Inflation, Asset Prices, and the Term Structure of Interest Rates in Monetary Economies

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This article offers a tractable monetary asset pricing model. In monetary economies, the price level, inflation, asset prices, and the real and nominal interest rates have to be determined simultaneously and in relation to each other. This link allows us to relate in closed form each of the dependent entities to the underlying real and monetary variables. Among other features of such economies, inflation can be partially nonmonetary and the real and nominal term structures can depend on fundamentally different risk factors. In one extreme, the process followed by the real term structure is independent of that followed by its nominal counterpart.

This article studies the endogenous and simultaneous determination of the price level, inflation, asset prices, and the term structure of interest rates, both real and nominal. The setup is an intertemporal monetary economy where there is fiat money for transaction purposes and where investors find it useful to hold cash balances. The objective is to relate, in the spirit of Cox, Ingersoll, and Ross (1985b) (henceforth,

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CIR), the price level, equity returns, and both the real and the nominal term structures to primitive economic variables. As in CIR, the closed-form solutions in continuous time give us a way to understand and predict how changes in the real and monetary variables will affect the money, the stock, and the bond markets.

Our approach provides a way of integrating asset pricing theory with models from monetary economics. In the theory of asset prices and the term structure of interest rates, powerful tools from continuous-time mathematics have been used to derive closed-form formula that parsimoniously relate asset prices and interest rates to underlying economic variables. Premier examples include the consumption-based CAPM of Breeden (1979) and Cox, Ingersoll and Ross (1985a) and the term structure models of Constantinides (1992), CIR, Longstaff and Schwartz (1992), and Sun (1992). However, in most such models, money often plays no role and all prices are denominated in units of the numeraire consumption good. Consequently, when building term structure models of *nominal* interest rates, for instance, authors often choose to assume some *exogenously given* processes for the price level and the expected inflation rate [see, for example, CIR, Pennacchi (1991), and Sun (1992)] presumably with the understanding that these processes are in some way due to the existence of money. While such models have helped gain some understanding of the determinants of asset prices and interest rates, we do not yet know whether the assumed processes for the price level and inflation can be consistent with any general equilibrium in which these variables are endogenized. Overall, when money plays no direct role, the analysis can only be partial equilibrium in nature.

Some models in monetary economics that address asset pricing issues, however, do assume a role for money and endogenize the price level and inflation together with stock prices [see, for example, Boyle (1990), Danthine and Donaldson (1986), Foresi (1990), Lee (1989, 1992), LeRoy (1984), Lucas (1982), Marshall (1992), and Stulz (1986)]. These studies are typically motivated by citing the following empirical findings:

- Real stock returns are negatively correlated with inflation, expected or unexpected. Stock price levels are negatively correlated with the price level of consumption. In addition, real returns to nominally risk-free bonds are also negatively correlated with inflation [see, for instance, Fama (1981), Fama and Gibbons (1982), and Marshall (1992)].
- Real stock returns are positively correlated with money growth [Marshall (1992)]. Nominal stock prices are negatively related to the contemporaneous velocity of money, while real stock prices

are positively correlated with the velocity of money three quarters ahead [Friedman (1988)].
- Assets that are positively correlated with inflation earn a lower risk premium [Chen, Roll, and Ross (1986)].

The negative correlations between asset returns and inflation are contradictory to the traditional view that equity shares should be usable as hedges against inflation. When inflation is viewed as a purely monetary phenomenon and when the role of money is assumed away, the positive correlations between asset returns and money growth are also puzzling to the theory of finance. As shown by Boyle (1990), Danthine and Donaldson (1986), Foresi (1990), LeRoy (1984), and Stulz (1986), once money is introduced with the role that it contributes to investors' utility, negative correlations between inflation and stock returns become an equilibrium property. By assuming that fiat money facilitates consumption transactions, Marshall (1992) also endogenizes this property as part of economic equilibrium. Based on simulations, he shows that real returns will be positively related to money growth. In existing monetary asset pricing models, however, the focus is typically on the determination of stock market prices, not on that of the term structure of interest rates, real or nominal.

This article follows a tradition of monetary economics in that real cash balances directly enter the period utility of consumption function [e.g., Brock (1974, 1975)]. Feenstra (1986) has shown that this way of incorporating a role for money is equivalent to assuming that money facilitates consumption transactions. In our framework, the empirical facts listed above are also consistent with general equilibrium. However, our analysis differs from the existing monetary models in several ways. First, the existing models are generally cast on a discrete-time setup, which renders analytical solutions difficult to obtain. Here, as in Sun (1992), we start with a discrete-time economy and then take it to the continuous-time limit, which allows us to solve in closed form for all dependent variables in question. Among other things, these solutions show the exact relations of the financial markets to the underlying economy and make the theoretical predictions amenable for empirical applications. Second, we provide a complete analysis of the joint determination of the price level, inflation, equity prices, and the term structures. In this sense, our study contains most elements from Boyle (1990), Danthine and Donaldson (1986), Marshall (1992), and Stulz (1986), on the relations between inflation, asset returns, and money growth, and from Bosshardt (1987), Constantinides (1992), CIR, Longstaff and Schwartz (1992), Pennacchi (1991), and Sun (1992), on the nominal and real term structures of interest rates.

In particular, we show that the nominal and real term structures can have completely different properties, including their having fundamentally different risk structures. For instance, in some cases, the nominal term structure is perfectly correlated with the money supply process, while at the same time the real term structure is perfectly correlated with the production output process. In those cases, if monetary shocks are independent of output shocks, the process followed by the real term structure will be independent of that followed by its nominal counterpart. This finding is in sharp contrast with those in the existing term structure literature. For example, in CIR and Sun (1992), the real term structure has technological shocks as its single risk factor, while its nominal counterpart has two risk factors: technological and inflationary shocks. Given their results, one would not have considered the possibility that the real and the nominal term structures may follow independent processes. Third, our framework is general enough that for a reasonable class of stochastic processes for the underlying real and monetary variables, closed-form solutions for the commodity price level, stock prices, and the real and nominal term structures exist. This feature should be especially useful for empirical implementations.

This article is organized as follows. Section 1 formally outlines a dynamic monetary economy in which production output and money supply are the two underlying state variables. Section 2 offers general formulas that relate the equilibrium risk premium, the price level, inflation, and both the nominal and real interest rates to real activity and money supply. Section 3 examines a simple economy in which both money growth and output growth follow an i.i.d. process. This simple example serves to illustrate some of the basic properties of our general model. Section 4 explores a more realistic but more complex economy in which output growth and money growth still follow autonomous processes. There, we give a detailed analysis of the solutions, compare the real with the nominal term structure, and contrast our results with those from the existing literature. The last section concludes the article. The Appendix contains the proof of each result.

1. A Dynamic Monetary Economy with Uncertainty

Consider a representative-agent monetary economy where (1) a single perishable consumption good exists and (2) the agent's period utility function depends on both consumption and cash balances. This economy shares one crucial feature with Brock (1974), Danthine and Donaldson (1986) and Stulz (1986) in that a person's real cash balance is an argument in the direct utility function. Feenstra (1986) shows that this way of modeling the role of money is equivalent to incorporating money into liquidity costs in the budget constraint [see Marshall (1992) and the references therein]. Initially we assume that consumption, money demand, and portfolio adjustment decisions take place at discrete time intervals of length Δt; later we take the model to its continuous-time limit. Specifically, the infinitely lived agent chooses consumption, money demand, and portfolio holdings at each point of time so as to maximize his expected lifetime utility:

$$\max_{c_t, M_t^d:\ t=0,\ \Delta t,\ 2\Delta t,\ \ldots,\ \infty} \sum_{t=0}^{\infty} e^{-\rho\, t}\, E_0 \left\{ u\left(c_t, \frac{M_t^d}{P_t^c} \right) \right\} \Delta t, \tag{1}$$

where E_t is the expectation conditional on all time t information, c_t denotes the consumption flow during the interval $[t, t + \Delta t)$, M_t^d is the nominal money demand from time $(t - \Delta t)$ to t, P_t^c is the price of the consumption good, and ρ represents the discount factor. We adopt the convention that upper- and lowercase letters denote, respectively, the nominal and the real variable of the corresponding economic entity. Further assume that the utility function is twice continuously differentiable and concave in both real money demand and consumption, that is, $u_c > 0$, $u_m > 0$, $u_{cc} < 0$, $u_{mm} < 0$, $u_{cm} < 0$, $u_{cc}u_{mm} - (u_{cm})^2 > 0$, where $m_t \equiv \left(\frac{M_t^d}{P_t^c}\right)$ is the real money demand and subscripts on u denote the corresponding partial derivatives.

The intertemporal budget constraint for the problem in Equation (1) is constructed as follows. First, there is one equity share traded whose holder is entitled to all the output of a single production technology. This technology produces the sole consumption good. Its output in terms of units of the good, y_t, is governed by

$$\frac{\Delta y_t}{y_t} = \frac{y_{t+\Delta t} - y_t}{y_t} = \mu_{y,t}\ \Delta t + \sigma_{y,t}\ B_{y,t}\sqrt{\Delta t}, \tag{2}$$

where $\mu_{y,t}$ and $\sigma_{y,t}$ are, respectively, the conditional expected value and standard deviation of output growth per unit time, and $\{B_{y,t}\colon\ t = 0, \Delta t, \cdots\}$ is an i.i.d., standard normal process. The terms $\mu_{y,t}$ and $\sigma_{y,t}$ can be time varying, as in Cox, Ingersoll, and Ross (1985b) and Sun (1992). Let the time t nominal price of the equity share be denoted by $P_{z,t}$.

Second, also traded are one real bond and one nominal bond, both risk-free, and $(N-2)$ financial assets. The risk-free bonds are issued at each time t and mature at $(t+\Delta t)$, except that the real bond pays one unit of the good, whereas the nominal bond pays one dollar cash, at maturity. The real (the nominal) interest rate at time t, denoted by r_t (by R_t), is simply the real (nominal) rate of return on the real (nominal) bond. We refer to the real and the nominal bonds as financial assets

1 and 2, respectively. At time t, the nominal cum dividend price for every other financial asset i is denoted by $P_{i,t}$ for $i = 3, \dots, N$.

Finally, at each decision time t, consumption c_t, cash demand M_t^d, equity holding z_t (shares), and financial holdings $\alpha_t = (\alpha_{1,t}, \dots, \alpha_{N,t})'$ where $\alpha_{i,t}$ for $i = 1, \dots, N$ is the number of units of financial asset i held from $(t - \Delta t)$ to t, must satisfy

$$\begin{aligned} M_t^d + (P_{z,t} + P_t^c\, y_t\, \Delta t)\; z_t + P_t^c\, \alpha_{1,t} + \alpha_{2,t} + \sum_{i=3}^{N} P_{i,t}\; \alpha_{i,t} \\ = \; P_t^c\; c_t\; \Delta t + M_{t+\Delta t}^d + P_{z,t}\; z_{t+\Delta t} \\ + P_t^c\, \frac{\alpha_{1,t+\Delta t}}{1 + r_t \Delta t} + \frac{\alpha_{2,t+\Delta t}}{1 + R_t \Delta t} + \sum_{i=3}^{N} P_{i,t}\; \alpha_{i,t+\Delta t}, \end{aligned} \tag{3}$$

where the right-hand side is the time t "expenses," while the left-hand side is the total nominal value of everything carried over from the past.

