varying the persistence parameter ζ , while keeping the maturity, k , of the long-term bond to be 20 years or 29 years. We set $\beta = 0.998$ and σ_ε^2 to 0.15 or 0.40. First, a higher level of shock uncertainty σ_ε^2 translates into a higher $\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right]$ and $\text{Var} \left[\frac{M_{t+1}^T}{M_t^T} \right]$. Second, there is a disparity between $\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right]$ and $L \left[\frac{M_{t+1}^P}{M_t^P} \right]$, which is most pronounced at higher ζ . Lastly, the calculations do not materially change in response to our input for the long-term bond maturity.

[Fig. 1 about here.]

While omitted to save space, the basic message, namely, that there are intrinsic differences between the two dispersion measures, also obtains under conditioning information.

3.2. Under what situations, the framework based on the variance-measure may be preferable?

To appreciate what the variance-measure-based bounds potentially add beyond the L -measure-based bounds, the following aspects merit discussion.

As can be inferred from the properties of the L -measure (see Appendix A in Alvarez and Jermann (2005)), the bounds they derive are designed for the situation in which the SDFs correctly price *at most*

three assets: the long-term discount bond, the risk-free bond, and the equity portfolio, with no obvious way to generalize to the dimension of asset space beyond three.

We offer bounds that are derived under the condition that the permanent and the transitory components of the SDFs correctly price a finite number of assets. Our evidence in Section 5.1 puts on a firmer footing the notion that the variance bound on the permanent component is also considerably sharper. In fact, the bound gets tighter when the dimension of the asset space is increased. The novelty of the variance-measure-based bounds is that they exploit the information in both the average returns and the variance-covariance matrix of asset returns.

Even in the setting of the long-term discount bond, the risk-free bond, and the equity portfolio, some conceptual differences between the two treatments can be highlighted. Among a set of $\frac{M_{t+1}^P}{M_t^P}$ that correctly price asset returns, we denote by $\frac{M_{t+1}^{*P}}{M_t^{*P}}$ the permanent component of SDFs with the lowest variance. It is

$$\frac{M_{t+1}^{*P}}{M_t^{*P}} = 1 + \begin{pmatrix} 1 - E\left(\frac{R_{t+1,1}}{R_{t+1,\infty}}\right) \\ 1 - E\left(\frac{R_{t+1}}{R_{t+1,\infty}}\right) \end{pmatrix}' \left(\text{Var} \left[\begin{pmatrix} \frac{R_{t+1,1}}{R_{t+1,\infty}} \\ \frac{R_{t+1}}{R_{t+1,\infty}} \end{pmatrix} \right] \right)^{-1} \begin{pmatrix} \frac{R_{t+1,1}}{R_{t+1,\infty}} - E\left(\frac{R_{t+1,1}}{R_{t+1,\infty}}\right) \\ \frac{R_{t+1}}{R_{t+1,\infty}} - E\left(\frac{R_{t+1}}{R_{t+1,\infty}}\right) \end{pmatrix}. \quad (22)$$

Hence, in the three-asset setting, equation (22) can be viewed as the solution to the problem:

$$\min_{\frac{M_{t+1}^P}{M_t^P}} \text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] \text{ subject to } E\left(\frac{M_{t+1}^P}{M_t^P} \frac{R_{t+1,1}}{R_{t+1,\infty}}\right) = 1, \quad E\left(\frac{M_{t+1}^P}{M_t^P} \frac{R_{t+1}}{R_{t+1,\infty}}\right) = 1, \text{ and } E\left(\frac{M_{t+1}^P}{M_t^P}\right) = 1. \quad (23)$$

In general, the variance of $\frac{M_{t+1}^{*P}}{M_t^{*P}}$ equals the lower bound on the variance of the permanent component of SDFs. Therefore, our analysis makes it explicit that, regardless of the probability distribution of $\frac{M_{t+1}^P}{M_t^P}$, our results pertain to bounds on variance. The message worth conveying is that the variance of the permanent (and transitory) component from an asset pricing model is denominated in the same units of riskiness as the bounds recovered from the data, a trait that our framework shares also with the studies of Hansen and Jagannathan (1991), Bernardo and Ledoit (2000), and Cochrane and Saa-Requejo (2000).

Exhibiting a specific distributional property, the Alvarez and Jermann (2005) lower bound is the L -measure of the permanent component: $E(\log(R_{t+1}/R_{t+1,\infty}))$. Yet it is not possible to find an analytical expression for the permanent component of SDFs, namely, $\frac{\tilde{M}_{t+1}^P}{\tilde{M}_t^P}$ such as $L\left[\frac{\tilde{M}_{t+1}^P}{\tilde{M}_t^P}\right] = E(\log(R_{t+1}/R_{t+1,\infty}))$.

To establish this argument, we use the definition of the L -measure, whereby

$$L \left[\frac{\tilde{M}_{t+1}^P}{\tilde{M}_t^P} \right] = \log(1) - E \left(\log \left(\frac{\tilde{M}_{t+1}^P}{\tilde{M}_t^P} \right) \right), \quad (\text{since } E \left(\frac{\tilde{M}_{t+1}^P}{\tilde{M}_t^P} \right) = 1). \quad (24)$$

Therefore, we can at most deduce that $E \left(\log \left(\frac{\tilde{M}_{t+1}^P}{\tilde{M}_t^P} \right) \right) = E \left(\log \left(\frac{R_{t+1,\infty}}{R_{t+1}} \right) \right)$, and it may not be possible to recover $\frac{\tilde{M}_{t+1}^P}{\tilde{M}_t^P}$ analytically in terms of asset returns. It seems that the lower bound on the L -measure of the permanent component of SDFs does not satisfy the definition of the L -measure.

Apart from the aforementioned, the framework based on the variance-measure could assume economic significance when diagnosing asset pricing models. Suppose a calibration approach yields similar magnitudes for average asset returns, but different asset variances for two asset pricing models. The models are indistinguishable according to the L -measure, as the bounds are operationalized through average returns. At the same time, the variance-measure-based bounds can help to achieve a sharper differentiation across asset pricing models that happen to generate different implications for average asset returns and return variances.

In sum, the bounds on the permanent and the transitory components of SDFs can impose a set of testable restrictions, and therefore help to narrow the search for the least misspecified asset pricing model.

3.3. Lessons from a comparison with Alvarez and Jermann (2005) bounds in the data dimension

Still, some empirical questions remain with respect to observed data in the financial markets: What is gained by generalizing the L -measure-based setup in Alvarez and Jermann (2005), when applied to the long-term discount bond, the risk-free bond, and the equity market? In what sense do our proposed bounds quantitatively differ from Alvarez and Jermann (2005)? How sensitive are our variance bounds when one surrogates $R_{t+1,\infty}$ by the return of a bond with a reasonably long maturity?

To facilitate these objectives, here we follow Alvarez and Jermann (2005) in the choice of three assets, the monthly sample period of 1946:12 to 1999:12 (637 observations), as well as expressing the point estimates from the L -measure and, hence, from the variance-measure in *annualized* terms in Table 1. Specifically, we rely on data (<http://www.econometricsociety.org/suppmat.asp?id=61&vid=73&iid=6&aid=643>) on the return of a long-term bond, the return of a risk-free bond, and the return of a single equity portfolio (optimal growth portfolio based on 10 CRSP Size-Decile portfolios and the equity market).

The maturity of a long-term bond is guided by data considerations and allowed to take a value of 20,

25, or 29 years. Also reported in Table 1 are the 90% confidence intervals (in square brackets), which are based on 50,000 random samples of size 637 from the data and a block bootstrap.³

There are three lessons that can be drawn. First, our exercise suggests that our bounds are broadly different from Alvarez and Jermann, irrespective of which bond maturity is employed in the construction of the bounds. Second, when the maturity of the bond is altered from 20 to 29 years, the bound on $L \left[\frac{M_{t+1}^P}{M_t^P} \right]$ varies little, whereas the bound σ_{pc}^2 on $Var \left[\frac{M_{t+1}^P}{M_t^P} \right]$ varies somewhat from 0.799 to 0.933. This differential sensitivity can be ascribed to the fact that σ_{pc}^2 (as in equation (6)) is determined by both average returns and the variance-covariance matrix of asset returns. In contrast, altering the bond maturity modifies the average return of the long-term bond only slightly and, thus, leaves the sample analog of equation (7) unchanged.

Finally, the discrepancy between the bounds on $L \left[\frac{M_{t+1}^P}{M_t^P} \right]$ and $\frac{1}{2} Var \left[\frac{M_{t+1}^P}{M_t^P} \right]$ is large in the data, implying that the permanent component *is far from* being distributed normally in logs. This finding suggests that higher-moments of the permanent component of SDFs may be relevant to asset pricing.

4. Eigenfunction problem and the transitory and permanent components of SDFs in asset pricing models

This section contributes by deriving an analytical solution to the eigenfunction problem of Hansen and Scheinkman (2009) to determine the transitory and permanent components of the SDF. Featured is a strand of asset pricing models that have drawn considerable support in their ability to depict stylized properties of aggregate equity market returns and risk-free bond returns.

Specifically, we focus on models in (a) the long-run risk class (Bansal and Yaron (2004)), (b) the external habit persistence class (Campbell and Cochrane (1999)), and (c) the rare consumption disasters class (Rietz (1988) and Barro (2006)). Within each class we adopt a generalization of the SDF that invokes departure from log-normality, the purpose of which is to enrich the setting for researching the broader relevance of bounds in Propositions 1, 2, and 3.

The asset pricing models delineated next, along with those of Bansal and Yaron (2004) and Campbell

³We sample the return observations in blocks of consecutive observations instead of individual observations, to account for any dependency in the returns data. The data is divided into k^* non-overlapping blocks of length l^* , where sample size $T^* = k^*l^*$. Then we generate $b = 50,000$ random samples of size T^* from the original data, where the sampling is based on 12 blocks. We use each of the generated samples to obtain the estimates of variance bounds, generically denoted by v_b , for $b = 1, \dots, 50,000$. Following Davison and Hinkley (1997), the 90% bootstrap confidence interval for the variance bound v is obtained as $\left[2v - q \left(1 - \frac{q}{2} \right), 2v - q \left(\frac{q}{2} \right) \right]$, where $a = 0.10$, and $q \left(\frac{q}{2} \right)$ is the quantile such that 2500 of the bootstrap statistics $\{v_b\}_{b=1, \dots, 50,000}$ are less than or equal to $q \left(\frac{q}{2} \right)$.

and Cochrane (1999), are at the core of the empirical investigation. Throughout most of this analysis, we strive to adopt notations close to those in the original model formulations.

4.1. Solution to the eigenfunction problem for a model incorporating long-run risk

To determine the transitory and permanent components of the SDF through the eigenfunction problem, consider the modification of the long-run risk model proposed in Kelly (2009). The distinguishing attribute is that the model incorporates heavy-tailed shocks to the evolution of (log) nondurable consumption growth g_{t+1} , which are governed by a tail risk state variable Λ_t .

$$g_{t+1} = \mu + x_t + \sigma_g \sigma_t z_{g,t+1} + \sqrt{\Lambda_t} W_{g,t+1}, \quad x_{t+1} = \rho_x x_t + \sigma_x \sigma_t z_{x,t+1}, \quad (25)$$

$$\sigma_{t+1}^2 = \bar{\sigma}^2 + \rho_\sigma (\sigma_t^2 - \bar{\sigma}^2) + \sigma_\sigma z_{\sigma,t+1}, \quad \Lambda_{t+1} = \bar{\Lambda} + \rho_\Lambda (\Lambda_t - \bar{\Lambda}) + \sigma_\Lambda z_{\Lambda,t+1}, \quad (26)$$

$$z_{g,t+1}, z_{x,t+1}, z_{\sigma,t+1}, z_{\Lambda,t+1} \sim \text{i.i.d } \mathcal{N}(0, 1), \quad W_{g,t+1} \sim \text{Laplace}(0, 1). \quad (27)$$

Following Bansal and Yaron (2004), x_t is a persistently varying component of the expected consumption growth rate, and σ_t^2 is the conditional variance of consumption growth with unconditional mean $\bar{\sigma}^2$.

The z shocks are standard normal and independent. In addition to gaussian shocks, the consumption growth depends on non-gaussian shocks W_g , where the W_g shocks are Laplace-distributed variables with mean zero and variance 2, and independent. W_g shocks are independent of z shocks. The model maintains the tradition of Epstein and Zin (1991) recursive utility.

Proposition 4 *The transitory and permanent components of the SDF in the model of Kelly (2009) are*

$$\frac{M_{t+1}^T}{M_t^T} = \mathfrak{v} \exp(-c_1(x_{t+1} - x_t) - c_2(\sigma_{t+1}^2 - \sigma_t^2) - c_3(\Lambda_{t+1} - \Lambda_t)), \quad \text{and} \quad \frac{M_{t+1}^P}{M_t^P} = \frac{M_{t+1}}{M_t} / \frac{M_{t+1}^T}{M_t^T}, \quad (28)$$

where \mathfrak{v} is defined in (C29), and the coefficients c_1 , c_2 , and c_3 are defined in (C31)–(C33). The expression for $\frac{M_{t+1}}{M_t}$ is presented in (C15) of Appendix C.

Proof: See Appendix C. ■

Equation (28) is obtained by solving the eigenfunction problem of Hansen and Scheinkman (2009, Corollary 6.1):

$$E_t \left(\frac{M_{t+1}}{M_t} \frac{1}{M_{t+1}^e} \right) = \frac{\mathfrak{v}}{M_t^e}, \quad (29)$$

where the parameter \mathfrak{v} is the dominant eigenvalue. The conjectured M_{t+1}^e that satisfies (29) determines

$$M_t^T = v^t M_t^e \text{ and } M_t^P = M_t / M_t^T.$$

While the transitory component of the SDF is lognormally distributed, the permanent component of the SDF, and the SDF itself, are not lognormally distributed. The non-gaussian shocks W_g are meant to amplify the tails of the permanent component of the SDF and the SDF. Equation (28) makes it explicit how the sources of the variability in the transitory and permanent components of the SDF can be traced back to (i) the specification of the preferences, and (ii) the dynamics of the fundamentals.

Both the transitory and the permanent components of the SDF embed market volatility, and are linked to consumption volatility dynamics. Related to this feature, Beeler and Campbell (2009) make the observation that persistence in consumption growth volatility is central to generating realistic patterns across stock and bond markets.

