

TRANSPORTATION, TRANSSHIPMENT, AND ASSIGNMENT PROBLEMS

- There are m supply points with items available to be shipped to n demand points. Specifically, plant i can ship at most S_i items, and demand point j requires at least D_j items.
- The cost of shipping each unit from plant i to demand point j is c_{ij} .
- The objective is to select a routing plan that minimizes total transportation costs.

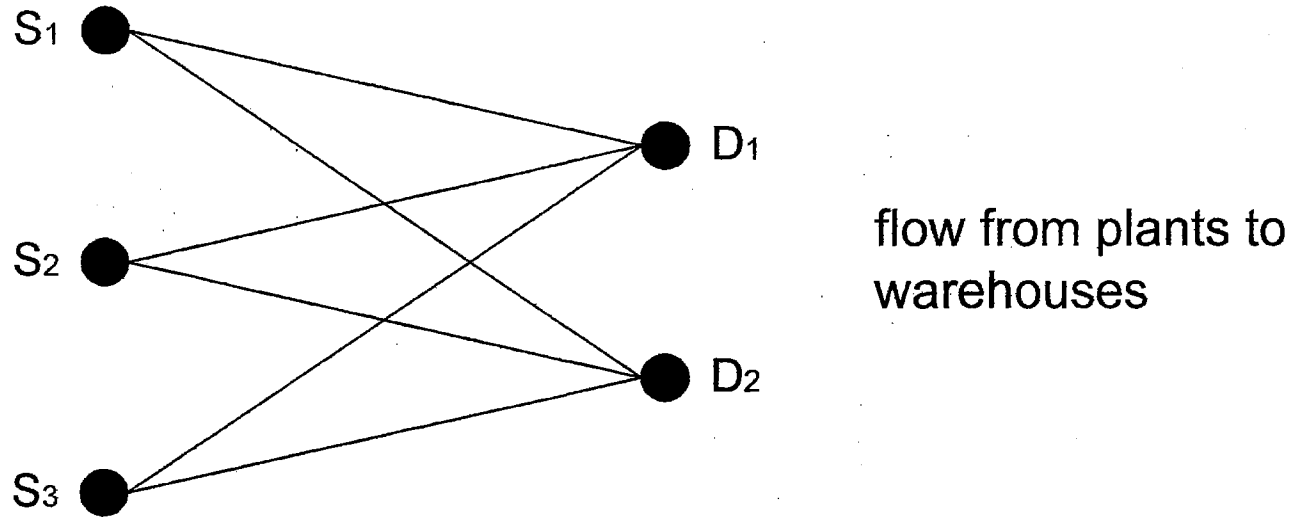
$$\min \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

$$\text{s. t.} \quad \sum_{j=1}^n x_{ij} \leq S_i \quad \text{for } i = 1, \dots, m \quad \text{supply}$$

$$\sum_{i=1}^m x_{ij} \geq D_j \quad \text{for } j = 1, \dots, n \quad \text{demand}$$

$$x_{ij} \geq 0 \quad \text{for all } i, j$$

TRANSPORTATION PROBLEMS



- The following result explains why we formulate the transportation problem as a linear programming problem (LP).

Among all the optimal solutions to the transportation model, there is at least one in which each x_{ij} is *integer-valued* provided the S_i and D_j are all positive integers, which we assume. The regular simplex method will find such a solution.

TRANSPORTATION PROBLEMS

- For the model to possess a feasible solution, total supply must be at least as large as total demand,

$$\sum_{i=1}^m S_i \geq \sum_{j=1}^n D_j.$$

- In analyzing a standard transportation model and devising an optimizing algorithm, however, it is convenient to assume that total supply equals total demand.

Create a fictitious destination with a requirement of $\sum_i S_i - \sum_j D_j$.

Denote this destination as fict, set $c_{i,\text{fict}} = 0$ and $x_{i,\text{fict}}$ is the slack capacity at plant i .

- A special case of the transportation problem is the *assignment problem*.

THE ASSIGNMENT PROBLEM

Each of n tasks can be performed by any of n agents. The cost of task i being accomplished by agent j is c_{ij} . Assign one agent to each task to minimize the total cost.

$$\min \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$$

$$\text{s.t. } \sum_{j=1}^n x_{ij} = 1 \quad \text{for all } i$$

$$\sum_{i=1}^n x_{ij} = 1 \quad \text{for all } j$$

$$x_{ij} \in \{0, 1\}$$

LP AND THE TRANSPORTATION PROBLEM

- The simplex solution to an LP problem always has at most ℓ positive variables, where ℓ is the number of independent constraints.
- Convince yourself that there are $m + n - 1$ independent constraints in the transportation problem.
- The standard simplex instructions are paraphrased next.