Monetary policy in this economy is such that the resulting money supply, M_t^s, follows a stochastic process over time:

$$\frac{\Delta M_t^s}{M_t^s} = \frac{M_{t+\Delta t}^s - M_t^s}{M_t^s} = \mu_{M,t}\; \Delta t + \sigma_{M,t}\; B_{M,t} \sqrt{\Delta t}, \tag{4}$$

where $\{B_{M,t} : t = 0, \Delta t, \dots\}$ is again an i.i.d., standard normal process, and $\mu_{M,t}$ and $\sigma_{M,t}$ are, respectively, the conditional expected value and standard deviation of money growth rate per unit time. In addition, we allow the money supply and output processes to be correlated.

2. Equilibrium Dynamics for Inflation and Asset Prices

In the assumed representative-agent economy, optimal consumption, money demand and portfolio holdings, $\{c_t, M_t^d, z_t, \alpha_t\}$, and prices have to adjust at each t such that in general equilibrium

$$\begin{aligned} c_t &= y_t, \\ M_t &\equiv M_t^s = M_t^d, \\ z_t &= 1, \\ \alpha_{i,t} &= 0 \qquad \forall i = 1, \dots, N \end{aligned}$$

for each t. Using these market clearing conditions, we can write the first-order conditions for the representative agent's problem in Equation (1) as follows:

$$u_c(y_t, m_t) \; = \; e^{-\rho \Delta t} E_t \left\{ u_c(y_{t+\Delta t}, m_{t+\Delta t}) \; (1 + r_t \Delta t) \right\} \tag{5}$$

$$u_c(y_t, m_t) = e^{-\rho\Delta t} E_t \left\{ u_c(y_{t+\Delta t}, m_{t+\Delta t})\ (1 + R_t \Delta t)\ \frac{P^c_t}{P^c_{t+\Delta t}} \right\} \tag{6}$$

$$\begin{aligned} u_c(y_t, m_t) &= e^{-\rho\Delta t} E_t \left\{ u_c(y_{t+\Delta t}, m_{t+\Delta t})\ \frac{p_{i,t+\Delta t}}{p_{i,t}} \right\} \\ &\quad i = 3, \ldots, N \end{aligned} \tag{7}$$

$$\begin{aligned} u_c(y_t, m_t) &= e^{-\rho\Delta t} E_t \left\{ \left[u_c(y_{t+\Delta t}, m_{t+\Delta t}) \right.\right. \\ &\qquad \left.\left. + \, u_m\left(y_{t+\Delta t}, m_{t+\Delta t}\right) \Delta t \right] \left(\frac{P^c_t}{P^c_{t+\Delta t}} \right) \right\}, \end{aligned} \tag{8}$$

where $m_t \equiv \frac{M_t}{P^c_t}$ is the real cash balance and $p_{i,t} \equiv \frac{P_{i,t}}{P^c_t}$ the real price of asset i at time t. Furthermore, Equation (7) has to hold for the equity share when $\frac{p_{i,t+\Delta t}}{p_{i,t}}$ is replaced by $\frac{p_{z,t+\Delta t} + y_{t+\Delta t} \cdot \Delta t}{p_{z,t}}$, where $p_{z,t} \equiv \frac{P_{z,t}}{P^c_t}$ is the real price of the equity share. Economically, the above equations mean the following. First, by Equation (8), the agent in equilibrium should be indifferent between holding P^c_t dollars more cash at time t and consuming one extra unit of the good, because both will result in the same marginal utility. This establishes the link between the price level and monetary policy. Second, by Equations (5), (6), and (8), the agent should also be indifferent between holding P^c_t dollars more cash and investing one extra unit of the good in the real or nominal risk-free bond, which fixes the link between interest rates and money supply. Third, by Equations (7) and (8), the agent should get the same amount of marginal utility, whether he holds P^c_t dollars more cash or simply invests it in any risky asset i. These two equations ensure that asset prices are in line with the equilibrium money supply and vice versa. Therefore, the above equations together provide the key links between the good market, the asset markets, and monetary policy.

In addition to the first-order conditions, two transversality conditions must be satisfied:

$$e^{-\rho T} E_t \left\{ \frac{u_c(y_T, m_T)}{u_c(y_t, m_t)}\ p_{i,T} \right\} \to 0, \quad as\ \ T \to \infty, \tag{9}$$

and

$$e^{-\rho T} E_t \left\{ \frac{u_c(y_T, m_T)}{u_c(y_t, m_t)}\ \frac{1}{p^c_T} \right\} \to 0, \quad as\ \ T \to \infty. \tag{10}$$

These conditions ensure the existence of an interior optimum. To briefly see this, first, suppose that Equation (9) were violated. Then, for each *little* amount of consumption reduction at time t, the agent would harvest a *large* marginal utility from investment-generated distant future consumption, which means that he would continue reducing current consumption without bound or, in the case that a bound exists, until the lower bound is reached. Second, suppose that Equation (10) did not hold. Then, the agent would want to continue reducing current consumption and hold as much cash as possible because, for each little reduction in current consumption, the agent could receive a large marginal utility from the increased future money services.

To characterize equilibrium relations, further assume, as is standard in the literature [e.g., Grossman and Shiller (1982), Merton (1971)], that the real asset prices follow a vector diffusion process:

$$\frac{\Delta p_{i,t}}{p_{i,t}} = \mu_{i,t}\ \Delta t + \sigma_{i,t} B_{i,t}\ \sqrt{\Delta t}, \tag{11}$$

where $\mu_{i,t}$ and $\sigma_{i,t}$ are, respectively, the conditional expected value and the standard deviation of the real rate of return per unit time on asset i, and $\{B_{i,t} : t = 0, \Delta t, \cdots\}$ is an i.i.d., standard normal process. The parameters $\mu_{i,t}$ and $\sigma_{i,t}$ can depend on the time t state of the economy. Now, we may prove:

Theorem 1. *In the continuous-time limit, equilibrium risk premiums for risky assets and the real equity price are given below:*

(i) The expected risk premium on asset i satisfies

$$\begin{aligned}\mu_{i,t} - r_t \ &= \ -\frac{c_t u_{cc}}{u_c}\ \mathrm{cov}_t\left(\frac{dp_{i,t}}{p_{i,t}}, \frac{dy_t}{y_t}\right) \\ &\quad - \frac{m_t u_{cm}}{u_c}\ \mathrm{cov}_t\left(\frac{dp_{i,t}}{p_{i,t}}, \frac{dm_t}{m_t}\right),\end{aligned} \tag{12}$$

where asset i can be any financial asset or the equity share and $\mathrm{cov}_t(\cdot, \cdot)$ *is the time t conditional covariance operator divided by dt;*

(ii) The real price for the equity share is

$$p_{z,t} = E_t \int_t^\infty e^{-\rho(s-t)} \frac{u_c(y_s, m_s)}{u_c(y_t, m_t)}\, y_s\, ds. \tag{13}$$

The equilibrium real price for the stock market, $p_{z,t}$, is thus equal to the total discounted value of all future dividend incomes. This conclusion is a standard result, and hence no further comment is necessary.

According to Equation (12), both production risk and monetary risk in this economy matter for the purpose of asset valuation. The expected risk premium on a risky asset is linear in its covariance with both production risk and monetary risk, and assets command different risk premiums based on their hedging capabilities against the two state variables. Intuitively, since investors care about not only consumption but also cash balances, fluctuations in the monetary sector can bring about changes in the real sector, and vice versa. As a result, both production and monetary risks should be compensated in equilibrium. Clearly, Breeden's (1979) consumption-based capital asset pricing model obtains when $u_{cm} = 0$. Stulz's (1986) pricing equation [Equation (8)] is also a special case of Equation (12). In his framework, the period utility function is $u(c, m) = \phi \ln(c) + (1 - \phi) \ln(m)$, which yields $u_{cm} = 0$.

In the empirical literature, Chen, Roll, and Ross (1986), for example, have shown that assets that are positively correlated with inflation command a lower risk premium. To see that this can occur in our rational expectations equilibrium, rewrite Equation (12) as

$$\begin{aligned}\mu_{i,t} - r_t \;=\; & -\frac{c_t u_{cc}}{u_c}\ \mathrm{cov}_t\left(\frac{dp_{i,t}}{p_{i,t}}, \frac{dy_t}{y_t}\right) \\ & -\frac{m_t u_{cm}}{u_c}\left\{\mathrm{cov}_t\left(\frac{dp_{i,t}}{p_{i,t}}, \frac{dM_t}{M_t}\right) - \mathrm{cov}_t\left(\frac{dp_{i,t}}{p_{i,t}}, \frac{dP^c_t}{P^c_t}\right)\right\}, \\ & \forall i.\end{aligned}$$

Suppose that for some asset i, its covariance with inflation is positive. Then, *ceteris paribus*, the expected risk premium on that asset will be lower (recall that $u_{cm} < 0$), since such assets are desirable for inflation hedging by risk averse investors. Analogously, risk premium tends to be higher for assets that are positively correlated with the growth rate of nominal money supply or aggregate real activity.

Theorem 2. *In the continuous-time limit, equilibrium interest rates and the price level are characterized as follows:*

(i) The real interest rate is

$$\begin{aligned}r_t \;=\; & \rho - \frac{y_t u_{cc}}{u_c}\frac{1}{dt}E_t\left\{\frac{dy_t}{y_t}\right\} - \frac{1}{2}\frac{y_t^2 u_{ccc}}{u_c}\,\mathrm{var}_t\left\{\frac{dy_t}{y_t}\right\} \\ & -\frac{m_t u_{cm}}{u_c}\frac{1}{dt}E_t\left\{\frac{dm_t}{m_t}\right\} - \frac{1}{2}\frac{m_t^2 u_{cmm}}{u_c}\,\mathrm{var}_t\left\{\frac{dm_t}{m_t}\right\} \\ & -\frac{y_t m_t u_{ccm}}{u_c}\ \mathrm{cov}_t\left(\frac{dy_t}{y_t}, \frac{dm_t}{m_t}\right),\end{aligned} \tag{14}$$

where $\mathrm{var}_t(\cdot)$ *is the conditional time t variance operator divided by dt;*

(ii) The nominal interest rate at time t

$$R_t = \frac{u_m(y_t, m_t)}{u_c(y_t, m_t)}; \tag{15}$$

(iii) The commodity price level at time t

$$\frac{1}{P^c_t} = E_t \int_t^{\infty} e^{-\rho\,(s-t)}\,\frac{u_m(y_s, m_s)}{u_c(y_t, m_t)}\,\frac{1}{P^c_s}\,ds, \tag{16}$$

and the expected inflation rate, denoted by π_t,

$$\begin{aligned}\pi_t &\equiv \frac{1}{dt}E_t\left\{\frac{dP_t^c}{P_t^c}\right\}\\ &= R_t - r_t + \mathrm{var}_t\left(\frac{dP_t^c}{P_t^c}\right) - \frac{y_t u_{cc}}{u_c}\,\mathrm{cov}_t\left(\frac{dP_t^c}{P_t^c}, \frac{dy_t}{y_t}\right)\\ &\quad -\frac{m_t u_{cm}}{u_c}\,\mathrm{cov}_t\left(\frac{dP_t^c}{P_t^c}, \frac{dm_t}{m_t}\right), \qquad (17)\end{aligned}$$

where $\frac{dP_t^c}{P_t^c}$ *is the continuous-time limit of* $\frac{P_{t+\Delta t}^c - P_t^c}{P_t^c}$ *as* $\Delta t \to 0$.

The real interest rate in Equation (14) differs from the now-standard interest rate equations in Breeden (1986) and CIR, in that money supply may also affect the real interest rate. Note that even in cases where the utility functions are such that $u_{cm} = u_{ccm} = u_{cmm} = 0$ [e.g., the additive log utility function in Stulz (1986)], monetary policy can still influence real interest rates. For example, the output process y_t can be a function of M_t.