4.2. Solution to the eigenfunction problem for a model incorporating external habit persistence

Bekaert and Engstrom (2010) propose a variant of the Campbell and Cochrane (1999) model where (i) the dynamics of consumption growth g_t consists of two fat-tailed skewed distributions, and (ii) the SDF is

$$\frac{M_{t+1}}{M_t} = \beta \exp(-\gamma g_{t+1} + \gamma(q_{t+1} - q_t)), \quad (30)$$

where β is the time preference parameter, γ is the curvature parameter, and $q_t \equiv \log\left(\frac{C_t}{C_t - H_t}\right) = \log\left(\frac{1}{S_t}\right)$. For external habit H_t and consumption C_t , the variable S_t is the surplus consumption ratio and, hence, q_t represents the log of the *inverse* surplus consumption ratio (see also Santos and Veronesi (2010) and Borovicka, Hansen, Hendricks, and Scheinkman (2011)). Uncertainty in this economy is described by

$$g_{t+1} = \bar{g} + x_t + \sigma_{gp} \omega_{p,t+1} - \sigma_{gn} \omega_{n,t+1}, \quad x_t = \rho_x x_{t-1} + \sigma_{xp} \omega_{p,t} + \sigma_{xn} \omega_{n,t}, \quad (31)$$

$$q_{t+1} = \mu_q + \rho_q q_t + \sigma_{qp} \omega_{p,t+1} + \sigma_{qn} \omega_{n,t+1}, \quad \omega_{p,t+1} = ge_{t+1} - p_t \text{ and } \omega_{n,t+1} = be_{t+1} - n_t, \quad (32)$$

$$p_t = \bar{p} + \rho_p (p_{t-1} - \bar{p}) + \sigma_{pp} \omega_{p,t}, \quad n_t = \bar{n} + \rho_n (n_{t-1} - \bar{n}) + \sigma_{nn} \omega_{n,t}, \quad (33)$$

$$ge_{t+1} \sim \text{Gamma}(p_t, 1), \quad be_{t+1} \sim \text{Gamma}(n_t, 1), \quad (34)$$

where p_t and n_t are the conditional mean of the good environment and bad environment shocks denoted by ge_{t+1} and be_{t+1} , respectively. The distinguishing attribute of the model is that it incorporates elements of external habit persistence together with long-run consumption risk.

Proposition 5 *The transitory and permanent components of the SDF in the model of Bekaert and Engstrom (2010) are*

$$\frac{M_{t+1}^T}{M_t^T} = \mathfrak{v} \exp(\gamma(q_{t+1} - q_t) - c_2(x_{t+1} - x_t) - c_3(p_{t+1} - p_t) - c_4(n_{t+1} - n_t)), \quad \text{and} \quad \frac{M_{t+1}^P}{M_t^P} = \frac{M_{t+1}}{M_t} / \frac{M_{t+1}^T}{M_t^T}, \quad (35)$$

where \mathfrak{v} is defined in (D5), the coefficients c_2 through c_4 are defined in (D6)–(D7), and $\frac{M_{t+1}}{M_t}$ is as in (30).

Proof: See Appendix D. ■

Besides generating the statistically observed risk-free return, the equity premium, and moments of consumption growth, among the noteworthy model features are its ability to generate time-variation in risk premiums and consistency with risk-neutralized equity return moments. One implication of the solution (35) is that the model embeds a transitory component of the SDF which comoves with changes in q_t . In contrast, the permanent component is detached from variations in q_t .

4.3. Solution to the eigenfunction problem for a model incorporating rare disasters

We consider a version of the asset pricing model of Rietz (1988) and Barro (2006). Uncertainty about consumption growth is modeled following Backus, Chernov, and Martin (2011) (see also Gabaix (2009), Gourio (2010), and Wachter (2011)) as

$$\log(g_{t+1}) = w_{t+1} + z_{t+1}, \quad \text{with } w_{t+1} \sim \text{i.i.d. } \mathcal{N}(\mu, \sigma^2), \quad z_{t+1} | J_{t+1} \sim \text{i.i.d. } \mathcal{N}(\theta J_{t+1}, \delta^2 J_{t+1}), \quad \text{and} \quad (36)$$

$$J_{t+1} \text{ is i.i.d. Poisson random variable with density } \frac{\omega^j}{j!} e^{-\omega} \text{ for } j \in \{0, 1, 2, \dots\}, \text{ and mean } \omega. \quad (37)$$

The distinguishing attribute of this model is that z_{t+1} produces sporadic crashes in consumption growth and serves as a device to produce a fat-tailed distribution of consumption growth. In the model, (w_{t+1}, z_{t+1}) are mutually independent over time, and the SDF is of the form (under the assumption of time-separable power utility):

$$\frac{M_{t+1}}{M_t} = \beta g_{t+1}^{-\gamma} = \exp(\log(\beta) - \gamma w_{t+1} - \gamma z_{t+1}), \quad (38)$$

where β is time preference parameter, and γ is the coefficient of relative risk aversion.

Proposition 6 *The transitory and permanent components of the SDF in the rare disasters model are*

$$\frac{M_{t+1}^T}{M_t^T} = \frac{1}{R_{t+1,\infty}}, \quad \text{and} \quad \frac{M_{t+1}^P}{M_t^P} = R_{t+1,\infty} \exp(\log(\beta) - \gamma w_{t+1} - \gamma z_{t+1}), \quad (39)$$

where $R_{t+1,\infty}^{-1} = \exp(\log(\beta) - \gamma\mu + \frac{1}{2}\gamma^2\sigma^2) \sum_{j=0}^{\infty} \frac{e^{-\omega j}}{j!} \exp(-\gamma\theta j + \frac{1}{2}\gamma^2\delta^2 j)$ is a constant.

Proof: See Appendix E. ■

Within the setting of (36)–(37), the model inherits the property that $\frac{M_{t+1}}{M_t}$ and $\frac{M_{t+1}^P}{M_t^P}$ are not lognormally distributed. It can be shown that $E\left(\frac{M_{t+1}^P}{M_t^P}\right)^\ell = (\beta R_{t+1,\infty})^\ell \exp\left(\omega\left(e^{0.5\gamma^2\delta^2\ell^2 - \gamma\theta\ell} - 1\right) + 0.5\gamma^2\sigma^2\ell^2 - \gamma\mu\ell\right)$, for $\ell = 2, 3, \dots$, which furnishes the moments of the permanent component, while maintaining $E\left(M_{t+1}^P\right) = M_t^P$.

5. Empirical application to asset pricing models

To lay the groundwork for the empirical examination, the analysis of this section starts by highlighting the tightness of our lower bound on the variance of the permanent component in the context of a finite number of assets. Then we elaborate on the performance of asset pricing models under our metrics of evaluation, including the lower bound restrictions on the permanent and transitory components of SDFs.

5.1. Description of the set of asset returns and the tightness of the variance bound

Stepping outside of three-asset setting in Table 1, recall that \mathbf{R}_{t+1} is the return vector that is correctly priced by the SDF along with $R_{t+1,\infty}$ (see equation (3)). Whereas our variance bound on the permanent component σ_{pc}^2 (see equation (6)) exhibits dependence on $E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)$ and $Var\left[\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right]$, tractable expressions are not yet available for L -measure-based bounds when there are more than three assets.

Two questions are pertinent to our development and to our empirical comparison of asset pricing models: (1) Does the variance bound σ_{pc}^2 get incrementally sharper when the SDF is required to correctly price more assets? (2) How does the tightness of the new bound fare relative to Alvarez and Jermann (2005)?

To answer these questions, we consider a set of monthly equity and bond returns over the period 1932:01 to 2010:12. Our choice of the start date circumvents missing observations. The source of risk-free bond and equity returns is the data library of Kenneth French, while the source of intermediate and long-term government bond returns is Morningstar (Ibbotson). Real returns are computed by deflating nominal

returns by the Consumer Price Index inflation. For our illustration, we consider four different \mathbf{R}_{t+1} :

- (i) SET A: Risk-free bond, equity market, intermediate government bond, and 25 Fama-French equity portfolios sorted by size and book-to-market;
- (ii) SET B: Risk-free bond, equity market, intermediate government bond, ten size-sorted, and six size and book-to-market, sorted equity portfolios;
- (iii) SET C: Risk-free bond, equity market, intermediate government bond, and six size and book-to-market, sorted equity portfolios; and,
- (iv) SET D: Risk-free bond, equity market, and intermediate government bond.

[Fig. 2 about here.]

The lower bound on the permanent component, σ_{pc}^2 , displayed in Figure 2, imparts two conclusions. First, σ_{pc}^2 declines from SET A to SET D, suggesting that expanding the number of assets in \mathbf{R}_{t+1} leads to a bound that is intrinsically tighter. Second, the estimates of σ_{pc}^2 generated from SET A through D are sharper relative to Alvarez and Jermann (2005, equation (4)), which is represented by the entry marked AJ. Analogous departures are manifested over the 1946:01 to 2010:12 subsample.

The gist of this exercise is that $\sigma_{pc}^2/2$ based on the return properties of SET A (SET D) is about 15 (four) times sharper than the L -measure-based lower bound on the permanent component, thereby substantiating its incremental value in asset pricing applications.

5.2. A setting for model evaluation, granted that each model calibrates to some data attributes

Turning to the themes of our study, we exploit the exactly solved eigenfunction problems in Propositions 4, 5, and 6 to provide the building blocks for our empirical study in a few ways:

- Our interest is in assessing the ability of a model to produce realistic permanent and transitory components of the SDF and whether their variance respects the proposed lower bounds. We rely on a statistic generated from a simulation procedure and the associated p -value;
- Analytical solutions to the eigenfunction problem facilitate a quantitative implication regarding the return of a long-term bond, whereby

$$E(r_{t+1,\infty}) = E(R_{t+1,\infty} - 1) = E\left(\frac{M_t^T}{M_{t+1}^T} - 1\right), \quad \text{Var}[r_{t+1,\infty}] = E\left(\frac{M_t^T}{M_{t+1}^T} - E\left(\frac{M_t^T}{M_{t+1}^T}\right)\right)^2. \quad (40)$$

The return moments of a long-term bond implicit in an asset pricing model are, thus, computable, and could be benchmarked to those from a suitable proxy to ascertain their plausibility. Campbell and Viceira (2001), among others, provide an impetus to develop asset pricing models that also generate reasonable return behavior for the long-term bond.

The bounds yardstick, when combined with (40), could serve as a differentiating diagnostic when competing asset pricing models (i) each calibrate closely to the mean and standard deviation of consumption growth, and (ii) offer conformity with the historical real return of risk-free bond and the real return of equity market.

Additionally, and equally relevant, we extract the implication of each model for the comovement between the permanent and transitory components of the SDF, in a manner to be described shortly via equations (41)–(42), and examine its merit relative to the one imputed from the data.

Primitive parameters are chosen consistently according to the models of Kelly (2009), Bekaert and Engstrom (2010), and Backus, Chernov, and Martin (2011), as displayed in Tables Appendix-I through Appendix-III, respectively, together with those of Bansal and Yaron (2004) and Campbell and Cochrane (1999). Even though we refrain from presenting the full-blown solution to the eigenfunction problem for the last two models to save space, our objective in implementing all five models is to offer a unifying picture of how each asset pricing model performs under our yardsticks of evaluation.

While the conditional variances are amenable to closed-form characterization, the unconditional variances are tractable only via simulations, except for the rare disaster model given i.i.d. uncertainties. Accounting for this feature, each model is simulated using the dynamics of consumption growth and other state variables, for instance, as in (25)–(27), for the long-run risk model of Kelly (2009), over a single simulation run of 360,000 months (30,000 years). Then we build the time series of model-specific $\left\{ \frac{M_{t+1}^P}{M_t^P} \right\}$ and $\left\{ \frac{M_{t+1}^T}{M_t^T} \right\}$, according to the solution of the eigenfunction problem, and we calculate the unconditional moments. Using a single simulation run to infer the population values for the entities of interest is consistent with, among others, the approach of Campbell and Cochrane (1999) and Beeler and Campbell (2009).

5.3. Models could appeal to utility specifications and dynamics of fundamentals that magnify the permanent component of SDFs

Our thrust is to examine whether models produce sensible dynamics for the permanent component of the SDF. Such an analysis can enable insights into how economic fundamentals are linked to SDFs,

and how the performance of an asset pricing model could be improved by altering the properties of the permanent component of SDFs.

At the outset, we report, for each model, the variance of the permanent component of the SDF, $Var [M_{t+1}^P/M_t^P]$, in Panel A of Table 2. Reported in the final column is the lower bound σ_{pc}^2 calculated based on SET A, along with the 90% confidence intervals, shown in square brackets, from a block bootstrap, when sampling is done with 15 blocks. It is helpful to think in terms of σ_{pc}^2 from SET A, since this set corresponds to a universe of equity portfolios whose return properties are the subject of much scrutiny in the empirical asset pricing research (e.g., Malloy, Moskowitz, and Vissing-Jorgensen (2009), Koijen, Lustig, and Van Nieuwerburgh (2010), and references therein).

Our implementations reveal that the monthly $Var [M_{t+1}^P/M_t^P]$ implied by the models of Kelly (2009) and Bansal and Yaron (2004) is 0.0374 and 0.0342, respectively, while that of the models of Bekaert and Engstrom (2010) and Campbell and Cochrane (1999) is 0.0280 and 0.0234, respectively. The most pronounced value of 0.0580 is obtained under the model of Backus, Chernov, and Martin (2011). We also utilize the closed-form expression for $Var [M_{t+1}^P/M_t^P]$ in the case of the rare disasters model to confirm that our single simulation run can reliably approximate the population variance.

Going further, we formulate the restriction: $\sigma_{pc}^2 - Var [M_{t+1}^P/M_t^P] \leq 0$ for a candidate asset pricing model, which allows one to elaborate on whether a model respects the lower bound (beyond eye-balling estimates; see also Cecchetti, Lam, and Mark (1994)). Then inference regarding this restriction can be drawn via repeated simulations (e.g., Patton and Timmermann (2010)). For this purpose, we rely on a finite-sample simulation of 948 months (1932:01 to 2010:12), and we choose 200,000 replications. The proportion of the replications satisfying $\sigma_{pc}^2 - Var [M_{t+1}^P/M_t^P] \leq 0$ can be interpreted as a p -value for a one-sided test. This p -value is shown in curly brackets in Table 2, and a low p -value indicates rejection. Pertinent to this exercise, our evidence reveals that the variance of the permanent component of the SDF from each model fails to meet the lower bound restriction of 0.1254 *per month*. Importantly, the reported p -values provide some support for the contention that the model based variance of the permanent component are reliably lower than σ_{pc}^2 . A likewise conclusion emerges when σ_{pc}^2 is computed from SET B and SET C (not reported). For example, the highest p -value of 0.099 (0.164) corresponds to the model with rare disasters for SET A (SET C).

While the models differ markedly in their capacity to generate a volatile permanent component, it is noteworthy that each asset pricing model parametrization reasonably mimics the equity premium and the real risk-free return, while simultaneously calibrating closely to the first-two moments of consumption

growth (see Panels B through C of Table 2). Thus, there appears to be a tension, within a model, between matching the sample average of equity returns and risk-free returns, versus generating a minimum volatility of the permanent component stipulated by theory and as inferred from the data. We further expand on this point when discussing the implication of the models for long-term bond returns.

While the backbone of the Kelly (2009) model is to incorporate tails in nondurable consumption growth, it tends to elevate the variance of the permanent component, modestly relative to Bansal and Yaron (2004). Specifically, in the formulation of Kelly (2009), the distributions of log SDF and the log of the permanent component of the SDF are symmetric with fat tails. From the documented results, we infer that long-run risk models could accentuate the variability in the permanent component of SDFs by incorporating more flexible tail properties in the SDF and in the permanent component of the SDF. We recognize nonetheless that there is insufficient evidence favoring the presence of skewness in nondurable consumption growth. Thus, an avenue to enrich long-run risk models is to incorporate durable consumption in the dynamics of the real economy. In the spirit of our results, Yang (2010) provides evidence that durable consumption growth is left-skewed and exhibits time-varying volatility.⁴

Judging by our results, the approach in Bekaert and Engstrom (2010) does not appear to substantially improve upon the variance of the permanent component relative to Campbell and Cochrane (1999). This finding is somewhat unexpected, given a flexible modeling of consumption growth as well as the term structure of interest rates, in conjunction with a modified process for marginal utility.