TRANSPORTATION ALGORITHM

HIGH LEVEL DESCRIPTION

- Step 1. Select a set of $m + n - 1$ routes that provides an initial basic feasible solution.
- Step 2. Check whether the solution is improved by introducing a nonbasic variable. If so, go to Step 3; otherwise stop.
- Step 3. Determine which routes leave the basis when the variable that you selected in Step 2 enters.
- Step 4. Adjust the flows of the other basic routes. Return to Step 2.

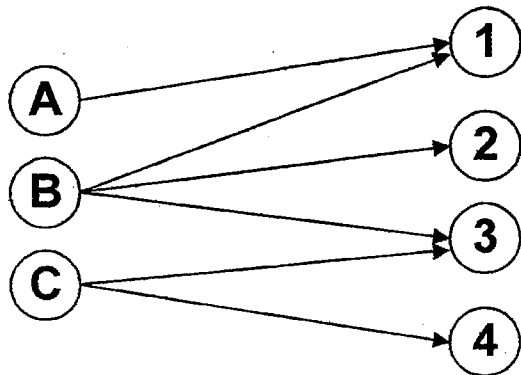
TRANSPORTATION ALGORITHM

- We need to address the following.
 1. Obtaining an initial solution.
 2. Optimization criterion to be used.
 3. Procedure to improve the current nonoptimal solution.

- First, we consider obtaining a basic feasible solution using the Northwest Corner Rule.

THE NORTHWEST CORNER RULE

	1	2	3	4	Supply
A	8 35	6	10	9	35 0
B	9 10	12 20	13 20	7	50 40 20 0
C	14	9	16 10	5 30	40 30 0
Demand	45 10 0	20 0	30 10 0	30 0	



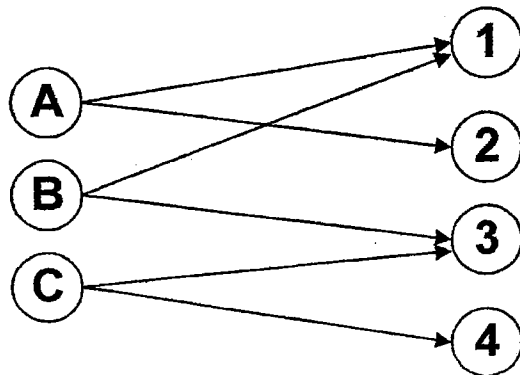
- Initial basis for Northwest Corner Method
- Note connection with spanning trees

Minimum Matrix Method

	1	2	3	4	Supply
A	8	6	10	9	35
B	9	12	13	7	50
C	14	9	16	5	40
Demand	45	20	30	30	

THE MINIMUM MATRIX METHOD

	1	2	3	4	Supply
A	8 15	6 20	10	9	35 15 0
B	9 30	12	13 20	7	50 20 0
C	14	9	16 10	5 30	40 10 0
Demand	45 30 0	20 0	30 10 0	30 0	



- Notice that the last arcs chosen are very expensive.
- Minimum Matrix Method often employs some of the least and most expensive arcs.

VOGEL'S APPROXIMATION

	1	2	3	4	Supply	Row Difference
A	8	6	10	9	35	2
B	9	12	13	7	50	2
C	14	9	16	5	40 30	4 ←
Demand	45	20	30	30 0		
Column Difference	1	3	3	2		

Remove column 4 and continue.

VOGEL'S APPROXIMATION — CONTINUED

	1	2	3	Supply	Row Difference
A	8	6	10	35	2
B	9	12	13	50	3
C	14	9 10	16	10 0	5 ←
Demand	45	20 10	30		
Column Difference	1	3	3		

Remove row C and continue.

VOGEL'S APPROXIMATION — CONTINUED

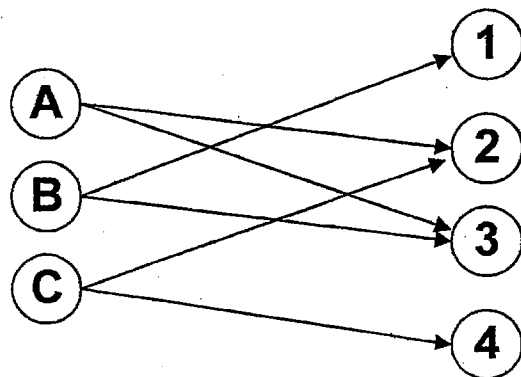
	1	2	3	Supply	Row Difference
A	8	6	10	35 25	2
B	9	12	13	50	3
Demand	45	40 0	30		
Column Difference	1	6	3		

↑

Remove column 2 and continue.