The nominal interest rate given in Equation (15) may be interpreted as follows. The right-hand side, $\frac{u_m}{u_c}$, is the marginal rate of substitution between consumption and real cash holding, and it can be viewed as the marginal benefit of holding one additional unit of cash balance, whereas the left-hand side, R_t, is the cost of doing so in terms of the interest foregone. Then, in equilibrium, the cost must equal the benefit.

Although the price level in Equation (16) is not yet solved for in closed-form, it provides the useful intuition that the real value of one dollar of cash, $\frac{1}{P_t^c}$, is equal to the total "discounted value" of future marginal benefits that the service from holding a dollar forever can bring. As will be clear later, this equation allows us to derive the endogenous price level, and nominal interest rates, in closed form and without having to solve any partial differential equations.

To appreciate Equation (17), first note that the Fisher identity does not hold in the type of economies in question. Second, rearrange Equation (17) as

$$\begin{aligned}\left\{R_t - \pi_t + \mathrm{var}_t\left(\frac{dP_t^c}{P_t^c}\right)\right\} - r_t &= \frac{c_t u_{cc}}{u_c}\,\mathrm{cov}_t\left(\frac{dP_t^c}{P_t^c}, \frac{dy_t}{y_t}\right)\\ &\quad + \frac{m_t u_{cm}}{u_c}\,\mathrm{cov}_t\left(\frac{dP_t^c}{P_t^c}, \frac{dm_t}{m_t}\right).\end{aligned}$$

Then, if we treat $\left\{R_t - \pi_t + \mathrm{var}_t\left(\frac{dP_t^c}{P_t^c}\right)\right\}$ as the "implied real rate" on the nominal bond, the left-hand side of the above equation becomes the risk premium of the nominal over the real bond and the right-hand side shows how this risk premium is determined in equilibrium. This reconfirms the assertion that, relative to a *real* instantaneous risk-free bond, a *nominal* instantaneous risk-free bond is a risky asset, and hence deserves a risk premium in real terms, which explains why it is generally hard to back out the real interest rate from its nominal counterpart. Equation (17) also closely resembles Equation (60) of CIR. In their case, the direct utility u would be replaced by their indirect utility of wealth, production output y_t by their real wealth term, and money supply M_t would be treated as a state variable.

To see the general relationship between the real and nominal term structures of interest rates, let $b(t, \tau)$ be the time t real price of a pure discount bond that pays one unit of consumption and $N(t, \tau)$ be the time t nominal price of a discount bond that pays one dollar, in τ periods. By the Euler equation in Equation (7), $b(t, \tau)$ and $N(t, \tau)$ must satisfy in equilibrium

$$b(t, \tau) = e^{-\rho\tau} E_t \left[\frac{u_c(y_{t+\tau}, m_{t+\tau})}{u_c(y_t, m_t)} \right] \tag{18}$$

$$\frac{N(t, \tau)}{P_t^c} = e^{-\rho\tau} E_t \left[\frac{u_c(y_{t+\tau}, m_{t+\tau})}{u_c(y_t, m_t)} \, \frac{1}{P_{t+\tau}^c} \right]. \tag{19}$$

Using the rule that $E_t(x \cdot z) = \mathrm{cov}_t(x, z) + E_t(x)\, E_t(z)$ for any two random variables x and z, we obtain from Equations (18) and (19)

$$\begin{aligned} \frac{N(t, \tau)}{P_t^c} &= e^{-\rho\tau} \mathrm{cov}_t \left(\frac{u_c(y_{t+\tau}, m_{t+\tau})}{u_c(y_t, m_t)}, \, \frac{1}{P_{t+\tau}^c} \right) \\ &\quad + b(t, \tau)\, E_t \left(\frac{1}{P_{t+\tau}^c} \right), \end{aligned} \tag{20}$$

which determines the equilibrium relationship between the nominal and the real yields to maturity for any default-free discount bond.

In the existing term structure literature, log utility functions are often used, based on tractability considerations [see, for instance, CIR, Longstaff (1989), Longstaff and Schwartz (1992), and Pennacchi (1991)]. From now on, as in Stulz (1986), we restrict attention to the log utility below:

$$u(c_t, m_t) = \phi \ln(c_t) + (1 - \phi) \ln(m_t) \qquad 0 \le \phi \le 1, \tag{21}$$

where ϕ is the expenditure share on consumption. Substitute this function into Equation (15) and obtain

$$R_t = \frac{1 - \phi}{\phi} \, \frac{y_t}{m_t} = \frac{1 - \phi}{\phi} \, v_t, \tag{22}$$

where $v_t \equiv \frac{y_t}{m_t}$ is the time t velocity of money. Then, the nominal interest rate is increasing in time t output, but is inversely related to real money demand. This is the case because, as the opportunity cost of holding money (i.e., the interest foregone) increases, the agent is more willing to substitute cash balances into interest-bearing assets and, consequently, real money demand will fall. In other words, the velocity of money comoves with the nominal interest rate. Another point that is worth remembering is that whenever the velocity of money v_t is constant over time, so will be the nominal interest rate R_t. Therefore, in order for the nominal interest rate, and hence the nominal term structure, to follow a stochastic process, the primitive processes for y_t and M_t have to be such that the resulting velocity of money follows some stochastic process.

Without more specific restrictions on the parameters of the production and money supply processes, it is difficult to say much more, based on Theorems 1 and 2, about the price process P_t^c, the expected inflation process π_t, or the term structure of interest rates. For this reason, we now turn to examining several cases in which closed-form expressions for these and other endogenous variables are obtainable. In what follows, we only examine properties of the economies in their continuous-time limit.

3. An Economy with I.I.D. Output and Money Growth Processes

In financial economics, geometric Brownian motions are often used to model the processes of many economic and financial variables. It thus makes sense to first examine the case in which real output and money supply are as given below:

$$\frac{dy_t}{y_t} = \mu_y \, dt + \sigma_y \, dw_{y,t} \tag{23}$$

$$\frac{dM_t}{M_t} = \mu_M \, dt + \sigma_M \, dw_{M,t}, \tag{24}$$

where μ_y, σ_y, μ_M, and σ_M are all positive constants, with $dw_{y,t} \equiv B_{y,t}\sqrt{dt}$, $dw_{M,t} \equiv B_{M,t}\sqrt{dt}$. This economy has a few distinctive features. For example, both output growth and money growth are i.i.d., monetary policy has no impact on real output, and neither is monetary policy accommodating to economic growth. We start with this simple economy because it nonetheless illustrates many important properties of the general model.

Theorem 3. *For this i.i.d. economy, equilibrium is characterized by the following:*

(i) The real price of the equity share and its dynamics are, respectively,

$$p_{z,t} = \frac{y_t}{\rho} \tag{25}$$

$$\frac{dp_{z,t}}{p_{z,t}} = \mu_y \, dt + \sigma_y \, dw_{y,t}; \tag{26}$$

(ii) The price level and the inflation rate are, respectively,

$$P_t^c = \frac{\phi}{1-\phi}\left[\rho + \mu_M - \sigma_M^2\right]\frac{M_t}{y_t} \tag{27}$$

$$\frac{dP_t^c}{P_t^c} = \pi_t dt + \sigma_M \, dw_{M,t} - \sigma_y \, dw_{y,t}, \tag{28}$$

where the expected inflation is constant: $\pi_t = \mu_M - (\mu_y - \sigma_y^2) - \sigma_y \sigma_M \rho_{y,M}$, *with* $\rho_{y,M}$ *being the correlation between the two Wiener processes, respectively, determined by* $dw_{y,t}$ *and* $dw_{M,t}$*;*

(iii) The real and nominal interest rates are constants: $r_t = \rho + \mu_y - \sigma_y^2$ *and* $R_t = \rho + \mu_M - \sigma_M^2$. *The nominal price of a* τ*-period nominal discount bond is* $N(t,\tau) = e^{-(\rho+\mu_M-\sigma_M^2)\,\tau}$, *while the real price of a real discount bond is* $b(t,\tau) = e^{-(\rho+\mu_y-\sigma_y^2)\,\tau}$.

By Equation (25), the real price of the stock market is proportional to real output. The real rate of return on equity is the same as the growth rate of output. Rearrange Equation (25) to get $\frac{p_{z,t}}{y_t} = \frac{1}{\rho}$. Then, if we treat y_t as the aggregate earnings flow on the stock market, the equilibrium price-to-earnings ratio is simply the reciprocal of the time preference parameter ρ. The more the average investor discounts future utility of consumption (i.e., the higher the coefficient ρ), the lower the real price level of the stock market and the lower the equilibrium price-to-earnings ratio.

Following Equation (28), the inflation process is driven by both real and monetary shocks. The expected inflation, π_t, is increasing in expected money growth and decreasing in expected output growth. The fact that π_t is a constant implies that *for the type of economies specified here, a stochastic expected inflation process will not be consistent with equilibrium.*

The real money demand is proportional to output but unrelated to nominal money supply:

$$m_t = \frac{M_t}{P_t^c} = \frac{1-\phi}{\phi\,(\rho + \mu_M - \sigma_M^2)}\, y_t.$$

Thus, any change in money supply will not affect the real money demand. However, the nominal price levels, P_t^c and $P_{z,t} = \frac{\phi}{\rho(1-\phi)} \times \left[\rho + \mu_M - \sigma_M^2\right] M_t$, are only proportional to money supply, which means that a doubling in money supply will double the nominal prices of both the good and the stock market, but will not affect any real prices. There is nonetheless one exception, that is, the nominal bond price $N(t, \tau)$ is only a function of τ and does not depend on money stock M_t. Consequently, as $M_t \to \infty$, both P_t^c and $P_{z,t}$ will tend to infinity but the nominal bond prices will stay invariant. In this economy nominal stock and bond prices will not move together. Since the nominal interest rate is a constant, so is the velocity of money [by Equation (22)].

The real interest rate for any maturity is linear and increasing in expected output growth but independent of monetary policy, whereas the nominal interest rate for any maturity is increasing in expected money growth and totally unrelated to production output. In other words, an increase or decrease in expected output growth will not affect the nominal interest rate. This is true because an increase in μ_y increases the real interest rate by exactly as much as it decreases the expected inflation rate, making the nominal interest rate independent of output growth. This feature is due to the fact that (1) monetary policy is detached from the real sector and (2) the average investor has the log utility function.[1]

As noted previously, the persistent negative correlation between inflation and real stock returns has been a puzzling empirical phenomenon. To see how such a negative correlation can arise within a general equilibrium, note from Equations (26) and (28) that

$$\operatorname{cov}_t\left(\frac{dp_{z,t}}{p_{z,t}}, \frac{dP_t^c}{P_t^c}\right) = \operatorname{cov}_t\left(\frac{dy_t}{y_t}, \frac{dM_t}{M_t}\right) - \operatorname{var}_t\left(\frac{dy_t}{y_t}\right),$$

which is clearly negative, unless both money growth is procyclical

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[1] When the investor's utility function is replaced by

$$u(c, m) = \frac{\left(c^{\phi}\, m^{1-\phi}\right)^{1-\gamma}}{1-\gamma},$$

the limit of which, as $\gamma \to 1$, is the log utility in Equation (21), all qualitative properties of the i.i.d. economy are as stated in Theorem 3, except that the nominal interest rates become

$$R_t = \rho + (\gamma - 1)\left(\mu_y - \frac{1}{2}\sigma_y^2\right) + \mu_M - \sigma_M^2 - \frac{1}{2}(\gamma - 1)^2 \sigma_y^2 - (\gamma - 1)\,\sigma_y\,\sigma_M\,\rho_{y,M}.$$

In this case, the nominal interest rates are increasing when $\gamma > 1$ and decreasing when $\gamma < 1$ in expected output growth.

and its covariance with output growth dominates the variance of the latter. Christiano (1991), Friedman and Schwartz (1982), Kydland and Prescott (1990), and Romer and Romer (1989) all document that money supply is procyclical and leads the business cycle, but the covariance between money growth and output growth typically does not exceed the variance of the latter. This means the right-hand side of the above equation, and hence the left-hand side, should be negative, a prediction that is supported by numerous studies such as Fama (1981), Geske and Roll (1983), and Marshall (1992). On the other hand, as Kaul (1987) observes, the years from 1926 to 1940 marked a period during which money supply was strongly procyclical and the short-run correlation between stock returns and inflation was positive, a phenomenon that is also in line with the prediction of the above equation.