What could be a rationale for the highest estimate of the variance of the permanent component of the SDF in an asset pricing model with rare disasters? This outcome deserves two comments. First, the high variability of the permanent component arises from occasional crashes in the consumption growth process, and this success comes at the expense of somewhat unrealistic consumption growth higher-moments. For example, the skewness of the annualized log consumption growth in the model with rare disasters is -11.02 , with a kurtosis of 145.06 , as discussed also in Backus, Chernov, and Martin (2011, Table III). Second, the model with rare disasters admits the most right-skewed and fat-tailed distribution of the permanent component among all of our models.

Regarding the rare disasters approach to modeling asset prices, one message may be worth highlighting: Lowering the severity and frequency of consumption crashes in the model may be desirable for bringing tail size and asymmetry of consumption growth more in line with the data. At the same time, one could

⁴Recent studies include, among others, Bansal, Dittmar, and Kiku (2009), Constantinides and Ghosh (2008), Croce, Lettau, and Ludvigson (2008), Kaltenbrunner and Lochstoer (2010), Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010), Lettau and Ludvigson (2004), and Zhou and Zhu (2009).

consider generalizations of power utility function, for instance, those encompassing external habit, that would then aid in further raising the volatility of the permanent component of the SDF.

In summary, although the list of models under consideration is far from exhaustive, they still embed different utility specifications and specify the long-run and short-run risk in distinct ways (see also Hansen (2009)). Yet, a common thread among the models is their inability to meet the lower bound restriction on the variance of the permanent component of SDFs. The larger lesson being that the SDF of the asset pricing models could be refined to accommodate a permanent component featuring a larger variance. Our metrics of assessment can provide a perspective on how the dynamics of consumption growth and fundamentals could be modified, or preferences could be generalized, to improve the working of asset pricing models.

5.4. *Success of models is confounded by their lack of consistency with aspects of the bond market*

The methods of this paper allow us to contemplate two additional questions: (i) Which asset pricing model conforms with the lower bound on the transitory component σ_{tc}^2 (equation (9) of Proposition 2)?, and (ii) What are the quantitative implications of each model for the behavior of the long-term bond returns?

It bears emphasizing that while the lower bound σ_{pc}^2 is independent of the mean of the permanent component by construction, the lower bound on the transitory component exhibits dependence on the estimate of the mean $E(M_{t+1}^T/M_t^T)$ across models. With this feature in mind, we report σ_{tc}^2 , in relation to the estimate of $E(M_{t+1}^T/M_t^T)$, which is then compared to $Var[M_{t+1}^T/M_t^T]$ produced by a model in Table 3. The crux of our finding is that models generate insufficient $Var[M_{t+1}^T/M_t^T]$ relative to the lower bound σ_{tc}^2 . For example, the variance of the transitory component for the Bansal and Yaron (Bekaert and Engstrom) model is 1.9×10^{-3} (3.5×10^{-5}) with a mean of 1.0147 (1.0002), whereas the comparable lower bound is substantially more elevated at 4.3×10^{-1} (7.7×10^{-3}).

In essence, this part of our inquiry suggests that the adopted parametrization of fundamentals may not adequately characterize the transitory component of the SDF. This conclusion is confirmed through the p -values that examine the restriction $\sigma_{tc}^2 - Var[M_{t+1}^T/M_t^T] \leq 0$. We again obtain this p -value for each model by appealing to a finite sample simulation with 200,000 replications. It is seldom that $Var[M_{t+1}^T/M_t^T]$ is greater than the minimum volatility restriction on the transitory component, and the reported p -values are all below 0.01. To recapitulate, our approach potentially identifies a source of the misalignment of asset pricing models with aspects of the bond market data.

Moving to the second question of interest, Table 4 summarizes the model implications for the expected

return, and standard deviation, of the long-term bond. The key point to note is that calibrations geared toward replicating the equity return and the risk-free return can miss basic aspects of the long-term bond market. For example, the Bansal and Yaron, and the Bekaert and Engstrom, models imply an expected annualized long-term bond return of -14.26% and -0.21% , respectively. Given the real return of our proxy for long-term bond averages 2.43% , there appears to be a gap between the prediction of the models and the data.

That the misspecified transitory component is a source of the incongruity of models with bond market data is also revealed through the standard deviation of long-term bond returns. Specifically, when the model fail to generate plausible dynamics of the transitory component, they can introduce a wedge between the volatility of long-term bond returns implied by a model versus the data counterpart. Here, it can be seen that the Bansal and Yaron (Bekaert and Engstrom) model implies an annualized standard deviation of 12.41% (2.36%), which deviates from the data value of 8.87% . While the extant literature has largely focused efforts on rationalizing equity return volatility (see, for instance, Schwert (1989) and Wachter (2011)), far less effort has been devoted to rationalizing bond return volatility.

The goal to study the behavior of the return of a long-term bond has the flavor of Beeler and Campbell (2009, Table IX), whereby we explore the possible misalignment of long-term (infinite maturity) bond returns across models using the transitory component and by solving the eigenfunction problem.⁵ Our push to consider the long-term bond return as one criterion in model assessment is also guided by Alvarez and Jermann (2005). Specifically, they argue that, absent the permanent component of the SDF, the maximum risk premium in the economy is reflected in the long-term bond return. The versatility of an asset pricing model also lies in its ability to generate credible properties of the long-term bond returns, as also elaborated in a different context by Campbell and Viceira (2001).

5.5. Models face challenges capturing the relation between the permanent and the transitory components implicit in the data

Motivating the mixed performance of the models so far, we further ask: What can be discerned about the relation between the transitory and the permanent components, given that they jointly price a given set of assets? In the analysis to follow, we address this question from two angles.

⁵When building models, matching the real return of a long-term bond can offer two advantages relative to matching the slope of the term structure of inflation-indexed bonds. First, for most of the models under consideration, there is no closed-form characterization of inflation-indexed bonds (but it could be calculated recursively). Second, the use of inflation-indexed bonds can entail concerns regarding a relatively short history (Piazzesi and Schneider (2006)).

We first examine the comovement between the transitory and the permanent components embedded in an asset pricing model. More concretely, the solution to the eigenfunction problem allows us to deduce the left-hand side below (using $E(M_{t+1}^P/M_t^P) = 1$):

$$\text{Cov} \left[\frac{M_{t+1}^P}{M_t^P}, \frac{M_{t+1}^T}{M_t^T} \right] = E \left(\frac{M_{t+1}}{M_t} \right) - E \left(\frac{1}{R_{t+1,\infty}} \right). \quad (41)$$

Equivalently, it imparts the following quantitative implication (the conditional expectation of the SDF is the inverse of the return of the risk-free bond):

$$\frac{\text{Cov} \left[\frac{M_{t+1}^P}{M_t^P}, \frac{M_{t+1}^T}{M_t^T} \right]}{\text{Var} \left[\frac{M_{t+1}^T}{M_t^T} \right]} = \frac{E \left(\frac{1}{R_{t+1,1}} \right) - E \left(\frac{1}{R_{t+1,\infty}} \right)}{\underbrace{\text{Var} \left[\frac{1}{R_{t+1,\infty}} \right]}_{\text{can be inputted from the data, and is model-free}}}. \quad (42)$$

Note that the left-hand side of (42) can be recovered as the slope coefficient from the OLS regression: $\frac{M_{t+1}^P}{M_t^P} = a_0 + b_0 \left(\frac{M_{t+1}^T}{M_t^T} \right) + \varepsilon_{t+1}$ in the model-specific simulations. The variance of M_{t+1}^T/M_t^T is a convenient normalization that enables the quantity on the right-hand side to be inputted from the data. Asset pricing theory is silent on the joint distribution of the permanent and the transitory components of SDFs.

Table 5 summarizes (i) the slope coefficient imputed from the data, (ii) the estimate of b_0 from the regression in a single simulation run, and (iii) 95th and 5th percentiles of the b_0 estimates from a finite sample simulation with 200,000 replications.

Observe, however, that the numerator on the right-hand side of equation (42), is to first-order, the expected return spread of the long-term bond over the risk-free bond (since $1/(1+x) \approx 1-x$). Thus, a fundamental trait of the data is that it supports a positive covariance between the permanent and the transitory components of SDFs. The value of b_0 implied from the data is 1.96.

A comparison of the slope coefficients obtained through our simulations elicits the observation that, on average, three (one) out of five models produce a negative (positive) slope coefficients whose magnitudes contrast the data counterpart. The 95th and 5th percentiles for the b_0 distribution in the models of Bansal and Yaron and Bekaert and Engstrom are negative, suggesting that $b_0 < 0$ appears to be an intrinsic feature of their models. Overall, this evidence illustrates the ambivalence of the models in replicating the association between the permanent and the transitory components implicit in the data.

In addition, recognize in our context that a statistic to gauge the performance of models in capturing the joint pricing of bonds and other assets is the lower bound on $\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right]$. Complementing the

picture from Tables 2 and 3, the results in Table 6 show that asset pricing models are unable to describe the behavior of joint pricing implicit in the bound. Specifically, $Var \left[\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right]$ in the models of Bansal and Yaron (Kelly) is estimated to be 0.0449 (0.0375), whereas the stipulated lower bounds, accounting for the estimate of the mean $E \left(\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right)$, are much higher. A likewise departure is evident in the external habit persistence class, where $Var \left[\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right]$ for the Bekaert and Engstrom (Campbell and Cochrane) is estimated to be 0.0298 (0.0234). The p -values that examine the restriction $\sigma_{pt}^2 - Var \left[\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right] \leq 0$ are typically small, refuting another restriction suggested by the theory.

The relation between the permanent and the transitory components of the SDF has information content for modeling asset prices. A particular lesson to be gleaned is that asset pricing models could be enriched to better describe the joint dynamics of the transitory and the permanent components of SDFs as also reflected in bond risk premium, a positive equity risk premium, and the risk premium on a broad spectrum of assets.

5.6. Models may reproduce some asset market phenomena, but find it onerous to satisfy bounds

Synthesizing elements of Tables 2, 3, and 6, we pose three additional questions in turn.

First, why is it that formulations from an asset pricing model can come close to duplicating the observed equity premium, and at the same time not satisfy the lower bound on the permanent component of the SDF for a broad set of asset returns? We expand on this seemingly contradictory observation by computing the lower bound on the permanent component when the SDF is required to satisfactorily price the long-term bond, the risk-free bond, and the market equity (returning to the set of assets in Alvarez and Jermann (2005)). For this set of assets, $\sigma_{pc}^2 = 0.0185$, and the minimum p -value examining the restriction $\sigma_{pc}^2 - Var \left[\frac{M_{t+1}^P}{M_t^P} \right] \leq 0$ is 0.314 across models. The nuance is that asset pricing models can meet the restrictions for a narrower set of assets, but not the broader set of assets that includes the 25 (6) Fama-French equity portfolios (as in SET A (SET C)).

As one enlarges the set of assets in \mathbf{R}_{t+1} , the permanent component of the SDF must be generalized to cope with the risk premiums on a wider array of assets. This prompts the next question: What is our incremental value beyond the L -measure in comparing asset pricing models? The analysis below (with p -values computed as before) speaks directly to the relevance of the tightness of our bounds in empirical applications:

	Kelly	Bansal-Yaron	Bekaert-Engstrom	Campbell-Cochrane	Rare Disasters	Lower bound
$L \left[\frac{M_{t+1}^P}{M_t^P} \right]$	0.0176	0.0171	0.0100	0.0115	0.0032	0.0041
p -value	{0.986}	{0.990}	{0.999}	{0.994}	{0.254}	

In particular, our results point to a scenario in which models generate high (low) p -values with the L -measure (variance-measure), as seen by comparing the entries for p -values in Table 2. It is the stringency of the lower variance bound, together with a set of empirically relevant assets in \mathbf{R}_{t+1} , as also demonstrated through Figure 2, that enhances the flexibility of the bounds approach to differentiate between competing asset pricing models.

Building on our observations, Lemma 1 in Appendix F finally asks whether an asset pricing model can satisfy the Hansen and Jagannathan (1991) bound for a broad set of assets, for instance, SET A, and yet fails the lower bound on the permanent component in Proposition 1. The Lemma establishes the relevance of a bound on $Cov\left[\left(\frac{M_{t+1}^P}{M_t^P}\right)^2, \left(\frac{M_{t+1}^T}{M_t^T}\right)^2\right]$, with the implication that further theoretical work is needed to get a firmer grasp of the joint distribution of M_{t+1}^P/M_t^P and M_{t+1}^T/M_t^T . More can be learned about the functioning of asset pricing models through the lens of permanent and transitory components of SDFs.

6. Conclusions and extensions

This paper presents a variance bounds framework in the context of permanent and transitory components of stochastic discount factors. Under this framework, the transitory component prices the long-term bond, whereas the permanent component prices other assets. Besides, the stochastic discount factors correctly prices the full set of assets while consistently characterizing the dependence between its transitory and permanent constituents. At the center of our approach are three theoretical results, one related to a lower bound on the variance of the permanent component, another on the lower bound on the variance of the transitory component, and also a lower bound on the variance of the ratio of the permanent to the transitory components of stochastic discount factors.

Instrumental to the tasks at hand, we establish the tightness of our variance bounds relative to Alvarez and Jermann (2005), and we show that our bounds can be useful in asset pricing applications. A specific conclusion is that bound implications for the permanent component of the stochastic discount factors in the setting of Alvarez and Jermann (2005) are considerably weaker than those reported in our context of generic set of assets. Our analysis furnishes bounds that incorporate information from average returns as well as the variance-covariance matrix of returns.

Combining the variance bounds with the eigenfunction problem offers guidance for asset price modeling in several ways. First, we present the solution to the eigenfunction problem for five asset pricing models in the class of long-run risk, external habit persistence, and rare disasters. This solution justifies

the calculation of the moments of the transitory and the permanent components, as well as all its mixed moments. Second, we corroborate that models face a particular impediment satisfying the lower bound restrictions imposed by our bounds, even when the models are successful in matching the equity premium and the return of the risk-free bond. Third, exploiting the solution to the eigenfunction problem, we find that the models are not compatible with the return properties of the long-term bond. Finally, while the data supports a positive comovement between the transitory and the permanent components, our analysis reveals that this feature is not easily reconciled within our parametrization of asset pricing models.

Our work could be extended. While our focus is directed toward primary assets, one could refine the analysis to include stochastic discount factors that also satisfactorily price out-of-the-money put options on the market as well as claims on market variance. We appreciate that as monthly out-of-the-money index put options often expire worthless, the put option returns exhibit relatively high volatility compared to index returns. Hence, incorporating such claims can impose further hurdles on asset pricing models.