VOGEL'S APPROXIMATION — CONTINUED

	1	3	Supply	Row Difference
A	8	10	25	2
B	9	13	5	4 ←
Demand	45	30		
Column Difference	1	3		



Total Cost

Northwest Corner Rule \$1180

Minimum Matrix Method \$1080

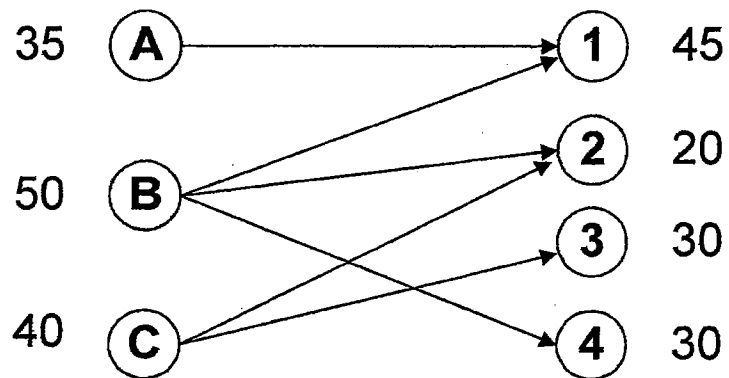
Vogel's Approximation \$1020

BASIC FEASIBLE SOLUTIONS

- The three procedures for finding an initial basic feasible solution resulted in 3 different bases. However, there are similarities:
 - (1) Each basis consists of exactly $m + n - 1$ arcs, one less than the number of nodes in the network.
 - (2) Ignore directions on arcs and we have a spanning tree.
- The fact that the basis corresponds to a spanning tree makes the solution of these problems by the simplex method very efficient.

THE TRIANGULARITY PROPERTY

- Suppose we are given the feasible basis shown below.



- We can read off the corresponding feasible solution. This is known as the triangularity property.
- As long as initial supplies and demands are integers, any basic feasible solution will be an integer solution.

OPTIMALITY CONDITIONS

- For the sample transportation problem that we have worked with, we show that Vogel's method gives the optimal solution.
- Formulate the primal LP problem in equality-constraint form. The dual LP becomes

$$\begin{aligned} \max \quad & \sum_{i=1}^m S_i V_i + \sum_{j=1}^n D_j W_j \\ & V_i + W_j \leq C_{ij} \quad \forall (i, j) \end{aligned}$$

- The theorem of complementary slackness gives us our optimality conditions.

$$\bar{C}_{ij} = V_i + W_j - C_{ij} \leq 0 \quad \text{if } X_{ij} = 0$$

$$\bar{C}_{ij} = V_i + W_j - C_{ij} = 0 \quad \text{if } X_{ij} > 0.$$

OPTIMALITY CONDITIONS APPLIED

- Apply optimality conditions to see if Vogel's solution is optimal.

$$\bar{C}_{ij} = V_i + W_j - C_{ij} \leq 0 \quad \text{if } X_{ij} = 0; \quad \bar{C}_{ij} = V_i + W_j - C_{ij} = 0 \quad \text{if } X_{ij} > 0.$$

	1	2	3	4	V_i
A	8 -2	6 0	10 0	9 -7	0
B	9 0	12 -3	13 0	7 -2	3
C	14 -5	9 0	16 -3	5 0	3
W_j	6	6	10	2	

Therefore, \$1020 is the best we can do.

	1	2	3	4	V_i
A	8 <u>0</u>	6 <u>0</u>	10 <u>0</u>	9 <u>0</u>	0
B	9 <u>0</u>	12 <u>0</u>	13 <u>0</u>	7 <u>0</u>	
C	14 <u>0</u>	9 <u>0</u>	16 <u>0</u>	5 <u>0</u>	
w_j					

TRANSPORTATION ALGORITHM ILLUSTRATION

	1	2	3	4	Supply
A	2	3	11	7	6
B	1	0	6	1	1
C	5	8	15	9	10
Demand	7	5	3	2	

TRANSPORTATION ALGORITHM — CONTINUED

	1		2		3		4		Supply
A	2	6	3		11		7		6
B	1		0	1	6		1		1
C	5	1	8	4	15	3	9	2	10
Demand		7		5		3		2	

Initial basic feasible solution via minimum matrix method.