Finally, the correlation between stock returns and money growth depends on that between output growth and money growth, as indicated below:

$$\operatorname{cov}_t\left(\frac{dp_{z,t}}{p_{z,t}}, \frac{dM_t}{M_t}\right) = \operatorname{cov}_t\left(\frac{dy_t}{y_t}, \frac{dM_t}{M_t}\right).$$

Then, when monetary policy is procyclical [as is the case based on existing studies, e.g., Stock and Watson (1989)], this correlation is positive, a prediction consistent with the conclusions of Marshall (1992) and Stulz (1986).

In summary, while this economy possesses many intuitively or empirically consistent properties, it has no lack of counterfactual features. For example, the expected inflation, the real and nominal interest rates, and the velocity of money are all constant over time; the real and nominal term structures are flat. This observation helps motivate our next section which offers an economy with more empirically plausible features.

4. A Model with More Realistic Money Supply and Output Processes

For this section, monetary policy still does not directly affect production output, nor does output determine money supply. The choice of processes for M_t and y_t is, however, based more on what is empirically plausible. To make the contrast with CIR and Sun (1992) clear, we adopt their processes for technology and output. That is, technology, denoted by x_t, and production output follow the processes

$$dx_t = \kappa_x (\theta_x - x_t)\, dt + \sigma_x \sqrt{x_t}\, dw_{x,t}, \qquad (29)$$

$$\frac{dy_t}{y_t} = (\mu_y + \eta_y x_t)\, dt + \sigma_y \sqrt{x_t}\, dw_{x,t}, \tag{30}$$

where all coefficients are assumed to be positive, θ_x is the long-run mean of x_t, and κ_x reflects the speed of adjustment to this mean. Note that production output is perfectly correlated with the technology process since both of their shock terms are driven by the standard Wiener process $w_{x,t}$. The square-root process, x_t, is what CIR use to characterize the underlying source of uncertainty in their models, and y_t is similar to the output process in Assumption 2 of Sun (1992). For this reason, further interpretation of the two processes is omitted here.

In an important contribution, Stock and Watson (1989) find that the M1 money supply process in the U.S. can be described as being stationary around a significant time trend. The continuous-time counterpart to their discrete-time model is, using our notation, as follows:[2]

$$d \ln(M_t) = \mu_M^* \, dt + d \ln\left(g_t\right) \qquad \mu_M^* > 0, \tag{31}$$

where the growth rate of g_t follows a stationary, zero-mean process. According to this specification, money supply has two parts: a deterministic time trend that grows at (exponential) rate μ_M^*, and stochastic deviations about that trend caused by monetary shocks. Here, g_t can also be thought of as the "detrended money supply." For our analysis, g_t is assumed to have the following dynamics:

$$\frac{dg_t}{g_t} = \kappa_g \left(\theta_g - g_t\right)\, dt + \sigma_g \sqrt{g_t}\, dw_{M,t} \qquad g_0 > 0, \tag{32}$$

where all the coefficients are positive, θ_g is the long-run mean of g_t, and κ_g measures the speed of adjustment. This mean-reverting process g_t is a reasonable specification for its counterpart in Equation (31) because (1) the steady-state growth rate of g_t is zero and (2) the growth process for g_t is stationary. As given in Equation (32), the

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[2] To be exact, the discrete-time model of Stock and Watson (1989) also has a quadratic time trend in the log of money supply:

$$\Delta \ln(M_t) = \left(\mu_M^* + \xi\, t\right) \Delta t + \Delta \ln(g_t) \qquad \xi > 0,$$

where ξ captures the deterministic quadratic time trend in log money supply. To simplify the closed-form solutions to follow, we chose to include only a linear time trend for the log of money supply. Solutions for the more general case that includes a quadratic trend are more complex and are available from the authors upon request. See Marshall (1992) for a simulation-based study of a monetary asset pricing model with a quadratic trend in the log of money supply.

probability that $g_t = \infty$ for any t is zero.[3] From Equations (31) and (32) and by Ito's lemma, the dynamics of money supply are described by

$$\frac{dM_t}{M_t} = \mu_{M,t}\, dt + \sigma_g \sqrt{g_t}\, dw_{M,t}, \tag{33}$$

where the expected money growth rate is

$$\mu_{M,t} \equiv \mu_M^* + \kappa_g(\theta_g - g_t). \tag{34}$$

Since g_t follows a mean-reverting process, so does the expected money growth process, $\mu_{M,t}$. Expected money growth will be high (low) when the monetary shock process, g_t, is low (high) relative to its long-run mean, θ_g. The volatility of money growth is, nonetheless, increasing in g_t. Note that the steady-state mean of money growth is positive and given by μ_M^*:

$$E^*\left(\mu_{M,t}\right) = \mu_M^* + \kappa_g[\theta_g - E^*(g)] = \mu_M^* > 0,$$

where $E^*(\cdot)$ is the steady-state expectation operator. For simplicity, assume that past technological disturbances cannot be used to forecast future monetary disturbances and vice versa, that is, $\text{cov}(dw_{x,s}, dw_{M,t}) = 0$ and $\text{cov}(dw_{x,t}, dw_{M,s}) = 0$ for all $s < t$. However, this does not rule out the possibility that contemporaneous technological and monetary disturbances are correlated: $\text{cov}(dw_{x,t}, dw_{M,t}) \neq 0$.

In the subsections to follow, we examine the price level, inflation, stock prices, and the real and nominal term structures for the economy with the assumed processes.

4.1 The price level, inflation, and stock prices

Theorem 4. *Let the representative agent's utility function be as in Equation (21). Then, equilibrium prices in the continuous-time limit are as follows:*

(i) The real price of equity is as given in Equation (25), with its dynamics being

$$\frac{dp_{z,t}}{p_{z,t}} = (\mu_y + \eta_y x_t)dt + \sigma_y \sqrt{x_t}\, dw_{x,t}; \tag{35}$$

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[3] This can be seen from the fact that $\frac{1}{g_t}$ follows a mean-reverting square-root process:

$$d\left(\frac{1}{g_t}\right) = \eta_0\left(\mu_0 - \frac{1}{g_t}\right)dt - \sigma_0\sqrt{\frac{1}{g_t}}\, dw_{M,t},$$

for some constants μ_0, η_0, and σ_0. For this process, it is known that the probability that $\frac{1}{g_t} = 0$ is zero.

(ii) The time t commodity price level is

$$P_t^c = \frac{\phi\,(1-\phi)^{-1}\,(\rho+\mu_M^*)\,(\rho+\mu_M^*+\kappa_g\theta_g)}{\rho+\mu_M^*+(\kappa_g+\sigma_g^2)\,g_t}\;\frac{M_t}{y_t} \qquad \forall t, \tag{36}$$

with the inflation process given by

$$\frac{dP_t^c}{P_t^c} = \pi_t\,dt + \frac{\sigma_g\sqrt{g_t}(\rho+\mu_M^*)}{\rho+\mu_M^*+(\kappa_g+\sigma_g^2)g_t}\,dw_{M,t} - \sigma_y\sqrt{x_t}\,dw_{x,t}, \tag{37}$$

where

$$\begin{aligned}\pi_t \;=\; & \mu_M^* + \kappa_g(\theta_g - g_t) - \Big[\mu_y + (\eta_y - \sigma_y^2)x_t\Big] \\ & - \frac{(\kappa_g+\sigma_g^2)\kappa_g g_t(\theta_g - g_t)}{\rho+\mu_M^*+(\kappa_g+\sigma_g^2)g_t} \\ & + \left[\frac{\kappa_g+\sigma_g^2}{\rho+\mu_M^*+(\kappa_g+\sigma_g^2)\,g_t}\right]^2 \sigma_g^2\,g_t^3 \\ & - \frac{\sigma_g^2\,g_t^2\,(\kappa_g+\sigma_g^2)\,g_t}{\rho+\mu_M^*+(\kappa_g+\sigma_g^2)\,g_t} \\ & - \frac{\rho+\mu_M^*}{\rho+\mu_M^*+(\kappa_g+\sigma_g^2)\,g_t}\,\sigma_y\,\sigma_g\,\rho_{y,M}\,\sqrt{g_t x_t}. \end{aligned} \tag{38}$$

A number of points can be made based on Theorem 4. First, the real price of the stock market is the same as in the i.i.d. case, so the discussion on $p_{z,t}$ following Theorem 3 applies here as well. The stock return dynamics are, however, different from before. By Equation (35), the real equity returns are increasing in expected production growth. Positive technological changes will have a positive impact on the stock market, at least in the immediate future.

Second, the commodity price level, P_t^c, is a decreasing convex function of output and a strictly increasing function of money supply. That is, as output rises, the price level declines, but at an increasing speed. This is empirically plausible. To see this, suppose that there is a sudden increase in output, and fix money supply and other things in the economy. Then, the fixed amount of fiat money has to be divided among more units of the consumption good, which lowers the nominal price level. As output increases further, this negative impact of output on the price level becomes stronger. The commodity price level is also an increasing, concave function of the expected money growth rate ($\mu_{M,t}$). Among other features, the price level is increasing in both the steady-state money growth rate (μ_M^*) and the subjective discount factor (ρ).

Third, by Equations (25) and (36), the nominal stock price is again totally unrelated to real output:

$$P_{z,t} = p_{z,t} P_t^c = \frac{\phi (\rho + \mu_M^*) (\rho + \mu_M^* + \kappa_g \theta_g)}{(1-\phi) \rho} \times \frac{M_t}{\rho + \mu_M^* + (\kappa_g + \sigma_g^2) g_t} \qquad \forall t. \tag{39}$$

This is the case because, *ceteris paribus*, the doubling of output, for instance, doubles the real stock price but halves the nominal commodity price. In effect, the nominal stock price becomes invariant to output shocks. Among other things, the nominal stock price is increasing in money supply. Its comparative statics with respect to other parameters are the same as the corresponding ones for the commodity price level. As in the i.i.d. case, when $M_t \to \infty$, $P_t^c \to \infty$, and $P_{z,t} \to \infty$.[4]

Fourth, by Equations (37) and (38), the dependence of inflation on both monetary and technological shocks is just as in the i.i.d. case: positive monetary shocks have a positive effect, while positive technological shocks have a negative impact on (unanticipated) inflation. Furthermore, expected inflation increases in expected money growth and decreases in expected output growth. Since both the technological shock (x_t) and the monetary shock (g_t) are mean reverting processes, expected inflation is also mean reverting. Although the steady-state distribution for expected inflation is not known in closed form, we can examine the behavior of π_t in the deterministic counterpart of the subject economy where $\sigma_g = \sigma_y = \sigma_x = 0$. In that case, the deterministic steady-state inflation is positive for plausible parameter values:

$$\lim_{t\to\infty} \pi_t = \mu_M^* - \left(\mu_y + \eta_y \theta_x\right).$$

Thus, the long-run expected inflation depends crucially on the long-run money and output growth rates.