Finally, when an asset pricing model is required to comply with the bound restrictions imposed by the returns data, it could offer an alternative way to recover parameters of preferences together with those governing fundamentals. Achieving identification through the information contained in the low frequency ingredients of stochastic discount factors could refine the quest for a better understanding of asset returns.

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Appendix A: Proofs of unconditional bounds

In the results that follow, we provide proofs of Propositions 1, 2, and 3.

Proof of Proposition 1. The proof is by construction. Recognize that

$$E \left(\left(\frac{M_{t+1}^P}{M_t^P} - E \left(\frac{M_{t+1}^P}{M_t^P} \right) \right) \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right) \right) \right) = \mathbf{1} - E \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right), \quad (\text{A1})$$

where the right-hand side of (A1) is obtained by noting that $E \left(\frac{M_{t+1}^P}{M_t^P} \right) = 1$ and $R_{t+1,\infty} = (M_{t+1}^T/M_t^T)^{-1}$.

Denote

$$\Omega \equiv E \left(\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right) \right) \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right) \right)' \right) \quad \text{and} \quad \mathbf{B} \equiv \mathbf{1} - E \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right). \quad (\text{A2})$$

Multiply the right-hand side of (A1) by $\mathbf{B}'\Omega^{-1}$ to obtain:

$$\mathbf{B}'\Omega^{-1}\mathbf{B} = E \left(\left(\frac{M_{t+1}^P}{M_t^P} - E \left(\frac{M_{t+1}^P}{M_t^P} \right) \right) \left(\mathbf{B}'\Omega^{-1} \frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E \left(\mathbf{B}'\Omega^{-1} \frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right) \right) \right), \quad (\text{A3})$$

$$= \text{Cov} \left[\frac{M_{t+1}^P}{M_t^P} - E \left(\frac{M_{t+1}^P}{M_t^P} \right), \mathbf{B}'\Omega^{-1} \frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E \left(\mathbf{B}'\Omega^{-1} \frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right) \right], \quad (\text{A4})$$

$$\leq \left(\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] \right)^{1/2} \times \left(\text{Var} \left[\mathbf{B}'\Omega^{-1} \frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right] \right)^{1/2}. \quad (\text{A5})$$

Given that $\text{Var} \left[\mathbf{B}'\Omega^{-1} \frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}} \right]$ equals $\mathbf{B}'\Omega^{-1}\mathbf{B}$, our application of the Cauchy-Schwarz inequality implies that $\left(\mathbf{B}'\Omega^{-1}\mathbf{B} \right)^{1/2} \leq \left(\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] \right)^{1/2}$. Thus, we have established the bound in equation (6) of Proposition 1. ■

Proof of the lower bound on the size of the permanent component in equation (8). By an elementary calculation

$$\text{Var} \left[\frac{M_{t+1}}{M_t} \right] = \text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \frac{M_{t+1}^T}{M_t^T} \right] = E \left(\left(\frac{M_{t+1}^P}{M_t^P} \right)^2 \left(\frac{1}{R_{t+1,\infty}} \right)^2 \right) - \mu_m^2. \quad (\text{A6})$$

Thus, $\text{Var} \left[\frac{M_{t+1}}{M_t} \right] + \mu_m^2 = E \left(\left(\frac{M_{t+1}^P}{M_t^P} \right)^2 \left(\frac{1}{R_{t+1,\infty}} \right)^2 \right)$. It is instructive to note that if

$$E \left(\left(\frac{M_{t+1}^P}{M_t^P} \right)^2 \left(\frac{1}{R_{t+1,\infty}} \right)^2 \right) \leq E \left(\left(\frac{M_{t+1}^P}{M_t^P} \right)^2 \right), \quad (\text{A7})$$

we get $\text{Var} \left[\frac{M_{t+1}}{M_t} \right] + \mu_m^2 \leq E \left(\left(\frac{M_{t+1}^P}{M_t^P} \right)^2 \right) - \underbrace{\left(E \left(\frac{M_{t+1}^P}{M_t^P} \right) \right)^2}_{=1} + 1$. Therefore,

$$\text{Var} \left[\frac{M_{t+1}}{M_t} \right] \leq \text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] + 1 - \mu_m^2. \quad (\text{A8})$$

Inequality (A8) can be expressed as

$$\frac{1}{\text{Var} \left[\frac{M_{t+1}}{M_t} \right]} \geq \frac{1}{\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] + 1 - \mu_m^2}, \quad (\text{A9})$$

and

$$\frac{\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right]}{\text{Var} \left[\frac{M_{t+1}}{M_t} \right]} \geq \frac{\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right]}{\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] + 1 - \mu_m^2}. \quad (\text{A10})$$

Now the first derivative of the function $\mathcal{L}[u] = \frac{u}{u+1-\mu_m^2}$ is $\mathcal{L}'[u] = \frac{1-\mu_m^2}{(u+1-\mu_m^2)^2} > 0$ because $\mu_m^2 < 1$. Since $\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] \geq \sigma_{pc}^2$, it follows that $\mathcal{L} \left[\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] \right] \geq \mathcal{L} \left[\sigma_{pc}^2 \right]$. Hence,

$$\frac{\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right]}{\text{Var} \left[\frac{M_{t+1}}{M_t} \right]} \geq \frac{\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right]}{\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] + 1 - \mu_m^2} \geq \frac{\sigma_{pc}^2}{\sigma_{pc}^2 + 1 - \mu_m^2}, \quad (\text{A11})$$

which completes the proof of equation (8).

The remaining task is to examine equation (A7). If a pricing kernel is required to satisfy the condition $\text{Cov} \left(\left(\frac{M_{t+1}^P}{M_t^P} \right)^2, \left(\frac{M_{t+1}^T}{M_t^T} \right)^2 \right) < 0$, it is equivalent to $E \left(\left(\frac{M_{t+1}^P}{M_t^P} \right)^2 \left(\frac{1}{R_{t+1,\infty}^2} \right) \right) \leq E \left(\left(\frac{M_{t+1}^P}{M_t^P} \right)^2 \right) E \left(\frac{1}{R_{t+1,\infty}^2} \right)$. Additionally,

$$\frac{1}{R_{t+1,\infty}^2} = \frac{1}{(1+r_{t+1,\infty})^2} \simeq 1 - 2r_{t+1,\infty}, \quad (\text{A12})$$

which implies

$$E \left(\left(\frac{M_{t+1}^P}{M_t^P} \right)^2 \left(\frac{1}{R_{t+1,\infty}^2} \right) \right) \leq E \left(\left(\frac{M_{t+1}^P}{M_t^P} \right)^2 \right) (1 - 2E(r_{t+1,\infty})). \quad (\text{A13})$$

Since $0 < 1 - 2E(r_{t+1,\infty}) < 1$, (A13) is equivalent to (A7). ■

Proof of Proposition 2. For brevity, denote $\mu_{tc} \equiv E \left(\frac{M_{t+1}^T}{M_t^T} \right)$, which is the mean of the transitory component

of the SDF. Again the proof is by construction, whereby we recognize that

$$E \left(\left(\frac{M_{t+1}^T}{M_t^T} - E \left(\frac{M_{t+1}^T}{M_t^T} \right) \right) (R_{t+1,\infty} - E(R_{t+1,\infty})) \right) = 1 - \mu_{tc} E(R_{t+1,\infty}), \quad (\text{A14})$$

since $E \left(\frac{M_{t+1}^T}{M_t^T} R_{t+1,\infty} \right) = 1$. Multiply both sides of (A14) by $\frac{(1-\mu_{tc} E(R_{t+1,\infty}))}{\text{Var}[R_{t+1,\infty}]}$, and rearrange, to obtain

$$\frac{(1 - \mu_{tc} E(R_{t+1,\infty}))^2}{\text{Var}[R_{t+1,\infty}]} = \text{Cov} \left[\frac{M_{t+1}^T}{M_t^T}, R_{t+1,\infty} \times \frac{(1 - \mu_{tc} E(R_{t+1,\infty}))}{\text{Var}[R_{t+1,\infty}]} \right], \quad (\text{A15})$$

$$\leq \left(\text{Var} \left[\frac{M_{t+1}^T}{M_t^T} \right] \right)^{\frac{1}{2}} \left(\left(\frac{(1 - \mu_{tc} E(R_{t+1,\infty}))}{\text{Var}[R_{t+1,\infty}]} \right)^2 \text{Var}[R_{t+1,\infty}] \right)^{\frac{1}{2}}. \quad (\text{A16})$$

Equation (A16) is a consequence of the Cauchy-Schwarz inequality which, upon simplification and cancellation of terms, validates the bound in equation (9) of Proposition 2. ■

Proof of the upper bound on the size of the transitory component in equation (10). We exploit the well-known expression

$$\text{Var} \left[\frac{M_{t+1}}{M_t} \right] \geq \sigma_{HJ}^2, \quad \text{where } \sigma_{HJ}^2 \equiv \left(\mathbf{1} - \mu_m E(\mathbf{R}_{t+1}) \right)' (\text{Var}[\mathbf{R}_{t+1}])^{-1} \left(\mathbf{1} - \mu_m E(\mathbf{R}_{t+1}) \right), \quad (\text{A17})$$

as in equation (12) of Hansen and Jagannathan (1991). In conjunction with the fact that $\text{Var} \left[\frac{M_{t+1}^T}{M_t^T} \right] = \text{Var} \left[\frac{1}{R_{t+1,\infty}} \right]$, it follows that

$$\frac{1}{\sigma_{HJ}^2} \geq \frac{1}{\text{Var} \left[\frac{M_{t+1}}{M_t} \right]}, \quad \text{and therefore,} \quad \frac{\text{Var} \left[\frac{M_{t+1}^T}{M_t^T} \right]}{\text{Var} \left[\frac{M_{t+1}}{M_t} \right]} \leq \frac{\text{Var} \left[\frac{1}{R_{t+1,\infty}} \right]}{\sigma_{HJ}^2}. \quad (\text{A18})$$

This confirms equation (10). ■

Proof of Proposition 3. Observe that

$$E \left(\left(\left(\frac{M_{t+1}^P}{M_t^P} - E \left(\frac{M_{t+1}^P}{M_t^P} \right) \right) \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2} - E \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2} \right) \right) \right) \right) = \mathbf{1} - \mu_{pt} E \left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2} \right), \quad (\text{A19})$$

where recalling the notation $\mu_{pt} = E\left(\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T}\right)$. We denote

$$\Sigma \equiv E\left(\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2} - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right)\right)\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2} - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right)\right)'\right) \quad \text{and} \quad \mathbf{D} \equiv \mathbf{1} - \mu_{pt} E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right). \quad (\text{A20})$$

Multiply (A19) by $\mathbf{D}'\Sigma^{-1}$ and apply the Cauchy-Schwarz inequality to the left-hand side of (A19). The lower bound on the variance of the relative contribution of the permanent component of SDFs to the transitory component of SDFs can then be derived as: $\mathbf{D}'\Sigma^{-1}\mathbf{D} \leq \text{Var}\left[\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T}\right]$, as asserted. ■

Appendix B: Proofs of unconditional bounds that incorporate conditioning information

Proof of Propositions 1, 2, and 3 with conditioning variables. For tractability of exposition, we denote by z_t the set of conditioning variables. The variable z_t predicts \mathbf{R}_{t+1} . We note that

$$\begin{aligned} E\left(\left(\frac{M_{t+1}^P}{M_t^P} - E\left(\frac{M_{t+1}^P}{M_t^P}\right)\right)\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)\right) &= E\left(E_t\left(\frac{M_{t+1}}{M_t} z_t'\mathbf{R}_{t+1}\right)\right) - E\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right), \\ &= E\left(z_t'\mathbf{1}\right) - E\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right). \end{aligned} \quad (\text{B1})$$

Equation (B1) follows, since the permanent component of the pricing kernel is a martingale. Now, denote

$$\Theta \equiv E\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}} - E\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)^2 \quad \text{and} \quad \mathbf{H} \equiv E\left(z_t'\mathbf{1}\right) - E\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right). \quad (\text{B2})$$

Multiplying (B1) by $\mathbf{H}'\Theta^{-1}$ and applying the Cauchy-Schwarz inequality to (B1), we obtain the lower bound on the permanent component as

$$\mathbf{H}'\Theta^{-1}\mathbf{H} \leq \text{Var}\left[\frac{M_{t+1}^P}{M_t^P}\right]. \quad (\text{B3})$$

This ends the proof of Proposition 1 with conditioning variables. The same approach can be used to derive Propositions 2 and 3 with conditioning variables. ■

Appendix C: Solution to the eigenfunction problem in Kelly (2009)

Our end-goal is to present the permanent and the transitory components of the SDF. To make our proof self-contained, we first outline the expression for the SDF of this model. Then we proceed to the transitory component by solving the eigenfunction problem.

Under the dynamics of the real economy posited in equations (25)–(27) and recursive utility, the SDF is of the form (see also Bansal and Yaron (2004)),

$$\begin{aligned} \log\left(\frac{M_{t+1}}{M_t}\right) &= \theta \log(\beta) - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{M,t+1}, \\ &= \xi_t - \lambda_g \left(\sigma_g \sigma_t z_{g,t+1} + \sqrt{\Lambda_t} W_{g,t+1} \right) - \lambda_x \sigma_x \sigma_t z_{x,t+1} - \lambda_\sigma \sigma_\sigma z_{\sigma,t+1} - \lambda_\Lambda \sigma_\Lambda z_{\Lambda,t+1}, \end{aligned} \quad (C1)$$

with

$$\lambda_g = 1 - \theta + \frac{\theta}{\psi}, \quad \lambda_x = (1 - \theta) \kappa_1 A_x, \quad \lambda_\sigma = (1 - \theta) \kappa_1 A_\sigma, \quad \lambda_\Lambda = (1 - \theta) \kappa_1 A_\Lambda, \quad \text{and} \quad (C2)$$

$$A_x = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho_x}, \quad A_\Lambda = \frac{\theta \left(1 - \frac{1}{\psi}\right) \left(1 - \frac{1}{\psi}\right)}{1 - \kappa_1 \rho_x}, \quad A_\sigma = \frac{\theta}{2} \left(\frac{\left(1 - \frac{1}{\psi}\right)^2 \sigma_g^2 + \kappa_1^2 A_x^2 \sigma_x^2}{1 - \kappa_1 \rho_\sigma} \right), \quad (C3)$$

$$A_0 = \frac{\log(\beta) + \left(1 - \frac{1}{\psi}\right) \mu + \kappa_0 + \kappa_1 \left(A_\sigma \bar{\sigma}^2 (1 - \rho_\sigma) + A_\Lambda \bar{\Lambda} (1 - \rho_\Lambda) \right) + \frac{1}{2} \theta \kappa_1^2 \left(A_\sigma^2 \sigma_\sigma^2 + A_\Lambda^2 \sigma_\Lambda^2 \right)}{1 - \kappa_1}, \quad (C4)$$

and

$$\xi_t = \theta \log(\beta) - \frac{\theta}{\psi} (\mu + x_t) + (\theta - 1) E_t(r_{M,t+1}). \quad (C5)$$