Cost = 112

$$\bar{c}_{ij} = v_i + w_j - c_{ij}$$

2		3		11		7		0
	<u>0</u>							
1		0		6		1		
		<u>0</u>						
5		8		15		9		
	<u>0</u>		<u>0</u>		<u>0</u>		<u>0</u>	
								v_i w_j

TRANSPORTATION ALGORITHM – CONTINUED

$$V_i + W_j - c_{ij}$$

2	0	3	2	11	1	7	-1	0
1	-4	0	0	6	1	1	0	-5
5	0	8	0	15	0	9	0	3
	2		5		12		6	V_i
								W_j

Bring route (A, 2) into the basis

$6-\theta$	$+\theta$		
	1		
$1+\theta$	$4-\theta$	3	2

Deciding which variable leaves

TRANSPORTATION ALGORITHM – CONTINUED

2	4		
	1		
5		3	2

New basis cost = 104

$$V_i + W_j - c_{ij}$$

2	0	3	0	11	1	7	-1	0
1	-2	0	0	6	3	1	2	-3
5	0	8	-2	15	0	9	0	3
	2		3		12		6	V_i
								W_j

Bring route
(B, 3) into
the basis

TRANSPORTATION ALGORITHM — CONTINUED

$2-\theta$	$4+\theta$		
	$1-\theta$	$+\theta$	
$5+\theta$		$3-\theta$	2

Deciding which
variable leaves the
basis

1	5		
		1	
6		2	2

New basis cost = 101

TRANSPORTATION ALGORITHM – CONTINUED

$$V_i + W_j - c_{ij}$$

2	0	3	0	11	1	7	-1	0
1	-5	0	-3	6	0	1	-1	-6
5	0	8	-2	15	0	9	0	3
	2		3		12		6	V_i
								W_j

Bring route
(A, 3) into
the basis

$1-\theta$	5	$+\theta$	
		1	
$6+\theta$		$2-\theta$	2

Deciding which
variable leaves the
basis

TRANSPORTATION ALGORITHM — CONTINUED

	5	1	
		1	
7		1	2

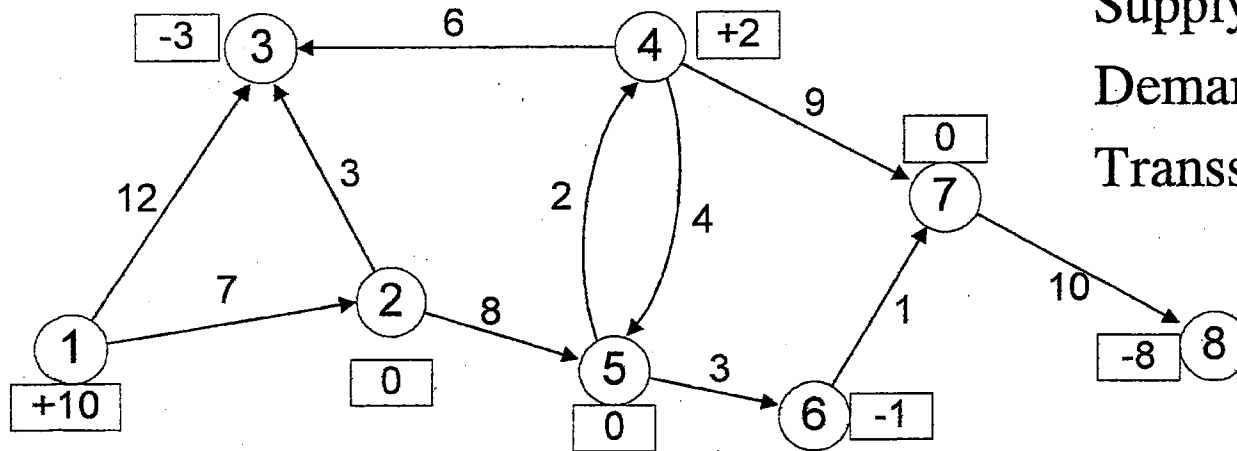
New basis cost = 100

$$V_i + W_j - c_{ij}$$

2	-1	3	0	11	0	7	-2	0
1	-5	0	-2	6	0	1	-1	-5
5	0	8	-1	15	0	9	0	4
1		3		11		5		V_i
								W_j

Optimal,
since all
 $V_i + W_j \leq c_{ij}$

THE TRANSSHIPMENT PROBLEM



Supply nodes: 1, 4

Demand nodes: 3, 6, 8

Transshipment nodes: 2, 5, 7

Equivalence to the transportation problem

	3	6	8	Supply
1	10 3	18 1	29 6	10
4	6	7	18 2	2
Demand	3	1	8	