The endogenously determined price level and expected inflation processes are in contrast with those assumed in, for example, Bosshardt (1987), CIR, Pennacchi (1991), Richard (1978), and Sun (1992). In these existing studies, expected inflation is typically taken to be an AR(1) process [e.g., Sun (1992)] or a mean-reverting square-root process [e.g., in Section 7 of CIR]. Clearly, from Equation (38), such exogenous processes are unlikely to be consistent with a general equilibrium.

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[4] As noted before, the probability that $g_t \to \infty$ is zero (even as $t \to \infty$). Therefore, it is only when $t \to \infty$ that M_t tends to infinity.

Fifth, the real money demand is linearly decreasing in expected money growth, $\mu_{M,t}$:

$$m_t = \frac{(1-\phi)\{\kappa_g(\rho+\mu_M^*)+(\kappa_g+\sigma_g^2)(\mu_M^*+\kappa_g\theta_g)-(\kappa_g+\sigma_g^2)\mu_{M,t}\}}{\phi\,\kappa_g\,(\rho+\mu_M^*)\,(\rho+\mu_M^*+\kappa_g\theta_g)}\,y_t. \tag{40}$$

This is because increases in $\mu_{M,t}$ lead to increases in expected inflation, which in turn results in agents decreasing their money balances. As shown later, higher expected inflation means higher nominal interest rates (and higher velocity of money), which increases investors' desire to substitute interest-bearing assets for cash (i.e., lower money demand).

Sixth, note from Equations (35) and (37) that

$$\begin{aligned} \mathrm{cov}_t\left(\frac{dp_{z,t}}{p_{z,t}}, \frac{dP_t^c}{P_t^c}\right) &= \mathrm{cov}_t\left(\frac{dy_t}{y_t}, \frac{dP_t^c}{P_t^c}\right) \\ &= \frac{\rho+\mu_M^*}{\rho+\mu_M^*+(\kappa_g+\sigma_g^2)\,g_t}\,\mathrm{cov}_t\left(\frac{dy_t}{y_t}, \frac{dM_t}{M_t}\right) \\ &\quad -\mathrm{var}_t\left(\frac{dy_t}{y_t}\right) \\ &\leq \mathrm{cov}_t\left(\frac{dy_t}{y_t}, \frac{dM_t}{M_t}\right) - \mathrm{var}_t\left(\frac{dy_t}{y_t}\right). \end{aligned} \tag{41}$$

Thus, this more realistic economy possesses the same empirically plausible correlation structure between inflation and stock returns as the i.i.d. economy in Section 3 does. The correlation structure between stock returns and money growth is also similar to that in the i.i.d. case. Thus, further discussion on these points is omitted here.

Finally, Friedman (1988) reports that for the period from 1961 to 1986, the velocity of money was negatively related to the nominal stock price and positively related to the three-quarter lagged real stock price. To see how his finding may be supported in this model, we have the velocity of money below:

$$v_t = \frac{\phi(1-\phi)^{-1}\,(\rho+\mu_M^*)\,(\rho+\mu_M^*+\kappa_g\theta_g)}{\rho+\mu_M^*+(\kappa_g+\sigma_g^2)\,g_t}, \tag{42}$$

which is strictly increasing in $\mu_{M,t}$: $\frac{dv_t}{d\mu_{M,t}} > 0$ (recall that $\mu_{M,t}$ is a decreasing and linear function of g_t). The velocity of money is thus decreasing in g_t, whereas by Equation (39) the nominal stock price is increasing in g_t. This implies that the nominal stock price is inversely related to the velocity of money. Next, the dynamics of v_t are described by

$$\begin{aligned}\frac{dv_t}{v_t} &= \left\{ \frac{\sigma_g^2\,(\kappa_g+\sigma_g^2)^2\,g_t^3}{\left(\rho+\mu_M^*+(\kappa_g+\sigma_g^2)\,g_t\right)^2} - \frac{\kappa_g\,(\kappa_g+\sigma_g^2)\,(\theta_g-g_t)\,g_t}{\rho+\mu_M^*+(\kappa_g+\sigma_g^2)\,g_t} \right\} dt \\ &\quad - \frac{\sigma_g\,(\kappa_g+\sigma_g^2)\,g_t^{\frac{3}{2}}}{\rho+\{\eta_M+\sigma_M^2\}\,g_t}\,dw_{M,t},\end{aligned}$$

which means positive monetary shocks lead to lower changes in the velocity of money. The covariance between real stock returns and changes in velocity is

$$\begin{aligned}\operatorname{cov}_t\left(\frac{dp_{z,t}}{p_{z,t}}, \frac{dv_t}{v_t}\right) &= \operatorname{cov}_t\left(\frac{dp_{z,t}}{p_{z,t}}, \frac{dy_t}{y_t}\right) + \operatorname{cov}_t\left(\frac{dp_{z,t}}{p_{z,t}}, \frac{dP_t^c}{P_t^c}\right) \\ &\quad - \operatorname{cov}_t\left(\frac{dp_{z,t}}{p_{z,t}}, \frac{dM_t}{M_t}\right) \\ &= -\frac{(\kappa_g+\sigma_g^2)\,g_t}{\rho+\mu_M^*+(\kappa_g+\sigma_g^2)\,g_t}\operatorname{cov}_t\left(\frac{dy_t}{y_t}, \frac{dM_t}{M_t}\right),\end{aligned}$$

which is positive when money growth is countercyclical and negative otherwise.

4.2 The term structure of real interest rates

Theorem 5. *In the continuous-time limit, the equilibrium real interest rate is*

$$r_t = \rho + \mu_y + (\eta_y - \sigma_y^2)\,x_t, \tag{43}$$

and the real price of a pure discount bond that pays one unit of the good in τ periods is

$$b(t,\tau) = \exp\left[-q_1(\tau) - q_2(\tau)x_t\right], \tag{44}$$

where

$$\begin{aligned}q_1(\tau) &\equiv (\rho+\mu_y)\,\tau + \frac{2\kappa_x\theta_x}{\sigma_x^2}\left\{\ln\left[1+\frac{(1-e^{-q_3\tau})(\kappa_x+\sigma_x\sigma_y-q_3)}{2\,q_3}\right]\right. \\ &\quad \left. + \frac{1}{2}\tau\left[q_3-(\kappa_x+\sigma_x\sigma_y)\right]\right\}\end{aligned}$$

$$q_2(\tau) \equiv \frac{2(\eta_y-\sigma_y^2)(1-e^{-q_3\tau})}{2q_3+\left[\kappa_x+\sigma_x\sigma_y-q_3\right]\,(1-e^{-q_3\tau})}$$

$$q_3 \equiv \sqrt{(\kappa_x + \sigma_x \sigma_y)^2 + 2\sigma_x^2(\eta_y - \sigma_y^2)}.$$

Note from this theorem that the real interest rate and the real bond prices do not depend on money supply. This fact is consistent with the observation made earlier that the real price of the stock market is not related to M_t. In either case, the nondependence on money supply is due to the fact that monetary policy here does not affect real output and both money supply and output follow an autonomous process of their own.

Next, the real spot rate is increasing in both the time preference parameter ρ and the expected output growth. Provided that $(\eta_y - \sigma_y^2) > 0$, the real interest rate is also increasing in the technology index x_t. By Ito's lemma, Equation (43) implies the following dynamics for r_t:

$$dr_t = \kappa_x\,(\theta_r - r_t)\,dt + \sigma_r\,\sqrt{x_t}\,dw_{x,t}, \tag{45}$$

with $\theta_r \equiv \rho + \mu_y + (\eta_y - \sigma_y^2)\,\theta_x$ and $\sigma_r \equiv (\eta_y - \sigma_y^2)\,\sigma_x$, which means that, like x_t, the real interest rate follows a mean-reverting, square-root process with the same speed of adjustment but a different long-run mean. Since technological shocks have opposite effects on inflation and the real interest rate, the real interest rate must be negatively correlated with inflation in this economy. The interest rate process in Equation (45) is virtually identical to the one given in Equation (17) of CIR.

The price of a real discount bond also depends solely on the technology index x_t, and it is decreasing in (1) the time discount factor ρ, (2) the invariant portion of output growth μ_y, (3) the long-run mean of the technology index, θ_x, and (4) the term to maturity. Using the definition for real yield to maturity [denoted by $r(t,\tau)$], $e^{-r(t,\tau)\,\tau} \equiv b(t,\tau)$, we obtain the term structure of real interest rates:

$$r(t,\tau) = \frac{q_1(\tau)}{\tau} + \frac{q_2(\tau)}{\tau} x_t. \tag{46}$$

Movements in the real term structure are thus completely driven by real shocks. The real bond price equation in Equation (44) resembles Equation (23) of CIR as well as the real bond price given in Corollary 1 of Sun (1992). Since both the real interest rate and the real term structure have the same functional forms with the corresponding ones in CIR and Sun (1992), all the discussion on the real term structure dynamics offered by them can be carried over to our context, and the repetition is omitted here.

4.3 The term structure of nominal interest rates

Theorem 6. *In the continuous-time limit, the equilibrium nominal interest rate is*

$$R_t = \frac{(\rho + \mu_M^*)\,(\rho + \mu_M^* + \kappa_g \theta_g)}{\rho + \mu_M^* + (\kappa_g + \sigma_g^2)\, g_t} \qquad \forall t, \tag{47}$$

and the nominal price for a pure discount bond that pays one dollar in τ *periods is*

$$N(t, \tau) = \frac{d_1(\tau) + \left[d_2(\tau) - d_3(\tau)\right] g_t}{\rho + \mu_M^* + (\kappa_g + \sigma_g^2)\, g_t}, \tag{48}$$

where $d_2(\tau) - d_3(\tau) > 0$ *and*

$$\begin{aligned} d_1(\tau) &\equiv (\rho + \mu_M^*) e^{-(\rho+\mu_M^*+\kappa_g\theta_g)\tau} > 0 \\ d_2(\tau) &\equiv \frac{\kappa_g + \sigma_g^2}{\kappa_g \theta_g} \left[\rho + \mu_M^* + \kappa_g \theta_g\right] e^{-(\rho+\mu_M^*)\tau} > 0 \\ d_3(\tau) &\equiv \frac{\kappa_g + \sigma_g^2}{\kappa_g \theta_g} [\rho + \mu_M^*]\, e^{-(\rho+\mu_M^*+\kappa_g\theta_g)\tau} > 0. \end{aligned}$$

Before examining properties of the nominal interest rates, observe the major difference between the real term structure implied by Theorem 5 and its nominal counterpart in Theorem 6: the real term structure is completely driven by technological shocks, x_t, whereas its nominal counterpart is driven by monetary shocks, g_t. In this economy with the log utility function, the real and the nominal term structures thus have fundamentally different factor structures/risk characteristics: one is perfectly correlated with the technology process x_t and the other is correlated with the monetary shock process g_t.[5] Particularly when monetary shocks are uncorrelated with real shocks, that is, when $\text{cov}_t\left(dx_t, dg_t\right) = 0$, the process followed by the real term structure will be independent of that followed by the nominal term structure. This finding is in sharp contrast with those in CIR and Sun (1992). In their partial equilibrium setups, the real term structure is virtually the same as implied by Theorem 5 and it follows a single-factor (real shocks) process, but their nominal term structures follow a two-factor process, driven by both the real shocks and inflationary shocks.

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[5] As noted in the previous section, this "detachment" of the nominal from the real term structure is due to the assumptions that (1) monetary policy and real output are not tied to one another and (2) the average investor has the log utility function. One can easily construct examples in which either one of the two assumptions does not hold and in which the real and the nominal term structures share some common risk factor(s). Such examples are not given in this article in order to save space.