The market return is

$$r_{M,t+1} = \kappa_0 + \kappa_1 w g_{t+1} - w g_t + g_{t+1}, \quad \text{with} \quad (C6)$$

$$w g_{t+1} = A_0 + A_x x_{t+1} + A_\sigma \sigma_{t+1}^2 + A_\Lambda \Lambda_{t+1}. \quad (C7)$$

Therefore,

$$r_{M,t+1} - E_t(r_{M,t+1}) = \kappa_1 A_\Lambda \sigma_\Lambda z_{\Lambda,t+1} + \sigma_g \sigma_t z_{g,t+1} + \sqrt{\Lambda_t} W_{g,t+1} + \kappa_1 A_x \sigma_x \sigma_t z_{x,t+1} + \kappa_1 A_\sigma \sigma_\sigma z_{\sigma,t+1}, \quad (C8)$$

with

$$E_t(r_{M,t+1}) = \kappa_0 + \kappa_1 (A_0 + A_x E_t(x_{t+1}) + A_\sigma E_t(\sigma_{t+1}^2) + A_\Lambda E_t(\Lambda_{t+1})) - wg_t + E_t(g_{t+1}), \quad (\text{C9})$$

where

$$E_t(x_{t+1}) = \rho_x x_t, \quad E_t(\sigma_{t+1}^2) = \bar{\sigma}^2 (1 - \rho_\sigma) + \rho_\sigma \sigma_t^2, \quad (\text{C10})$$

$$E_t(\Lambda_{t+1}) = \bar{\Lambda} (1 - \rho_\Lambda) + \rho_\Lambda \Lambda_t, \quad E_t(g_{t+1}) = \mu + x_t. \quad (\text{C11})$$

The next step is to show that the SDF depends on the market variance. In this regard,

$$\text{Var}_t[r_{M,t+1}] = \kappa_1^2 (A_\Lambda^2 \sigma_\Lambda^2 + A_\sigma^2 \sigma_\sigma^2) + (\sigma_g^2 + \kappa_1^2 A_x^2 \sigma_x^2) \sigma_t^2 + \text{Var}_t[W_{g,t+1}] \Lambda_t. \quad (\text{C12})$$

Now the moment generating function of the Laplace variable W_g is $E_t(\exp(sW_{g,t+1})) = \frac{1}{1-s^2}$. Therefore, $\text{Var}_t[W_{g,t+1}] = 2$, and the conditional variance of the market return is

$$\sigma_{M,t+1}^2 = \kappa_1^2 (A_\Lambda^2 \sigma_\Lambda^2 + A_\sigma^2 \sigma_\sigma^2) + (\sigma_g^2 + \kappa_1^2 A_x^2 \sigma_x^2) \sigma_{t+1}^2 + 2\Lambda_{t+1}, \quad (\text{C13})$$

and the innovation in the market variance is

$$\sigma_{M,t+1}^2 - E_t(\sigma_{M,t+1}^2) = (\sigma_g^2 + \kappa_1^2 A_x^2 \sigma_x^2) \sigma_\sigma z_{\sigma,t+1} + 2\sigma_\Lambda z_{\Lambda,t+1}. \quad (\text{C14})$$

We replace the conditional market variance in the SDF and get

$$\begin{aligned} \log\left(\frac{M_{t+1}}{M_t}\right) &= \xi_t + \mathcal{D}_1 (g_{t+1} - E_t(g_{t+1})) + \mathcal{D}_2 (x_{t+1} - E_t(x_{t+1})) + \mathcal{D}_3 (\sigma_{M,t+1}^2 - E_t(\sigma_{M,t+1}^2)) \\ &\quad + \mathcal{D}_4 (\Lambda_{t+1} - E_t(\Lambda_{t+1})), \end{aligned} \quad (\text{C15})$$

with ξ_t defined in (C5) and

$$\mathcal{D}_1 = -\lambda_g, \quad \mathcal{D}_2 = -\lambda_x, \quad \mathcal{D}_3 = -\frac{\lambda_\sigma}{(\sigma_g^2 + \kappa_1^2 A_x^2 \sigma_x^2)}, \quad \mathcal{D}_4 = \frac{2\lambda_\sigma}{(\sigma_g^2 + \kappa_1^2 A_x^2 \sigma_x^2)} - \lambda_\Lambda. \quad (\text{C16})$$

Expression (C15) for the SDF is imperative for recovering the permanent component of the SDF.

Moving to the transitory component of the SDF, define, for notational simplicity,

$$\mathbf{X}_t = \begin{pmatrix} x_t \\ \sigma_t^2 - \bar{\sigma}^2 \\ \Lambda_t - \bar{\Lambda} \end{pmatrix}. \quad (\text{C17})$$

The aim is to solve the eigenfunction problem $E_t \left(\frac{M_{t+1}}{M_t} \frac{1}{M_{t+1}^e} \right) = \frac{v}{M_t^e}$, or equivalently, $E_t \left(\frac{M_{t+1}}{M_t} e[\mathbf{X}_{t+1}] \right) = v e[\mathbf{X}_t]$. Using our time series assumptions,

$$\mathbf{X}_{t+1} = \Gamma \mathbf{X}_t + \Pi \mathbf{z}_{t+1}, \quad (\text{C18})$$

with

$$\Gamma = \begin{pmatrix} \rho_x & 0 & 0 \\ 0 & \rho_\sigma & 0 \\ 0 & 0 & \rho_\Lambda \end{pmatrix}, \quad \Pi = \begin{pmatrix} \sigma_x \sigma_t & 0 & 0 \\ 0 & \sigma_\sigma & 0 \\ 0 & 0 & \sigma_\Lambda \end{pmatrix}, \quad \mathbf{z}_{t+1} = \begin{pmatrix} z_{x,t+1} \\ z_{\sigma,t+1} \\ z_{\Lambda,t+1} \end{pmatrix}. \quad (\text{C19})$$

We conjecture that the solution is of the form

$$e[\mathbf{X}_{t+1}] = \exp(\mathbf{c}' \mathbf{X}_{t+1}), \quad (\text{C20})$$

with $\mathbf{c} = (c_1, c_2, c_3)$. Now,

$$\begin{aligned} \Psi &= E_t \left(\frac{M_{t+1}}{M_t} e[\mathbf{X}_{t+1}] \right) = E_t \left(\exp \left(\log \left(\frac{M_{t+1}}{M_t} \right) + \log(e[\mathbf{X}_{t+1}]) \right) \right) \\ &= E_t \left(\exp \left(\xi_t + \mathbf{c}' \Gamma \mathbf{X}_t - \lambda_g \sigma_g \sigma_t z_{g,t+1} - (\boldsymbol{\varphi}' - \mathbf{c}' \Pi) \mathbf{z}_{t+1} \right) \exp \left(\lambda_g \sqrt{\Lambda_t} W_{g,t+1} \right) \right). \end{aligned} \quad (\text{C21})$$

Since $W_{g,t+1}$ is independent of the z variables, it follows that

$$\Psi = \exp(\xi_t + \mathbf{c}' \Gamma \mathbf{X}_t) \Upsilon_1 \Upsilon_2, \quad (\text{C22})$$

with $\boldsymbol{\varphi}' = (\lambda_x \sigma_x \sigma_t, \lambda_\sigma \sigma_\sigma, \lambda_\Lambda \sigma_\Lambda)$, and

$$\Upsilon_1 = E_t \left(\exp \left(-\lambda_g \sigma_g \sigma_t z_{g,t+1} - (\boldsymbol{\varphi}' - \mathbf{c}' \Pi) \mathbf{z}_{t+1} \right) \right), \quad \Upsilon_2 = E_t \left(\exp \left(-\lambda_g \sqrt{\Lambda_t} W_{g,t+1} \right) \right). \quad (\text{C23})$$

We notice that

$$\Upsilon_1 = \exp\left(\frac{1}{2}(\lambda_g^2 \sigma_g^2 \sigma_t^2) + \frac{1}{2}(\boldsymbol{\phi}' - \mathbf{c}'\Pi)(\boldsymbol{\phi} - \Pi\mathbf{c})\right), \quad \Upsilon_2 = E_t\left(\exp\left(-\lambda_g \sqrt{\Lambda_t} W_{g,t+1}\right)\right) = \frac{1}{1 - \lambda_g^2 \Lambda_t}, \quad (\text{C24})$$

and, hence, rewrite (C22) as

$$\Psi = \exp\left(\xi_t + \mathbf{c}'\Gamma\mathbf{X}_t\right) \exp\left(\frac{1}{2}(\lambda_g^2 \sigma_g^2 \sigma_t^2) + \frac{1}{2}(\boldsymbol{\phi}' - \mathbf{c}'\Pi)(\boldsymbol{\phi} - \Pi\mathbf{c})\right) \exp\left(\log\left(\frac{1}{1 - \lambda_g^2 \Lambda_t}\right)\right). \quad (\text{C25})$$

Following Kelly (2009), we use the Taylor expansion of $\frac{1}{1 - \lambda_g^2 \Lambda_t}$ to get $\frac{1}{1 - \lambda_g^2 \Lambda_t} \approx \lambda_g^2 \Lambda_t$. Therefore, (C25) simplifies to

$$\begin{aligned} \Psi &= \exp\left(\xi_t + \mathbf{c}'\Gamma\mathbf{X}_t\right) \exp\left(\frac{1}{2}(\lambda_g^2 \sigma_g^2 \sigma_t^2) + \frac{1}{2}(\boldsymbol{\phi}' - \mathbf{c}'\Pi)(\boldsymbol{\phi} - \Pi\mathbf{c})\right) \exp(\lambda_g^2 \Lambda_t), \\ &= \exp\left(\xi_t + \mathbf{c}'\Gamma\mathbf{X}_t + \lambda_g^2 \Lambda_t + \frac{1}{2}\lambda_g^2 \sigma_g^2 \sigma_t^2 + \frac{1}{2}\boldsymbol{\phi}'\boldsymbol{\phi} - \boldsymbol{\phi}'\Pi\mathbf{c} + \frac{1}{2}\mathbf{c}'\Pi\Pi\mathbf{c}\right). \end{aligned} \quad (\text{C26})$$

This implies that

$$\begin{aligned} \mathbf{c}'\Gamma\mathbf{X}_t &= c_1 \rho_x x_t + c_2 \rho_\sigma (\sigma_t^2 - \bar{\sigma}^2) + c_3 \rho_\Lambda (\Lambda_t - \bar{\Lambda}), & \boldsymbol{\phi}'\boldsymbol{\phi} &= \lambda_x^2 \sigma_x^2 \sigma_t^2 + \lambda_\sigma^2 \sigma_\sigma^2 + \lambda_\Lambda^2 \sigma_\Lambda^2, \\ \boldsymbol{\phi}'\Pi\mathbf{c} &= \lambda_x \sigma_x^2 \sigma_t^2 c_1 + \lambda_\sigma \sigma_\sigma^2 c_2 + \lambda_\Lambda \sigma_\Lambda^2 c_3, & \mathbf{c}'\Pi\Pi\mathbf{c} &= c_1^2 \sigma_x^2 \sigma_t^2 + c_2^2 \sigma_\sigma^2 + c_3^2 \sigma_\Lambda^2, \end{aligned}$$

and we rewrite (C26) as

$$\Psi = \exp\left(\begin{array}{l} \xi_t + \mathbf{c}'\Gamma\mathbf{X}_t + \lambda_g^2 \Lambda_t + \frac{1}{2}\lambda_g^2 \sigma_g^2 \sigma_t^2 + \frac{1}{2}(\lambda_x^2 \sigma_x^2 \sigma_t^2 + \lambda_\sigma^2 \sigma_\sigma^2 + \lambda_\Lambda^2 \sigma_\Lambda^2) \\ -(\lambda_x \sigma_x^2 \sigma_t^2 c_1 + \lambda_\sigma \sigma_\sigma^2 c_2 + \lambda_\Lambda \sigma_\Lambda^2 c_3) + \frac{1}{2}(c_1^2 \sigma_x^2 \sigma_t^2 + c_2^2 \sigma_\sigma^2 + c_3^2 \sigma_\Lambda^2) \end{array}\right), \quad (\text{C27})$$

where

$$\begin{aligned} \xi_t &= \theta \log(\beta) + (\theta - 1)\kappa_0 + (\theta - 1)\kappa_1 A_0 - \frac{\theta}{\Psi} \mu + (\theta - 1)\kappa_1 A_\sigma \bar{\sigma}^2 (1 - \rho_\sigma) \\ &\quad + (\theta - 1)\mu + (\theta - 1)\kappa_1 A_\Lambda \bar{\Lambda} (1 - \rho_\Lambda) - (\theta - 1)(A_0 + A_\sigma \bar{\sigma}^2 + A_\Lambda \bar{\Lambda}) \\ &\quad + (\theta - 1)A_\Lambda \kappa_1 \rho_\Lambda \bar{\Lambda} + (\theta - 1)A_\sigma \kappa_1 \rho_\sigma \bar{\sigma}^2 \left(-\frac{\theta}{\Psi} + (\theta - 1)\kappa_1 A_x \rho_x + (\theta - 1) - (\theta - 1)A_x\right) x_t \\ &\quad + (\theta - 1)A_\sigma (\kappa_1 \rho_\sigma - 1)(\sigma_t^2 - \bar{\sigma}^2) + (\theta - 1)A_\Lambda (\kappa_1 \rho_\Lambda - 1)(\Lambda_t - \bar{\Lambda}). \end{aligned} \quad (\text{C28})$$

We can, therefore, express (C27) as $\Psi = \mathbf{v} \exp(\mathbf{c}' \mathbf{X}_t)$, with

$$\mathbf{v} = \exp \left(\begin{array}{l} \theta \log(\beta) + (\theta - 1) \kappa_0 + (\theta - 1) \kappa_1 A_0 - \frac{\theta}{\psi} \mu + (\theta - 1) \kappa_1 A_\sigma \bar{\sigma}^2 (1 - \rho_\sigma) \\ + (\theta - 1) \mu + (\theta - 1) \kappa_1 A_\Lambda \bar{\Lambda} (1 - \rho_\Lambda) - (\theta - 1) (A_0 + A_\sigma \bar{\sigma}^2 + A_\Lambda \bar{\Lambda}) \\ + (\theta - 1) A_\Lambda \kappa_1 \rho_\Lambda \bar{\Lambda} + (\theta - 1) A_\sigma \kappa_1 \rho_\sigma \bar{\sigma}^2 \\ + \frac{1}{2} (\lambda_\sigma^2 \bar{\sigma}^2 + \lambda_\Lambda^2 \bar{\Lambda}^2) - (\lambda_\sigma \bar{\sigma}^2 c_2 + \lambda_\Lambda \bar{\Lambda}^2 c_3) + \frac{1}{2} (c_2^2 \bar{\sigma}^2 + c_3^2 \bar{\Lambda}^2) + \lambda_g^2 \bar{\Lambda} \\ + (-\lambda_x \bar{\sigma}^2 c_1 + \frac{1}{2} c_1^2 \bar{\sigma}^2 + \frac{1}{2} \lambda_g^2 \bar{\sigma}^2 + \frac{1}{2} \lambda_x^2 \bar{\sigma}^2) \bar{\sigma}^2 \end{array} \right), \quad (\text{C29})$$

$$\mathbf{c} = \left(\begin{array}{l} c_1 \rho_x + (-\lambda_g + (\theta - 1) A_x (\kappa_1 \rho_x - 1)) \\ c_2 \rho_\sigma - \lambda_x \bar{\sigma}^2 c_1 + \frac{1}{2} c_1^2 \bar{\sigma}^2 + \frac{1}{2} \lambda_g^2 \bar{\sigma}^2 + \frac{1}{2} \lambda_x^2 \bar{\sigma}^2 + (\theta - 1) A_\sigma (\kappa_1 \rho_\sigma - 1) \\ c_3 \rho_\Lambda + \lambda_g^2 + (\theta - 1) A_\Lambda (\kappa_1 \rho_\Lambda - 1) \end{array} \right). \quad (\text{C30})$$

Solving (C30) for the individual components in $\mathbf{c} = (c_1, c_2, c_3)$ allows us to deduce that

$$c_1 = \frac{-\lambda_g + (\theta - 1) A_x (\kappa_1 \rho_x - 1)}{1 - \rho_x}, \quad (\text{C31})$$

$$c_2 = \frac{-\lambda_x \bar{\sigma}^2 c_1 + \frac{1}{2} c_1^2 \bar{\sigma}^2 + \frac{1}{2} \lambda_g^2 \bar{\sigma}^2 + \frac{1}{2} \lambda_x^2 \bar{\sigma}^2 + (\theta - 1) A_\sigma (\kappa_1 \rho_\sigma - 1)}{1 - \rho_\sigma}, \quad (\text{C32})$$

$$c_3 = \frac{\lambda_g^2 + (\theta - 1) A_\Lambda (\kappa_1 \rho_\Lambda - 1)}{1 - \rho_\Lambda}. \quad (\text{C33})$$

The transitory component of the SDF is $\frac{M_{t+1}^T}{M_t^T} = \mathbf{v} \exp(-\mathbf{c}' (\mathbf{X}_{t+1} - \mathbf{X}_t))$, and the permanent component is determined accordingly. ■

Appendix D: Solution to the eigenfunction problem in Bekaert and Engstrom (2010)

Write $q_{t+1} - \delta_q = \rho_q (q_t - \delta_q) + \sigma_{qp} \omega_{p,t+1} + \sigma_{qn} \omega_{n,t+1}$, with $\mu_q / (1 - \rho_q) \equiv \delta_q$. Denote the state vector as

$$\mathbf{Z}_t = \begin{pmatrix} q_t - \delta_q \\ x_t \\ p_t - \bar{p} \\ n_t - \bar{n} \end{pmatrix}. \quad (\text{D1})$$

We have

$$\mathbf{Z}_{t+1} = \mathbf{a} \mathbf{Z}_t + \mathbf{b} \mathbf{w}_{t+1}, \quad (\text{D2})$$

where

$$\mathbf{a} \equiv \begin{pmatrix} \rho_q & 0 & 0 & 0 \\ 0 & \rho_x & 0 & 0 \\ 0 & 0 & \rho_p & 0 \\ 0 & 0 & 0 & \rho_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} \equiv \begin{pmatrix} \sigma_{qp} & \sigma_{qn} \\ \sigma_{xp} & \sigma_{xn} \\ \sigma_{pp} & 0 \\ 0 & \sigma_{nn} \end{pmatrix} \quad \text{and} \quad \mathbf{w}_{t+1} \equiv \begin{pmatrix} \omega_{p,t+1} \\ \omega_{n,t+1} \end{pmatrix}. \quad (\text{D3})$$

In the eigenfunction problem $E_t \left(\frac{M_{t+1}}{M_t} e[\mathbf{Z}_{t+1}] \right) = v e[\mathbf{Z}_t]$, conjecture that $e[\mathbf{Z}_t] = \exp(\mathbf{c}'\mathbf{Z}_t)$. Using the fact that g_{t+1} and b_{t+1} follow gamma distributions (for some real number λ),

$$E_t \left(\exp(\lambda g_{t+1}) \right) = \exp(-p_t \log(1-\lambda)), \quad \text{and} \quad E_t \left(\exp(\lambda b_{t+1}) \right) = \exp(-n_t \log(1-\lambda)), \quad (\text{D4})$$

we manipulate a set of equations to yield the eigenvalue as

$$v = \exp(\log(\beta) - \gamma \bar{g} - a_0 \bar{p} - b_0 \bar{n} - \bar{p} \log(1-a_0) - \bar{n} \log(1-b_0)), \quad (\text{D5})$$

and

$$c_1 = -\gamma, \quad c_2 = -\frac{\gamma}{1-\rho_x}. \quad (\text{D6})$$

Moreover, c_3 and c_4 are a solution to the non-linear equations,

$$c_3(1-\rho_p) + a_0 + \log(1-a_0) = 0 \quad \text{and} \quad c_4(1-\rho_n) + b_0 + \log(1-b_0) = 0, \quad (\text{D7})$$

where setting $a_0 \equiv -\gamma \sigma_{gp} - \frac{\gamma \sigma_{xp}}{1-\rho_x} + c_3 \sigma_{pp}$ and $b_0 \equiv \gamma \sigma_{gn} - \frac{\gamma \sigma_{xn}}{1-\rho_x} + c_4 \sigma_{nn}$. This concludes our proof. \blacksquare

Appendix E: Solution to the eigenfunction problem in a rare disasters model

The main step is that the return from holding a long-term bond with maturity k from time t to $t+1$ is

$$\begin{aligned} R_{t+1,\infty} &= \lim_{k \rightarrow \infty} R_{t+1,k} = \lim_{k \rightarrow \infty} \frac{E_{t+1} \left(\frac{M_{t+2}}{M_{t+1}} \cdots \frac{M_{t+k}}{M_{t+k-1}} \right)}{E_t \left(\frac{M_{t+1}}{M_t} \frac{M_{t+2}}{M_{t+1}} \cdots \frac{M_{t+k}}{M_{t+k-1}} \right)}, \\ &= \frac{1}{E_t \left(\frac{M_{t+1}}{M_t} \right)} = \frac{1}{\beta E_t \left(g_{t+1}^{-\gamma} \right)}, \quad (\text{since } \left\{ \frac{M_{t+1}}{M_t} \right\} \text{ are independent}). \end{aligned} \quad (\text{E1})$$

Based on the evolution of w_{t+1} and z_{t+1} in (36)–(37), and the properties of the Poisson random variables

$$\beta E_t \left(g_{t+1}^{-\gamma} \right) = \beta \exp \left(-\gamma \mu + \frac{1}{2} \gamma^2 \sigma^2 \right) \sum_{j=0}^{\infty} \frac{e^{-\omega} \omega^j}{j!} \exp \left(-\gamma \theta j + \frac{1}{2} \gamma^2 \delta^2 j \right). \quad (\text{E2})$$

Combining (E1)–(E2) justifies the result in (39). ■

Appendix F: Restrictions implied by an asset pricing model such that it satisfies the volatility bound on the SDF, and yet fails to satisfy the lower bound on the variance of the permanent component of the SDF

The essence of the result is captured in the following Lemma. For brevity, let $\mu_{R_\infty} \equiv E \left(\frac{1}{R_{t+1, \infty}^2} \right)$.

Lemma 1 *Suppose the SDF implied from an asset pricing model can be decomposed into a permanent component and a transitory component. Suppose that the asset pricing model satisfies two restrictions:*

$$(a) \text{Var} \left[\frac{M_{t+1}}{M_t} \right] \geq \sigma_{HJ}^2, \text{ where } \sigma_{HJ}^2 \text{ is defined in Hansen and Jagannathan (1991, eq. (12)), and} \quad (\text{F1})$$

$$(b) \text{Cov} \left[\left(\frac{M_{t+1}^P}{M_t^P} \right)^2, \left(\frac{M_{t+1}^T}{M_t^T} \right)^2 \right] \leq \Delta_c \equiv \mu_m^2 \left(1 + \frac{\sigma_{HJ}^2}{\mu_m^2} \right) - \mu_{R_\infty} (1 + \sigma_{pc}^2). \quad (\text{F2})$$

If (a) and (b) hold, then $\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] \geq \sigma_{pc}^2$, where σ_{pc}^2 is defined in (6) of Proposition 1. Alternatively, if (a) and $\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] \leq \sigma_{pc}^2$ hold, then $\text{Cov} \left[\left(\frac{M_{t+1}^P}{M_t^P} \right)^2, \left(\frac{M_{t+1}^T}{M_t^T} \right)^2 \right] \geq \Delta_c$.

Proof: If an asset pricing model satisfies the Hansen and Jagannathan (1991) variance bound (i.e., restriction (a)), we have,

$$\text{Var} \left[\frac{M_{t+1}}{M_t} \right] = E \left(\left(\frac{M_{t+1}^P}{M_t^P} \right)^2 \left(\frac{M_{t+1}^T}{M_t^T} \right)^2 \right) - \mu_m^2, \quad (\text{F3})$$

$$= \text{Cov} \left[\left(\frac{M_{t+1}^P}{M_t^P} \right)^2, \left(\frac{M_{t+1}^T}{M_t^T} \right)^2 \right] + \left(E \left(\frac{M_{t+1}^T}{M_t^T} \right)^2 \right) \left(1 + \text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right] \right) - \mu_m^2. \quad (\text{F4})$$

Consider the SDF $\frac{M_{t+1}^*}{M_t^*}$, which displays the minimum variance property, namely, $\text{Var} \left[\frac{M_{t+1}^*}{M_t^*} \right] = \sigma_{HJ}^2$ (Hansen and Jagannathan (1991)). Consider also the permanent component of the SDF $\frac{M_{t+1}^{*P}}{M_t^{*P}}$, which displays the minimum variance property, namely, $\text{Var} \left[\frac{M_{t+1}^{*P}}{M_t^{*P}} \right] = \sigma_{pc}^2$ (see equation (22)). The SDF with the minimum

variance property admits a unique decomposition (e.g., Roman (2007, Theorem 9.15, page 220)):

$$\frac{M_{t+1}^*}{M_t^*} = \text{proj} \left[\frac{M_{t+1}^*}{M_t^*} \left| \frac{M_{t+1}^{*P}}{M_t^{*P}} \right. \right] + \varepsilon_{t+1}, \quad \text{with} \quad E \left(\varepsilon_{t+1} \frac{M_{t+1}^{*P}}{M_t^{*P}} \right) = 0, \quad (\text{F5})$$

where $\text{proj} \left[\frac{M_{t+1}^*}{M_t^*} \left| \frac{M_{t+1}^{*P}}{M_t^{*P}} \right. \right]$ represents the projection of $\frac{M_{t+1}^*}{M_t^*}$ onto $\frac{M_{t+1}^{*P}}{M_t^{*P}}$, i.e., $\text{proj} \left[\frac{M_{t+1}^*}{M_t^*} \left| \frac{M_{t+1}^{*P}}{M_t^{*P}} \right. \right] = \frac{E \left(\frac{M_{t+1}^*}{M_t^*} \frac{M_{t+1}^{*P}}{M_t^{*P}} \right)}{E \left(\frac{M_{t+1}^{*P}}{M_t^{*P}} \right)^2} \frac{M_{t+1}^{*P}}{M_t^{*P}}$.

For future use, define

$$\bar{b} \equiv \frac{E \left(\frac{M_{t+1}^*}{M_t^*} \frac{M_{t+1}^{*P}}{M_t^{*P}} \right)}{E \left(\frac{M_{t+1}^{*P}}{M_t^{*P}} \right)^2}, \quad \text{hence,} \quad \frac{M_{t+1}^*}{M_t^*} = \bar{b} \left(\frac{M_{t+1}^{*P}}{M_t^{*P}} \right) + \varepsilon_{t+1}. \quad (\text{F6})$$

Therefore,

$$E \left(\left(\frac{M_{t+1}^*}{M_t^*} \right)^2 \right) = \bar{b}^2 E \left(\left(\frac{M_{t+1}^{*P}}{M_t^{*P}} \right)^2 \right) + E(\varepsilon_{t+1}^2). \quad (\text{F7})$$

Rearranging,

$$\sigma_{HJ}^2 = \bar{b}^2 (\sigma_{pc}^2 + 1) + E(\varepsilon_{t+1}^2) - \mu_m^2. \quad (\text{F8})$$

Impose the condition that the asset pricing model explains the volatility bound. Then

$$\text{Var} \left[\frac{M_{t+1}^*}{M_t^*} \right] \geq \sigma_{HJ}^2 = \underbrace{\bar{b}^2 (\sigma_{pc}^2 + 1) + E(\varepsilon_{t+1}^2) - \mu_m^2}_{\text{from equation (F8)}}. \quad (\text{F9})$$

Combining (F4) and (F9), we arrive at

$$\text{Cov} \left[\left(\frac{M_{t+1}^p}{M_t^p} \right)^2, \left(\frac{M_{t+1}^T}{M_t^T} \right)^2 \right] + \left(E \left(\frac{M_{t+1}^T}{M_t^T} \right)^2 \right) \left(1 + \text{Var} \left[\frac{M_{t+1}^p}{M_t^p} \right] \right) - \mu_m^2 \geq \bar{b}^2 (\sigma_{pc}^2 + 1) + E(\varepsilon_{t+1}^2) - \mu_m^2. \quad (\text{F10})$$

Rearranging and using equation (F8),

$$\text{Var} \left[\frac{M_{t+1}^p}{M_t^p} \right] \geq \Delta_d + \sigma_{pc}^2, \quad (\text{F11})$$

recognizing that $\left(\frac{M_{t+1}^T}{M_t^T} \right)^2 = R_{t+1,\infty}^{-2}$ and letting

$$\Delta_d \equiv \left(E(1/R_{t+1,\infty}^2) \right)^{-1} (\mu_m^2 + \sigma_{HJ}^2) - (1 + \sigma_{pc}^2) - \left(E(1/R_{t+1,\infty}^2) \right)^{-1} \text{Cov} \left[\left(\frac{M_{t+1}^p}{M_t^p} \right)^2, \left(\frac{M_{t+1}^T}{M_t^T} \right)^2 \right]. \quad (\text{F12})$$

Based on (F11), we may deduce the following:

- If $Var \left[\frac{M_{t+1}^p}{M_t^p} \right] \leq \sigma_{pc}^2$, then $\Delta_d \leq 0$;
- Alternatively, if $\Delta_d \geq 0$, then $Var \left[\frac{M_{t+1}^p}{M_t^p} \right] \geq \sigma_{pc}^2$.