Their real and nominal term structures hence share one source of uncertainty (i.e., the real shocks). To briefly see the internal working of our model, note that the real interest rate in Equation (43) has the term $(\eta_y - \sigma_y^2)\, x_t$, while the expected inflation in Equation (38) has the term $-(\eta_y - \sigma_y^2)\, x_t$. This means that real shocks affect expected inflation and real interest rate with the same magnitude but in opposite directions, which results in a nominal interest rate completely unrelated to real shocks.

Having said the above, we now turn to analyzing the nominal term structure in some detail. In Equation (47), the nominal interest rate is a decreasing concave function of monetary shock, and with $g_0 > 0$, it has the economically plausible property of being positive with a probability of one. As the level of expected money growth increases, both the expected inflation rate and the nominal interest rate increase:

$$\frac{dR_t}{d\mu_{M,t}} = \frac{(\rho + \mu_M^*)\,(\rho + \mu_M^* + \kappa_g \theta_g)}{\Big(\rho + \mu_M^* + (\kappa_g + \sigma_g^2)\, g_t\Big)^2}\, \frac{1}{\kappa_g} > 0.$$

In addition, the nominal interest rate is increasing in (1) the subjective discount factor (ρ), (2) the steady-state mean money growth rate (μ_M^*), and (3) the long-run mean (θ_g) of the monetary shock index. It is also increasing in κ_g when $g_t < \theta_g$, and decreasing otherwise.

Applying Ito's lemma to Equation (47) gives

$$\begin{aligned} dR_t = {} & \left\{ \frac{\kappa_g}{\beta_0 \beta_2}\,(\beta_0 - \beta_1 R_t)\big[\beta_0 - (\beta_1 + \beta_2 \theta_g) R_t\big] - \frac{\sigma_g^2}{\beta_0^2 \beta_2}\,(\beta_0 - \beta_1 R_t)^3 \right\} dt \\ & - \frac{\sigma_g}{\beta_0 \sqrt{\beta_2}} \sqrt{R_t}\,(\beta_0 - \beta_1 R_t)^{3/2}\, dw_{M,t}, \end{aligned} \tag{49}$$

where

$$\begin{aligned} \beta_0 &\equiv (\rho + \mu_M^*)\,(\rho + \mu_M^* + \kappa_g \theta_g) > 0 \\ \beta_1 &\equiv \rho + \mu_M^* > 0 \\ \beta_2 &\equiv \kappa_g + \sigma_g^2 > 0. \end{aligned}$$

This mean-reverting process for R_t is similar in complexity to the single-factor interest rate process derived by Constantinides (1992), and it is not nested within any known diffusion process. The drift term is a nonlinear function of the nominal rate, and the variance term also follows a mean-reverting process and is sensitive to the level of the nominal rate. This is an important trait since the empirical work of Chan et al. (1992) demonstrates that the performance of any interest rate model depends critically on the variance of interest rate changes. Observe that since $g_t > 0$ with probability one, and hence $\beta_0 - \beta_1 R_t > 0$ with probability one, the mean reversion in the nominal interest rate process is determined by $[\beta_0 - (\beta_1 + \theta_g \beta_2) R_t]$. The drift in dR_t is positive when $[\beta_0 - (\beta_1 + \theta_g \beta_2) R_t] > 0$, and is negative otherwise.

Since g_t has a long-run steady-state distribution, so will R_t. Even though the exact steady-state distribution of R_t is not known in closed form, its deterministic steady-state value can be calculated for the case in which $\sigma_g = 0$. As $\beta_0 - \beta_1 R_t > 0$, the interest rate dynamics correspond to the deterministic logistic model that has the property that as $t \to \infty$, $R_t \to \frac{\beta_0}{\beta_1 + \beta_2 \theta_g} = \rho + \mu_M^*$. Thus, the subjective discount rate and the long-run mean money growth rate are two critical determinants of the long-run nominal interest rate. In some sense, this is consistent with Friedman's (1989) observation that countries that have had the slowest rate of growth in the quantity of money have experienced low interest rates.

As can be checked, the nominal price of the τ-period bond satisfies both the boundary and the transversality conditions, since $N(t, \tau) \to 1$ as $\tau \to 0$ and $N(t, \tau) \to 0$ as $\tau \to \infty$. Note that

$$\begin{aligned}\frac{\partial N(t,\tau)}{\partial \mu_{M,t}} &= \frac{\partial N(t,\tau)}{\partial g_t} \frac{\partial g_t}{\partial \mu_{M,t}} \\ &= \frac{(\rho + \mu_M^*)(\kappa_g + \sigma_g^2)(\rho + \sigma_M^* + \kappa_g \theta_g) e^{-(\rho + \mu_M^*)\tau}(1 - e^{-\kappa_g \theta_g \tau})}{\left[\rho + \sigma_M^* + (\kappa_g + \sigma_g^2) g_t\right]^2} \\ &\quad \times \frac{-1}{\kappa_g} < 0,\end{aligned}$$

which says that higher expected money growth means lower nominal bond prices. This is true because higher expected money growth leads to higher expected inflation, which in turn makes the nominal bonds less valuable. It is straightforward to verify that the nominal bond prices are decreasing in the term to maturity and hence forward rates are positive:

$$\begin{aligned}f(t,\tau) &\equiv -\frac{\partial[\ln\{N(t,\tau)\}]}{\partial \tau} \\ &= \frac{\rho + \mu_M^* + \kappa_g \theta_g}{\kappa_g \theta_g} \\ &\quad \times \frac{d_1(\tau) + (\rho + \mu_M^*)(\kappa_g + \sigma_g^2)(1 - e^{-\kappa_g \theta_g \tau}) e^{-(\rho + \mu_M^*)\tau} g_t}{d_1(\tau) + [d_2(\tau) - d_3(\tau)] g_t} > 0,\end{aligned} \tag{50}$$

where $f(t, \tau)$ is the τ period ahead forward rate.

The nominal yield to maturity for a nominal discount bond, denoted by $R(t, \tau)$, is

$$\begin{aligned} R(t, \tau) &\equiv -\frac{1}{\tau} \ln\left[N(t, \tau)\right] \\ &= -\frac{1}{\tau}\left(\ln\left[d_1(\tau) + \{d_2(\tau) - d_3(\tau)\}\, g_t\right]\right. \\ &\qquad \left. - \ln\left[\rho + \mu_M^* + \{\kappa_g + \sigma_g^2\}\, g_t\right]\right). \end{aligned} \tag{51}$$

Indeed, the nominal term structure has the monetary shock process, g_t, as its sole risk factor. The dependence of the term structure on monetary shocks is nonlinear. We can further note the following features. First, as $\tau \to \infty$, $R(t, \tau) \to \rho + \mu_M^*$, which means the upper tail of the nominal yield curve is virtually flat. This also implies that yields on very long-term bonds are less volatile than those of short-term bonds. Second, as $\tau \to 0$, $R(t, \tau) \to R_t$. Finally, the yield to maturity is a nonlinear function of the term to maturity. As a result, the nominal term structure can take different shapes, depending on the parameters governing the evolution of money supply.

Using Ito's lemma, we obtain the geometric term premium for a nominal discount bond as

$$\begin{aligned} TP(t, \tau) &\equiv E_t\left\{\frac{1}{dt}\{d \ln[N(t, \tau)]\} - R_t\right\} \\ &= \left[\frac{d_2(\tau) - d_3(\tau)}{d_1(\tau) + [d_2(\tau) - d_3(\tau)]\, g_t} - \frac{\kappa_g + \sigma_g^2}{\rho + \mu_M^* + (\kappa_g + \sigma_g^2) g_t}\right] \\ &\quad \times \kappa_g g_t \{\theta_g - g_t\} \\ &\quad - \frac{1}{2}\left\{\left[\frac{d_2(\tau) - d_3(\tau)}{d_1(\tau) + [d_2(\tau) - d_3(\tau)]\, g_t}\right]^2\right. \\ &\qquad \left. - \left[\frac{\kappa_g + \sigma_g^2}{\rho + \mu_M^* + (\kappa_g + \sigma_g^2) g_t}\right]^2\right\} \sigma_g^2 g_t^3 \\ &\quad + f(t, \tau) - \frac{(\rho + \mu_M^*)\,(\rho + \mu_M^* + \kappa_g \theta_g)}{\rho + \mu_M^* + (\kappa_g + \sigma_g^2)\, g_t}, \end{aligned} \tag{52}$$

which is nonlinear in both the term to maturity and the monetary shock process. Depending on the parameter values and the level of monetary shocks, the term premium can be positive or negative, which is in contrast with the result in CIR that the term premium is uniformly positive. Here the term premium need not be an increasing function of τ.

5. Concluding Remarks

In the Fisher identity, the nominal interest rate equals the real interest rate plus inflation. As a somewhat immediate implication of this theory, the nominal interest rate process should be driven by both the factors affecting the real rate process and those affecting the inflation process. Presumably this is what underlies the results in, for example, CIR, Pennacchi (1991), and Sun (1992), that the real term structure has real shocks as its single factor, whereas the nominal term structure has both real and inflationary shocks as its risk factors. Consequently, one would think that the respective processes followed by the term structures could not be independent of one another. That, however, is not true in certain economies. As demonstrated by the preceding exercises, it is possible for real shocks to have opposite effects of the same magnitude, respectively, on the real term structure and inflation. The two effects then offset one another in the nominal interest rate process, resulting in a nominal term structure completely unrelated to real shocks. This explains why, in the case of Section 4, the real and nominal term structures are driven by separate risk factors, one by real shocks and the other by monetary shocks.

Depending on the nature of the processes followed by output and money supply, the real and nominal term structures *can* nonetheless share the same risk factors even within our framework. For instance, suppose that both output and money supply are functions of real shocks as well as some other state variables. Then, as one can see from our discussion, the two term structures will both be driven by real shocks and the other state variables. The key point here is that with the log utility function, state variables can affect the nominal term structure only through their impact on the money supply process, and they can influence the real term structure and real asset prices only through their impact on the output process. In this sense, our framework does allow one to derive richer dynamics for inflation, asset prices, and interest rates than the ones presented in Sections 3 and 4. Given the space constraint, we chose to present only the simple cases in these two sections so as to demonstrate the basic properties of our general model.

Our results regarding the correlation structure between stock returns, on the one hand, and inflation and money growth, on the other, hold for a large class of output and money supply processes as well as for more general utility functions.[6] For many types of monetary

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[6] With more general utility functions, closed-form solutions are sometimes hard to obtain. For instance, with the power utility function given in Footnote 1, it is difficult to solve for the term structure of interest rates in closed form when output and money supply follow such general processes as the ones given in Section 4.

economies, stock returns will be negatively correlated with inflation and positively correlated with money growth. Thus, incorporating a role for money not only helps one to understand the internal determination of inflation and asset prices, but also permits one to reconcile certain gaps between empirical facts and economic theory.

Appendix

Proof of Theorem 1. Subtracting Equation (5) from Equation (7) and dividing both sides by $u_c(y_t, m_t)$ result in

$$E_t\left\{\frac{u_c(y_{t+\Delta t}, m_{t+\Delta t})}{u_c(y_t, m_t)}\left(\frac{\Delta p_{i,t}}{p_{i,t}} - r_t\,\Delta t\right)\right\} = 0. \tag{53}$$

Taking the Taylor series of $u_c(y_{t+\Delta t}, m_{t+\Delta t})$ at (y_t, m_t) gives

$$E_t\left\{\frac{u_c(y_t, m_t) + u_{cc}(y_t, m_t)\Delta y_t + u_{cm}(y_t, m_t)\Delta m_t}{u_c(y_t, m_t)}\right.$$
$$\left.\left[(\mu_{i,t} - r_t)\,\Delta t + \sigma_{i,t}B_{i,t}\sqrt{\Delta t}\right]\right\} + O(\Delta t)^{\frac{3}{2}} = 0, \tag{54}$$

where $O(\Delta t)^{\frac{3}{2}}$ is a linear function of $(\Delta t)^{\frac{3}{2}}$ and higher-order terms, which are negligible. Now, taking $\Delta t \to 0$, applying Ito's multiplication rule, and rearranging the resulting equation leads to the desired equation in Equation (12).