To derive the condition under which $\Delta_d \geq 0$, observe that

$$\Delta_d \geq 0 \Leftrightarrow \Delta_c \geq Cov \left[\left(\frac{M_{t+1}^p}{M_t^p} \right)^2, \left(\frac{M_{t+1}^T}{M_t^T} \right)^2 \right], \quad (\text{F13})$$

where Δ_c is as defined in (F2), delivering the statement of the Lemma. Δ_c is computable from the returns data. ■

Table 1

Comparison of our variance bounds with the L -measure-based bounds in Alvarez and Jermann (2005), when $R_{t+1,\infty}$ is surrogated by the return of a bond with maturity of 20, 25, or 29 years

Here we follow Alvarez and Jermann (2005), both in the choice of *three* assets and the sample period. The bounds hinge on the return properties of long-term discount bond, risk-free bond, and single equity portfolio (optimal growth portfolio based on ten CRSP size-decile portfolios and the equity market). We proxy $R_{t+1,\infty}$ by bond returns with maturity ranging from 20 years to 29 years. The monthly data used in the construction of the bounds is from 1946:12 to 1999:12, and taken from <http://www.econometricsociety.org/suppmat.asp?id=61&vid=73&iid=6&aid=643>. Reported are the estimates (*annualized, in bold*) of the bounds from the data, along with the 90% confidence intervals (in square brackets). To obtain the confidence intervals, we create 50,000 random samples of size 637 (number of observations) from the data, where the sampling is based on 12 blocks. The displayed lower bound on $Var \left[\frac{M_{t+1}^P}{M_t^P} \right]$ and $Var \left[\frac{M_{t+1}^P}{M_t^P} \right] / Var \left[\frac{M_{t+1}}{M_t} \right]$ are, respectively, based on equations (6) and (8), while the upper bound on $Var \left[\frac{M_{t+1}^T}{M_t^T} \right] / Var \left[\frac{M_{t+1}}{M_t} \right]$ is based on equation (10). The expressions for the bounds on the L -measure are presented in Alvarez and Jermann (2005, Propositions 2 and 3).

	$R_{t+1,\infty}$ is proxied by the return of a bond with maturity:		
	20 years	25 years	29 years
Lower bound on $Var \left[\frac{M_{t+1}^P}{M_t^P} \right]$	0.799 [0.646, 0.942]	0.909 [0.710, 1.089]	0.933 [0.675, 1.151]
Lower bound on $L \left[\frac{M_{t+1}^P}{M_t^P} \right]$	0.201 [0.178, 0.226]	0.205 [0.179, 0.232]	0.217 [0.184, 0.251]
$L \left[\frac{M_{t+1}^P}{M_t^P} \right] - \frac{1}{2} Var \left[\frac{M_{t+1}^P}{M_t^P} \right]$ (Difference in bounds)	-0.198 [-0.274, -0.118]	-0.249 [-0.343, -0.145]	-0.249 [-0.364, -0.116]
Lower bound on $Var \left[\frac{M_{t+1}^P}{M_t^P} \right] / Var \left[\frac{M_{t+1}}{M_t} \right]$	0.884 [0.866, 0.904]	0.896 [0.878, 0.917]	0.898 [0.876, 0.923]
Lower bound on $L \left[\frac{M_{t+1}^P}{M_t^P} \right] / L \left[\frac{M_{t+1}}{M_t} \right]$	1.018 [0.964, 1.075]	1.039 [0.964, 1.117]	1.098 [0.977, 1.222]
Upper bound on $Var \left[\frac{M_{t+1}^T}{M_t^T} \right] / Var \left[\frac{M_{t+1}}{M_t} \right]$	0.039 [0.029, 0.047]	0.055 [0.041, 0.067]	0.102 [0.075, 0.125]
Upper bound on $L \left[\frac{M_{t+1}^T}{M_t^T} \right] / L \left[\frac{M_{t+1}}{M_t} \right]$	0.079 [0.066, 0.090]	0.111 [0.092, 0.129]	0.199 [0.165, 0.232]

Table 2

Assessing the restriction on the variance of the permanent component of SDFs from asset pricing models

Parameters in Tables Appendix-I through Appendix-III are employed to compute the variance of the permanent component of the SDF via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. Reported values are the averages from a single simulation run of 360,000 months (30,000 years). The reported lower bound σ_{pc}^2 on the permanent component (see equation (6) of Proposition 1) is based on the return properties of \mathbf{R}_{t+1} , corresponding to SET A, and the long-term bond. The maturity of the long-term bond is 20 years, as also in Alvarez and Jermann (2005, page 1993). The monthly data used in the construction of σ_{pc}^2 is from 1932:01 to 2010:12 (948 observations), with the 90% confidence intervals in square brackets. To compute the confidence intervals, we create 50,000 random samples of size 948 from the data, where the sampling in the block bootstrap is based on 15 blocks. Reported below the estimates of $Var[M_{t+1}^P/M_t^P]$ are the p -values, shown in curly brackets, which represent the proportion of replications for which model-based $Var[M_{t+1}^P/M_t^P]$ exceeds σ_{pc}^2 in 200,000 replications of a finite sample simulation over 948 months. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. Reported annualized mean, standard deviation, and first-order autocorrelation of consumption growth in Panel B are obtained by following the convention of aggregating monthly consumption series to annual. Shown in Panel C are the average real return of the risk-free bond and the average return of equity market, all based on a single simulation run. The 90% bootstrap confidence intervals on the return of risk-free bond and equity are [0.0013, 0.0048] and [0.0527, 0.1189], respectively.

	Long-run risk		External habit		Rare	Lower
	Kelly	Bansal-Yaron	Bekaert-Engstrom	Campbell-Cochrane	disasters	bound, σ_{pc}^2 (based on SET A)
<i>Panel A: Permanent component of the SDF, monthly</i>						
$Var[M_{t+1}^P/M_t^P]$	0.0374	0.0342	0.0280	0.0234	0.0580	0.1254
	{0.000}	{0.000}	{0.001}	{0.000}	{0.099}	[0.045, 0.131]
<i>Panel B: Consumption growth, annualized</i>						
Mean	0.0245	0.0189	0.0380	0.0189	0.0205	0.0341
Standard deviation	0.0494	0.0298	0.0175	0.0150	0.0298	0.0285
First-order autocorrelation	0.248	0.517	0.243	0.000	0.000	0.227
<i>Panel C: Real return of a risk-free bond and equity, average (annualized)</i>						
Real return of risk-free bond, $R_{t+1,1} - 1$	0.0240	0.0252	0.0040	0.0094	0.0200	0.0017
Real return of equity, $R_{t+1} - 1$	0.0603	0.0802	0.0583	0.0724	0.0593	0.0864

Table 3

Assessing the restriction on the variance of the transitory component of SDFs from asset pricing models

Parameters in Tables Appendix-I through Appendix-III are employed to compute $Var [M_{t+1}^T/M_t^T]$ via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. Reported values are the averages from a single simulation run of 360,000 months. The reported lower bound σ_{tc}^2 on the transitory component is based on equation (9) of Proposition 2. The maturity of the long-term bond is 20 years, and the real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. The monthly data used in the construction of σ_{tc}^2 is from 1932:01 to 2010:12 (948 observations), with the 90% confidence intervals in square brackets. To compute the confidence intervals, we create 50,000 random samples of size 948 from the data, where the sampling in the block bootstrap is based on 15 blocks. Reported below the estimates of $Var [M_{t+1}^T/M_t^T]$ are the p -values, shown in curly brackets, which represent the proportion of replications for which model-based $Var [M_{t+1}^T/M_t^T]$ exceeds σ_{tc}^2 in 200,000 replications of a finite sample simulation over 948 months. Owing to the fact that the transitory component of the SDF in the model with rare disasters is a constant and, hence, the variance of the transitory component is zero, which renders the finite sample simulation redundant. For this reason, its p -value entry is shown as “na.”

	Long-run risk		External habit		Rare disasters
	Kelly	Bansal-Yaron	Bekaert-Engstrom	Campbell-Cochrane	
$Var [M_{t+1}^T/M_t^T]$	1.7×10^{-8} {0.000}	1.9×10^{-3} {0.000}	3.5×10^{-5} {0.000}	2.2×10^{-6} {0.000}	0.0000 na
Lower bound, σ_{tc}^2	9.8 $\times 10^{-7}$ [0.0000, 0.0000]	4.3 $\times 10^{-1}$ [0.3404, 0.5026]	7.7 $\times 10^{-3}$ [0.0041, 0.0141]	7.8 $\times 10^{-4}$ [0.0005, 0.0016]	2.0 $\times 10^{-4}$ [0.0000, 0.0004]
$E (M_{t+1}^T/M_t^T)$	0.9980	1.0147	1.0002	0.9987	0.9983

Table 4

Implications of the transitory component of the SDF for the real return of long-term bond

Parameters in Tables Appendix-I through Appendix-III, in conjunction with the solution of the eigenfunction problems, are used to generate $\{\frac{M_{t+1}^T}{M_t^T}\}$ via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. Through a single simulation run of 360,000 months, we compute the population values corresponding to:

$$E(r_{t+1,\infty}) = E(R_{t+1,\infty} - 1) = E\left(\frac{M_t^T}{M_{t+1}^T} - 1\right), \quad \text{Var}[r_{t+1,\infty}] = E\left(\frac{M_t^T}{M_{t+1}^T} - E\left(\frac{M_t^T}{M_{t+1}^T}\right)\right)^2.$$

For comparison, also reported are the annualized mean and standard deviation of the real return of long-term bond for the monthly data over 1932:01 to 2010:12. The 90% confidence intervals are shown in square brackets, created from 50,000 random samples of size 948 from the data where the sampling in the block bootstrap is based on 15 blocks. The source of the monthly nominal return data is Morningstar, and the maturity of the long-term bond is 20 years. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation.

	$E(r_{t+1,\infty})$	$\sqrt{\text{Var}[r_{t+1,\infty}]}$
<i>Real return of long-term bond from models, annualized</i>		
Long-run risk, Kelly	0.0243	0.0002
Long-run risk, Bansal-Yaron	-0.1426	0.1241
<i>External habit, Bekaert-Engstrom</i>		
External habit, Bekaert-Engstrom	-0.0021	0.0236
<i>External habit, Campbell-Cochrane</i>		
External habit, Campbell-Cochrane	0.0159	0.0069
<i>Rare disasters</i>		
Rare disasters	0.0201	0.0000
<i>Real return of long-term bond in the data, annualized</i>		
1932:01-2010:12 sample	0.0243	0.0887
	[0.0081, 0.0404]	[0.0829, 0.0946]

Table 5

Implication of the models for the comovement between the permanent and the transitory components of SDFs

Parameters in Tables Appendix-I through Appendix-III, in conjunction with the solution of the eigenfunction problems, are used to generate $\{\frac{M_{t+1}^P}{M_t^P}\}$ and $\{\frac{M_{t+1}^T}{M_t^T}\}$ via simulations, respectively, for the models that incorporate long-run risk and external habit persistence. Through a single simulation run of 360,000 months, we perform the OLS regression $\frac{M_{t+1}^P}{M_t^P} = a_0 + b_0 \left(\frac{M_{t+1}^T}{M_t^T}\right) + \varepsilon_{t+1}$, thereby inferring the slope coefficient $b_0 = Cov\left[\frac{M_{t+1}^P}{M_t^P}, \frac{M_{t+1}^T}{M_t^T}\right] / Var\left[\frac{M_{t+1}^T}{M_t^T}\right]$. Further, we generate the distribution of b_0 in 200,000 replications of a finite sample simulation over 948 months, and report the values of the mean, the 95th percentile, and the 5th percentile. For comparison, reported also is the imputed value of $\frac{E(1/R_{t+1,1}) - E(1/R_{t+1,\infty})}{Var[1/R_{t+1,\infty}]}$ for the monthly data over 1932:01 to 2010:12. The 90% confidence intervals are shown in square brackets, created from 50,000 random samples of size 948 from the data where the sampling in the block bootstrap is based on 15 blocks. The source of monthly nominal returns data on the long-term bond is Morningstar, while the nominal risk-free returns is from the library of Kenneth French. The maturity of the long-term bond is 20 years. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. Since the transitory component in the model with rare disasters is a constant, the slope coefficient is identically zero, which renders the finite sample simulation redundant. For this reason, some entries are shown as “na.”

Estimate of b_0 in the regression:

$$\frac{M_{t+1}^P}{M_t^P} = a_0 + b_0 \left(\frac{M_{t+1}^T}{M_t^T}\right) + \varepsilon_{t+1}$$

	Single simulation run	Finite sample simulation		
	b_0 , population	(distribution of b_0)		
		Mean	95th	5th
Long-run risk, Kelly	-5.88	-5.88	57.68	-69.56
Long-run risk, Bansal-Yaron	-2.43	-2.44	-2.22	-2.65
External habit, Bekaert-Engstrom	-16.25	-15.55	-9.28	-22.73
External habit, Campbell-Cochrane	0.23	0.00	0.19	-0.28
Rare disasters	0.00	na	na	na
<i>Imputed from the data, b_0</i> 1932:01-2010:12 sample	1.96 [1.07, 3.96]			

Table 6

Assessing the restriction on the variance of the ratio of the permanent to the transitory components of SDFs from asset pricing models

Parameters in Tables Appendix-I through Appendix-III are employed to compute $Var \left[\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right]$ via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. Reported values are the averages from a single simulation run of 360,000 months. The reported lower bound σ_{pt}^2 on the ratio of the permanent to the transitory components is based on equation (11) of Proposition 3. The bound is based on the return properties of \mathbf{R}_{t+1} , corresponding to SET A, and the long-term bond. Monthly data used in the construction of σ_{pt}^2 is from 1932:01 to 2010:12 (948 observations), with the 90% confidence intervals in square brackets. To compute the confidence intervals, we create 50,000 random samples of size 948 from the data, where the sampling in the block bootstrap is based on 15 blocks. Reported below the estimates of $Var \left[\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right]$ are the p -values, shown in curly brackets, which represent the proportion of replications for which model-based $Var \left[\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right]$ exceeds σ_{pt}^2 in 200,000 replications of a finite sample simulation over 948 months. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation.

	Long-run risk		External habit		Rare disasters
	Kelly	Bansal-Yaron	Bekaert-Engstrom	Campbell-Cochrane	
$Var \left[\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right]$	0.0375 {0.000}	0.0449 {0.000}	0.0298 {0.001}	0.0234 {0.000}	0.0582 {0.094}
Lower bound, σ_{pt}^2	0.1309 [0.0481, 0.1381]	0.1842 [0.0868, 0.1982]	0.1237 [0.0429, 0.1296]	0.1264 [0.0448, 0.1327]	0.1324 [0.0491, 0.1399]
$E \left(\frac{M_{t+1}^P}{M_t^P} / \frac{M_{t+1}^T}{M_t^T} \right)$	1.0018	0.9916	1.0008	1.0012	1.0019

Table Appendix-I

Parametrization of asset pricing models incorporating long-run risk

The long-run risk model of Kelly (2009) is based on the dynamics depicted in (25)–(27), while the long-run risk model of Bansal and Yaron (2004) is based on

$$\begin{aligned} g_{t+1} &= \mu + x_t + \sigma_t z_{g,t+1}, & x_{t+1} &= \rho_x x_t + \sigma_x \sigma_t z_{x,t+1}, \\ \sigma_{t+1}^2 &= \bar{\sigma}^2 + \rho_\sigma (\sigma_t^2 - \bar{\sigma}^2) + \sigma_\sigma z_{\sigma,t+1}, & z_{g,t+1}, z_{x,t+1}, z_{\sigma,t+1} &\sim \text{i.i.d. } \mathcal{N}(0, 1). \end{aligned}$$

In both models, parameters chosen for the Epstein and Zin (1991) utility function, the mean consumption growth, the predictable consumption component, and the conditional volatility of the log consumption growth are generally in line with the studies of Beeler and Campbell (2009), Constantinides and Ghosh (2008), Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010), and Yang (2010). Parameters of the conditional volatility of the tails are chosen to plausibly mimic the process that governs the conditional volatility of the consumption growth in Bansal and Yaron (2004). The models are simulated at the monthly frequency.

	Parameter	Kelly	Bansal-Yaron
<i>Preferences</i>			
Time preference	β	0.982	0.980
Risk aversion	γ	11	10
Elasticity of intertemporal substitution	ψ	1.5	1.5
<i>Consumption growth dynamics, g_t</i>			
Mean	μ	0.0019	0.0015
Volatility parameter	σ_g	0.009	
<i>Long-run risk, x_t</i>			
Persistence	ρ_x	0.400	0.987
Volatility parameter	σ_x	0.009	0.044
<i>Consumption growth volatility, σ_t</i>			
Mean	$\bar{\sigma}$	0.0018	0.0078
Persistence	ρ_σ	0.450	0.978
Volatility parameter	σ_σ	0.0001	0.23×10^{-5}
<i>Tail risk, Λ_t</i>			
Mean	$\bar{\Lambda}$	0.012^2	
Persistence	ρ_Λ	0.45	
Volatility parameter	σ_Λ	0.17×10^{-5}	

Table Appendix-II

Parametrization of asset pricing models incorporating external habit persistence

The external habit persistence model of Bekaert and Engstrom (2010) is based on the dynamics depicted in equations (31)–(34), while in Campbell and Cochrane (1999), the dynamics of consumption growth and surplus consumption ratio are

$$g_{t+1} = \bar{g} + v_{t+1}, \quad s_{t+1} - \bar{s} = \phi(s_t - \bar{s}) + \lambda[s_t] v_{t+1}, \quad v_{t+1} \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2).$$

The SDF in Campbell and Cochrane (1999) is $\frac{M_{t+1}}{M_t} = \beta \exp(-\gamma \bar{g} - \gamma(\phi - 1)(s_t - \bar{s}) - \gamma(1 + \lambda[s_t])v_{t+1})$, where the sensitivity function $\lambda[s_t]$ is $\lambda[s_t] = \frac{1}{\bar{s}} \sqrt{1 - 2(s_t - \bar{s})} - 1$ if $s_t \leq s_{\max}$ and zero otherwise. As in Campbell and Cochrane, $s_{\max} = \bar{s} + \frac{1}{2}(1 - \bar{s}^2)$, $\bar{s} = \sigma \sqrt{\frac{\gamma}{1 - \phi}}$, and $\bar{s} = \log(\bar{S})$ is the log steady state. Our parameterizations are based on Bekaert and Engstrom (2010, Table 3) and Campbell and Cochrane (1999, Table 1). The models are simulated at the monthly frequency.

	Parameter	Bekaert-Engstrom	Campbell-Cochrane
<i>Preferences</i>			
Time preference	β	0.9970	0.9903
Risk aversion	γ	3.0690	2.0000
<i>Consumption growth dynamics, g_t</i>			
Mean	\bar{g}	0.0031	0.0015
Volatility parameter	σ		0.0043
<i>Long-run risk, x_t</i>			
Persistence	ρ_x	0.9467	
Volatility parameter	$\sigma_{xp} \times 10^3$	-0.0016	
Volatility parameter	$\sigma_{xn} \times 10^3$	-0.0140	
Volatility parameter	σ_{gp}	0.0012	
Volatility parameter	σ_{gn}	0.0052	
<i>p_t dynamics</i>			
Mean	\bar{p}	12.1244	
Persistence	ρ_p	0.9728	
Volatility parameter	σ_{pp}	0.4832	
<i>n_t dynamics</i>			
Mean	\bar{p}	0.6342	
Persistence	ρ_p	0.9810	
Volatility parameter	σ_{pp}	0.1818	
<i>Log of inverse consumption surplus, q_t</i>			
Mean	\bar{q}	1.0000	
Persistence	ρ_q	0.9841	
Volatility parameter	σ_{qp}	0.0005	
Volatility parameter	σ_{qn}	0.0479	
Steady state surplus consumption ratio	\bar{S}		0.0570
Persistence in consumption surplus ratio	ϕ		0.9884
Log of risk-free rate	$r^f \times 10^2$		0.0783

Table Appendix-III

Parametrization of an asset pricing model incorporating rare disasters

The dynamics for consumption growth g_{t+1} is as depicted in equations (36)–(37), and the simulations are performed at the monthly frequency. We first adopt annualized parameters from Backus, Chernov, and Martin (2011, Table 3, Column 2), which are then transformed to monthly by scaling μ by $1/12$, σ by $1/\sqrt{12}$, and ω by $1/12$ (see page 42 of Backus, Chernov, and Martin (2011, equation (39))). The annual time preference parameter β in the model is obtained by equating $1/R_{t+1,1} = E_t(M_{t+1}/M_t)$ and by setting the average log return of the risk-free bond, as outlined on their page 5, to 2%. Following Campbell and Cochrane (1999, Table 1), among others, the monthly time preference parameter is the annual time preference parameter raised to the power $1/12$. The leverage parameter λ captures the ratio of the volatility of dividend growth to consumption growth and is set to 2.60 following Wachter (2011, Table 1), and is used in the computation of expected return of equity.

		Annual	Monthly
<i>Preferences</i>			
Time preference	β	1.0448	$1.0448^{1/12}$
Risk aversion	γ	5.190	5.190
<i>Gaussian component of consumption growth, w_{t+1}</i>			
Mean	μ	0.023	$0.023/12$
Volatility	σ	0.01	$0.01/\sqrt{12}$
<i>Non-gaussian component of consumption growth, z_{t+1}</i>			
Mean of Poisson density	ω	0.010	$0.010/12$
Mean of z_{t+1} conditional on J_{t+1}	θ	-0.30	-0.30
Variance of z_{t+1} conditional on J_{t+1}	δ^2	0.15^2	0.15^2
Mapping between dividend and consumption, $D_t = C_t^\lambda$	λ	2.60	2.60

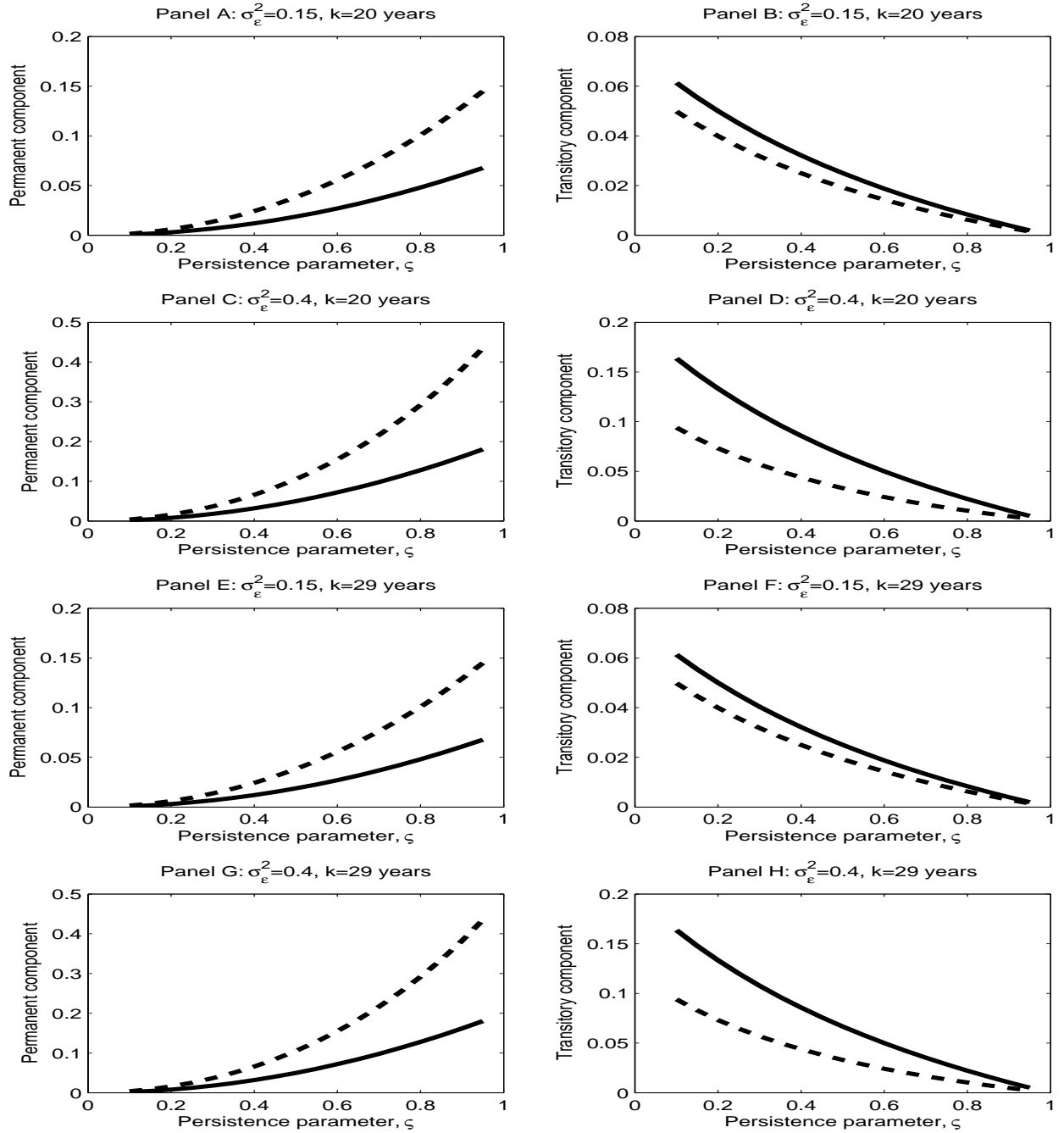


Fig. 1. Permanent and transitory components from the asset pricing model in Example 2

Plotted are the unconditional L -measure (solid black curve) and the variance-measure (solid dashed curve) corresponding to the permanent and the transitory components of SDF, when the pricing kernel process is $\log(M_{t+1}) = \log(\beta) + \varsigma \log(M_t) + \varepsilon_{t+1}$, with $\varepsilon_{t+1} \sim \mathcal{N}(0, \sigma_\varepsilon^2)$. The $L \left[\frac{M_{t+1}^P}{M_t^P} \right]$ and $L \left[\frac{M_{t+1}^T}{M_t^T} \right]$, for the permanent and the transitory component, are displayed in equation (19), while $\text{Var} \left[\frac{M_{t+1}^P}{M_t^P} \right]$ and $\text{Var} \left[\frac{M_{t+1}^T}{M_t^T} \right]$ are displayed in equations (20)–(21). For this illustration, we keep $\beta = 0.998$, $t=1200$ months, and the maturity, k , of the long-term discount bond is either 20 or 29 years.

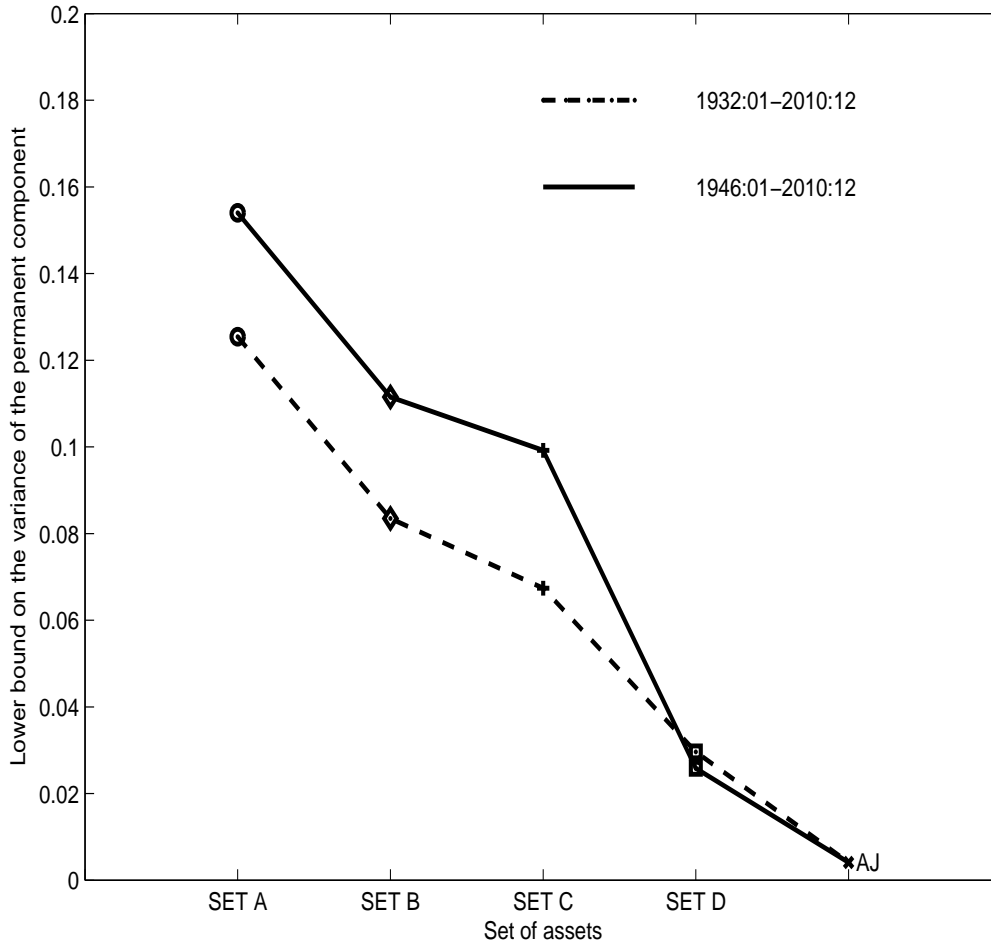


Fig. 2. **Tightness of variance bounds from Proposition 1 compared to Alvarez and Jermann (2005)**

Plotted are the unconditional variance bound σ_{pc}^2 on the permanent component of the SDF corresponding to SET A through SET D, as per equation (6) of Proposition 1. The Alvarez and Jermann (2005, equation (4)) lower bound is computed as the sample analog of $E(\log(R_{t+1}/R_{t+1,\infty}))$, and marked as AJ. SET A contains the risk-free bond, equity market, intermediate government bond, and 25 Fama-French equity portfolios sorted by size and book-to-market; SET B contains risk-free bond, equity market, intermediate government bond, ten size-sorted portfolios, and six size and book-to-market portfolios; SET C contains risk-free bond, equity market, intermediate government bond, and six size and book-to-market portfolios; and SET D contains risk-free bond, equity market, and intermediate government bond. Data on risk-free bond and equity returns are from the web site of Kenneth French, while returns of intermediate and long-term government bonds are from Morningstar (formerly, Ibbotson). Following Alvarez and Jermann (2005, page 1993), the maturity of the long-term bond is taken to be 20 years. The computations employ monthly returns over 1932:01 to 2010:12 sample and also over the 1946:01 to 2010:12 subsample. Real returns are computed by deflating nominal returns by the Consumer Price Index inflation.