Replacing the term $\frac{p_{i,t+\Delta t}}{p_{i,t}}$ in Equation (7) with $\frac{p_{z,t+\Delta t}+y_{t+\Delta t}\cdot\Delta t}{p_{z,t}}$ and rearranging the resulting equation leads to

$$p_{z,t} = E_t\left\{e^{-\rho\Delta t}\frac{u_c(y_{t+\Delta t}, m_{t+\Delta t})}{u_c(y_t, m_t)}\left(y_{t+\Delta t}\Delta t + p_{z,t+\Delta t}\right)\right\}.$$

Applying this equation iteratively, we have

$$p_{z,t} = E_t\sum_{j=1}^{\infty} e^{-\rho\,(j\,\Delta t)}\frac{u_c(y_{t+j\,\Delta t}, m_{t+j\,\Delta t})}{u_c(y_t, m_t)}\,y_{t+j\,\Delta t}\,\Delta t$$

whose continuous-time limit, as $\Delta t \to 0$, is Equation (13), provided that the transversality condition holds. ■

Proof of Theorem 2. This proof involves the repeated use of the Taylor series and Ito's lemma. Since these are standard techniques, in what follows we provide the main steps in order to save space.

First, to get the real interest rate, take the Taylor series of $u_c(y_{t+\Delta t}, m_{t+\Delta t})$ in Equation (5) at (y_t, m_t) and obtain

$$\begin{aligned} u_c(y_t, m_t)(1+\rho\Delta t) \;=\; & E_t\left\{\left[u_c(y_t, m_t) + u_{cc}(y_t, m_t)\Delta y_t \right.\right. \\ & + u_{cm}(y_t, m_t)\Delta m_t + \frac{1}{2}u_{ccc}(y_t, m_t)(\Delta y_t)^2 \\ & + \frac{1}{2}u_{cmm}(y_t, m_t)(\Delta m_t)^2 \\ & \left.\left. + u_{ccm}(y_t, m_t)\Delta y_t\, \Delta m_t\right](1+r_t\Delta t)\right\} \\ & + O(\Delta t)^{\frac{3}{2}}, \end{aligned}$$

or,

$$\begin{aligned} r_t \;=\; & \rho - \frac{1}{\Delta t}E_t\left\{\frac{y_t u_{cc}}{u_c}\frac{\Delta y_t}{y_t} + \frac{1}{2}\frac{y_t^2 u_{ccc}}{u_c}\left(\frac{\Delta y_t}{y_t}\right)^2 + \frac{m_t u_{cm}}{u_c}\frac{\Delta m_t}{m_t} \right. \\ & \left. + \frac{1}{2}\frac{m_t^2 u_{cmm}}{u_c}\left(\frac{\Delta m_t}{m_t}\right)^2 + \frac{y_t m_t u_{ccm}}{u_c}\frac{\Delta y_t}{y_t}, \frac{\Delta m_t}{m_t}\right\} + O(\Delta t)^{\frac{3}{2}}. \end{aligned}$$

Taking $\Delta t \to 0$ and applying Ito's lemma to the above equation yields Equation (14).

Next, to derive the nominal interest rate, subtract Equation (6) from Equation (8) and obtain

$$R_t\; E_t\left[\frac{u_c(y_{t+\Delta t}, m_{t+\Delta t})}{u_c(y_t, m_t)}\frac{P_t^c}{P_{t+\Delta t}^c}\right] = E_t\left[\frac{u_m(y_{t+\Delta t}, m_{t+\Delta t})}{u_c(y_t, m_t)}\frac{P_t^c}{P_{t+\Delta t}^c}\right]. \tag{55}$$

Note that using the Taylor approximation, we have

$$\frac{P_t^c}{P_{t+\Delta t}^c} = 1 - \frac{\Delta P_t^c}{P_t^c} + \left(\frac{\Delta P_t^c}{P_t^c}\right)^2 + O(\Delta t)^{\frac{3}{2}}. \tag{56}$$

Taking the Taylor series of $u_c(y_{t+\Delta t}, m_{t+\Delta t})$ at (y_t, m_t) and using the above fact, we have

$$\begin{aligned} E_t\left[\frac{u_c(y_{t+\Delta t}, m_{t+\Delta t})}{u_c(y_t, m_t)}\frac{P_t^c}{P_{t+\Delta t}^c}\right] \;=\; & E_t\left\{\left[1 + \frac{y_t u_{cc}}{u_c}\frac{\Delta y_t}{y_t} + \frac{m_t u_{cm}}{u_c}\frac{\Delta m_t}{m_t}\right]\right. \\ & \left.\left(1 - \frac{\Delta P_t^c}{P_t^c}\right) + O(\Delta t)^{\frac{3}{2}}\right\} \\ \to\; & 1 \qquad \text{as } \Delta t \to 0. \end{aligned}$$

Similarly,

$$E_t\left[\frac{u_m(y_{t+\Delta t}, m_{t+\Delta t})}{u_c(y_t, m_t)}\frac{P_t^c}{P_{t+\Delta t}^c}\right] \to \frac{u_m(y_t, m_t)}{u_c(y_t, m_t)} \qquad \text{as } \Delta t \to 0.$$

In light of these facts, we conclude that Equation (55) implies Equation (15) as $\Delta t \to 0$.

Rewrite Equation (8) as follows:

$$\frac{1}{P_t^c} = e^{-\rho\Delta t} E_t\left\{\frac{u_m(y_{t+\Delta t}, m_{t+\Delta t})}{u_c(y_t, m_t)}\frac{1}{P_{t+\Delta t}^c}\,\Delta t + \frac{u_c(y_{t+\Delta t}, m_{t+\Delta t})}{u_c(y_t, m_t)}\frac{1}{P_{t+\Delta t}^c}\right\}. \tag{57}$$

Applying this equation iteratively for times t, $t+\Delta t$, $t+2\Delta t, \ldots,$ and imposing the transversality condition, we arrive at

$$\frac{1}{P_t^c} = E_t\left\{\sum_{j=1}^{\infty} e^{-\rho\,(j\,\Delta t)}\frac{u_m(y_{t+j\,\Delta t}, m_{t+j\,\Delta t})}{u_c(y_t, m_t)}\frac{1}{P_{t+j\,\Delta t}^c}\,\Delta t\right\},$$

whose continuous-time counterpart, as $\Delta t \to 0$, is Equation (16).

Only Equation (17) remains to be shown. Divide Equation (6) by Equation (5), rearrange the terms, and obtain

$$\begin{aligned} &E_t\left\{\frac{u_c(y_{t+\Delta t}, m_{t+\Delta t})}{u_c(y_t, m_t)}(1 + r_t\Delta t)\right\} \\ &\qquad = E_t\left\{\frac{u_c(y_{t+\Delta t}, m_{t+\Delta t})}{u_c(y_t, m_t)}(1 + R_t\Delta t)\frac{P_t^c}{P_{t+\Delta t}^c}\right\}. \end{aligned}$$

Again, using the Taylor series and substituting Equation (56) into the above gives, after some rearrangement,

$$\begin{aligned} (R_t - r_t)\,\Delta t \;=\; & E_t\left\{\frac{u_c + y_t u_{cc}\frac{\Delta y_t}{y_t} + m_t u_{cm}\frac{\Delta m_t}{m_t}}{u_c}\left[\frac{\Delta P_t^c}{P_t^c} - \left(\frac{\Delta P_t^c}{P_t^c}\right)^2\right]\right\} \\ & + O(\Delta t)^{\frac{3}{2}}, \end{aligned}$$

whose continuous-time limit (as $\Delta t \to 0$) is Equation (17). ■

Proof of Theorem 3. Substitute the log utility in Equation (21) into Equation (13) and obtain

$$p_{z,t} = E_t\left[\int_t^{\infty} e^{-\rho(s-t)}\left(\frac{y_t}{y_s}\right) y_s\,ds\right] = y_t\int_t^{\infty} e^{-\rho(s-t)}\,ds, \tag{58}$$

which gives Equation (25). Applying Ito's lemma to Equation (25) results in the dynamics for the equity price as in Equation (26). With the log utility, the commodity price level can be derived from Equation (16) (Theorem 2) as

$$\frac{1}{P_t^c\, y_t} = \frac{1-\phi}{\phi} \int_t^\infty e^{-\rho(s-t)} E_t\left[\frac{1}{M_s}\right] ds. \tag{59}$$

Stochastically integrate Equation (24) to derive the following rule for money supply,

$$M_s = M_t \exp\left[\left(\mu_M - \frac{1}{2}\sigma_M^2\right)(s-t) + \sigma_M \int_t^s d\omega_{M,\ell}\right].$$

The conditional expectation can then be written as

$$E\left[\frac{1}{M_s} \mid M_t\right] = M_t^{-1} \exp\left[-(\mu_M - \sigma_M^2)(s-t)\right].$$

Substituting this expression into Equation (59) results in Equation (27). Applying Ito's lemma verifies the inflation dynamics in Equation (28).

The expressions for the nominal and real interest rate can be similarly derived. From Equations (19) and (27),

$$\begin{aligned} N(t,\tau) &= e^{-\rho\tau} E_t\left[\frac{u_c[y_{t+\tau}, m_{t+\tau}]}{u_c[y_t, m_t]} \frac{P_t^c}{P_{t+\tau}^c}\right] = e^{-\rho\tau}\, y_t\, P_t^c\, E_t\left[\frac{1}{y_{t+\tau} P_{t+\tau}^c}\right] \\ &= e^{-\rho\tau}\, E_t\left[\frac{M_t}{M_{t+\tau}}\right] = e^{-(\rho+\mu_M-\sigma_M^2)\tau} \end{aligned}$$

which is the desired expression for the nominal bond price. The nominal interest rate is obtained by taking $R_t = \lim_{\tau\to 0} -\frac{1}{\tau}\ln[N(t,\tau)]$. ■

Proof of Theorem 4. The equity price equation can be similarly derived as in the case of Theorem 3. The dynamics for $p_{z,t}$ then follow from applying Ito's lemma to the equity price equation. With the log utility, the commodity price level can be derived from Equation (16) (Theorem 2) as

$$\begin{aligned} \frac{1}{P_t^c\, y_t} &= \frac{1-\phi}{\phi} \int_t^\infty e^{-\rho(s-t)} E_t\left[\frac{1}{M_s}\right] ds \\ &= \frac{1-\phi}{\phi} \int_t^\infty e^{-\rho\,(s-t)-\mu_M^*\, s} E_t\left[\frac{1}{g_s}\right] ds, \end{aligned} \tag{60}$$

noting that $M_s = e^{\mu_M^*\, s} g_s$ and assuming that $M_0 = g_0$. Now observe that by Ito's lemma,

$$d\left(\frac{1}{g_t}\right) = \left[(\kappa_g + \sigma_g^2) - \frac{\kappa_g \theta_g}{g_t}\right] dt - \frac{\sigma_g}{\sqrt{g_t}} dw_{M,t}.$$

Multiply both sides by $e^{\kappa_g \theta_g t}$ and rearrange to get

$$d\left(\frac{e^{\kappa_g\theta_g t}}{g_t}\right) = (\kappa_g + \sigma_g^2)e^{\kappa_g\theta_g t}dt - e^{\kappa_g\theta_g t}\frac{\sigma_g}{\sqrt{g_t}}dw_{M,t}.$$

Note that since $\mathrm{cov}(dw_{x,s'}, dw_{M,s}) = 0$ for all $s' < s$, all the information generated by the process x_t up to time t, $\{w_{x,s'} : s' < t\}$, is independent of future monetary shocks, $dw_{M,s}$, for all $s \geq t$. Thus, $E_t\left(\frac{1}{g_s}\right) = E\left[\frac{1}{g_s} \mid g_t\right]$. Then,

$$\begin{aligned} E\left[\frac{1}{g_s} \mid g_t\right] &= E\left[e^{-\kappa_g\theta_g s}\left(\int_t^s d\left\{\frac{e^{\kappa_g\theta_g \ell}}{g_\ell}\right\} + \frac{e^{\kappa_g\theta_g t}}{g_t}\right) \mid g_t\right] \\ &= E\left[e^{-\kappa_g\theta_g s}\left(\int_t^s (\kappa_g + \sigma_g^2)e^{\kappa_g\theta_g \ell}d\ell \right.\right. \\ &\qquad \left.\left. - \int_t^s e^{\kappa_g\theta_g \ell}\frac{\sigma_g}{\sqrt{g_\ell}}dw_{M,\ell} + \frac{e^{\kappa_g\theta_g t}}{g_t}\right) \mid g_t\right] \\ &= \frac{1}{g_t}e^{-\kappa_g\theta_g(s-t)} + \frac{\kappa_g + \sigma_g^2}{\kappa_g\theta_g}\left(1 - e^{-\kappa_g\theta_g(s-t)}\right). \end{aligned} \tag{61}$$

Substituting Equation (61) into Equation (60) and integrating the resulting expression leads to Equation (36). With $\rho > 0$, $\kappa_g > 0$, $\theta_g > 0$, and $\mu_M^* > 0$, the same sequence of steps also verifies the transversality condition that is required for interior optimum, that is, as $T \to \infty$, $e^{-\rho T}E_t\left[\frac{1}{P_T^c y_T}\right] \to 0$. Finally, applying Ito's lemma to the price level in Equation (36) gives the inflation dynamics in Equation (37). ■

Proof of Theorem 5. Given the dynamics for y_t in Equation (30), we have

$$y_{t+\tau} = y_t\, e^{\mu_y \tau} \times \exp\left[\left(\eta_y - \frac{1}{2}\sigma_y^2\right)\int_t^{t+\tau} x_s ds + \sigma_y \int_t^{t+\tau} \sqrt{x_s}dw_{x,s}\right].$$

Substituting the log utility in Equation (21) into Equation (18) and using the above expression, we obtain

$$\begin{aligned} b(t,\tau) &= \exp[-\rho\,\tau]\,E_t\left(\frac{y_t}{y_{t+\tau}}\right) \\ &= \exp[-\rho\tau - \mu_y\tau]\,E_t \\ &\quad \times \left\{\exp\left[-\left(\eta_y - \frac{1}{2}\sigma_y^2\right)\int_t^{t+\tau} x_s ds - \sigma_y\int_t^{t+\tau}\sqrt{x_s}dw_{x,s}\right]\right\} \\ &\equiv \exp[-\rho\tau - \mu_y\tau]\,\hat{b}(t,\tau). \end{aligned}$$

According to Richard (1978, p. 49, Equation (57)), this $\hat{b}(t, \tau)$ is a solution to the following PDE [also see Friedman (1975) or Vasicek (1977)]:

$$\frac{1}{2}\sigma_x^2 x_t \hat{b}_{xx} + \left[\kappa_x(\theta_x - x_t) - \sigma_x\sigma_y x_t\right] \hat{b}_x - \hat{b}_\tau - (\eta_y - \sigma_y^2)\, x_t\, \hat{b} = 0.$$

Via a standard separation-of-variables technique, the unique solution subject to the boundary condition, $\hat{b}(t + \tau, 0) = 1$, is the real bond price formula in Theorem 5. Equation (43) is then obtained by taking $r_t = \lim_{\tau \to 0} -\frac{1}{\tau} \ln[b(t, \tau)]$. ■

Proof of Theorem 6. Substituting Equation (42) into Equation (22) gives the desired expression for the nominal interest rate R_t. From Equation (19),

$$\begin{aligned} N(t, \tau) &= e^{-\rho\tau} E_t\left[\frac{u_c[y_{t+\tau}, m_{t+\tau}]}{u_c[y_t, m_t]} \frac{P_t^c}{P_{t+\tau}^c}\right] \\ &= e^{-\rho\tau} y_t P_t^c E_t\left[\frac{1}{y_{t+\tau} P_{t+\tau}^c}\right]. \end{aligned} \qquad (62)$$

By the result in Equation (36),

$$\frac{1}{P_{t+\tau}^c y_{t+\tau}} = \frac{(1-\phi)e^{-\mu_M^*(t+\tau)}}{\phi(\rho + \mu_M^*)(\rho + \mu_M^* + \kappa_g\theta_g)} \left[(\kappa_g + \sigma_g^2) + \frac{\rho + \mu_M^*}{g_{t+\tau}}\right].$$

Substituting this into Equation (62), respectively, for $\frac{1}{P_t^c y_t}$ and $\frac{1}{P_{t+\tau}^c y_{t+\tau}}$, and taking the conditional expectation of $\frac{1}{g_{t+\tau}}$ as in the proof of Theorem 4, we arrive at the desired expression for the nominal bond price. ■

References

Bosshardt, D., 1987, “A Model of Intertemporal Discount Rates in the Presence of Real and Inflationary Autocorrelations,” *Journal of Finance*, 42, 1049–1070.

Boyle, G., 1990, “Money Demand and the Stock Market in a General Equilibrium Model With Variable Velocity,” *Journal of Political Economy*, 98, 1039–1053.

Breeden D., 1979, “An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities,” *Journal of Financial Economics*, 7, 265–296.

Breeden, D., 1986, “Consumption, Production, Inflation and Interest Rates: A Synthesis,” *Journal of Financial Economics*, 16, 3–39.

Brock, W., 1974, “Money and Growth: The Case of Long Run Perfect Foresight,” *International Economic Review*, 3, 750–777.

Brock, W., 1975, “A Simple Perfect Foresight Monetary Model,” *Journal of Monetary Economics*, 1, 130–150.

Chan, K., A. Karolyi, F. Longstaff, and A. Saunders, 1992, "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate," *Journal of Finance*, 47, 1209–1227.

Chen, N.-F., R. Roll, and S. Ross, 1986, "Economic Forces and the Stock Market," *Journal of Business*, 59, 383–403.

Christiano, L., 1991, "Modelling the Liquidity Effect of a Money Shock," *Quarterly Review*, Federal Reserve Bank of Minneapolis, Winter, 3–34.

Constantinides, G., 1992, "A Theory of the Nominal Term Structure of Interest Rates," *Review of Financial Studies*, 5, 531–552.

Cox, J., J. Ingersoll, and S. Ross, 1985a, "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, 53, 363–384.

Cox, J., J. Ingersoll, and S. Ross, 1985b, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, 385–407.

Danthine, J., and J. Donaldson, 1986, "Inflation and Asset Prices in an Exchange Economy," *Econometrica*, 54, 585–605.

Fama, E., 1981, "Stock Returns, Real Activity, Inflation, and Money," *American Economic Review*, 71, 545–565.

Fama, E., 1982, "Inflation, Output, and Money," *Journal of Business*, 2, 201–231.

Fama, E., and M. Gibbons, 1982, "Inflation, Real Returns and Capital Investment," *Journal of Monetary Economics*, 9, 297–323.

Feenstra, R., 1986, "Functional Equivalence Between Liquidity Costs and the Utility of Money," *Journal of Monetary Economics*, 17, 271–291.

Foresi, S., 1990, "Valuation of Nominal Securities," working paper, New York University.

Friedman, A., 1975, *Stochastic Differential Equations and Applications*, Academic Press, New York.

Friedman, M., 1988, "Money and the Stock Market," *Journal of Political Economy*, 96, 221–245.

Friedman, M, 1989, "Quantity Theory of Money," in J. Eatwell, M. Milgate, and P. Newman (eds.), *The New Palgrave*, W. W. Norton, New York.

Friedman, M., and A. Schwartz, 1982, *Monetary Trends in the United States and the United Kingdom: Their Relation to Income, Prices, and Interest Rates, 1867–1975*, University of Chicago Press, Chicago.

Geske, R., and R. Roll, 1983, "The Monetary and Fiscal Linkages between Stock Returns and Inflation," *Journal of Finance*, 38, 1–33.

Grossman, S., and R. Shiller, 1982, "Consumption Correlatedness and Risk Measurement in Economics with Non-Traded Assets and Heterogeneous Information," *Journal of Financial Economics*, 10, 195–210.

Kaul, G., 1987, "Stock Returns and Inflation: The Role of the Monetary Sector," *Journal of Financial Economics*, 18, 253–276.

Kydland, F., and E. Prescott, 1990, "Business Cycles: Real Facts and a Monetary Myth," *Quarterly Review*, Federal Reserve Bank of Minneapolis, Winter, 3–18.

Lee, B., 1989, "Solving, Estimating, and Testing a Nonlinear Stochastic Equilibrium Model, with an Example of the Asset Returns and Inflation Relationship," *Journal of Economic Dynamics and Control*, 13, 499–531.

Lee, B., 1992, "Causal Relations Among Stock Returns, Interest Rates, Real Activity, and Inflation," *Journal of Finance*, 47, 1591–1604.

LeRoy, S., 1984, "Nominal Prices and Interest Rates in General Equilibrium: Money Shocks," *Journal of Business*, 177–195.

Longstaff, F., 1989, "A Nonlinear General Equilibrium Model of the Term Structure of Interest Rates," *Journal of Financial Economics*, 23, 195–224.

Longstaff, F., and E. Schwartz, 1992, "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model," *Journal of Finance*, 47, 1259–1282.

Lucas, R., 1982, "Interest Rates and Currency Prices in a Two-Country World," *Journal of Monetary Economics*, 10, 335–359.

Marshall, D., 1992, "Inflation and Asset Returns in a Monetary Economy," *Journal of Finance*, 47, 1315–1342.

Merton, R., 1971, "Optimal Consumption and Portfolio Rules in a Continuous Time Model," *Journal of Economic Theory*, 3, 373–413.

Pennacchi, G., 1991, "Identifying the Dynamics of Real Interest Rates and Inflation: Evidence Using Survey Data," *Review of Financial Studies*, 4, 53–86.

Richard, S., 1978, "An Arbitrage Model of the Term Structure of Interest Rates," *Journal of Financial Economics*, 6, 33–57.

Romer, C., and D. Romer, 1989, "Does the Monetary Policy Matter? A New Test in the Spirit of Friedman and Schwartz," in O. Blanchard and S. Fischer (eds.), *Macroeconomic Annual*, MIT Press, Cambridge, Mass.

Stock, J., and M. Watson, 1989, "Interpreting the Evidence on Money-Income Causality," *Journal of Econometrics*, 40, 161–181.

Stulz, R., 1986, "Asset Pricing and Expected Inflation," *Journal of Finance*, 41, 209–223.

Sun, T., 1992, "Real and Nominal Interest Rates: A Discrete-Time Model and Its Continuous Time Limit," *Review of Financial Studies*, 5, 581–611.

Vasicek, O., 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177–188.