Can Information and Inventories be Complements?

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This paper models Information Systems (IS) in a supply chain, enabling the study of the interaction between material and information flows. The optimal dynamic production and distribution policies under demand uncertainty are derived as a function of the firm's IS. Since the supply chain model is quite general (with a non-parametric modeling of IS), the model and solution can be specialized to study different issues. This is illustrated through applications. Further, the effects of the informativeness and timing of the firm's IS on its optimal policies, sales and inventories are analyzed. It is shown that, contrary to conventional wisdom, information and inventories can be complements under quite general conditions; i.e., better information can lead to higher inventories. The drivers of this relationship are studied.

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1 Motivation: The Economics of Supply Chains

In analytical models within the economics tradition, the firm is often approximated as a production function. While useful to derive insights into a variety of economic phenomena, a drawback of this simplification is that it abstracts away from the complex linkages among the firm's various internal activities. In contrast, the 'supply-chain' perspective adopted by some researchers emphasizes modeling of the firm's *internal* processes explicitly, including its production and distribution operations. This paper demonstrates why such a supply-chain perspective is indispensable, in a concrete setting.

The specific context we focus on is the relationship between demand information and inventories. Conventional wisdom, buttressed by analytical models that, implicitly or explicitly, treat the firm as a 'black box' (i.e., as a production function), suggests that information and inventories are substitutes. We show that the received wisdom is heavily predicated on treating the firm as a 'black box', and is invalidated in a wide range of situations when one takes a supply-chain perspective.

Recently, some researchers (notably, Anand and Goyal (2008)) have thrown the spotlight on *information flows* in supply chain management. Anand and Goyal (2008) argue that while "the importance of material flow management for a profit-maximizing firm has been well-articulated in the supply chain literature", this needs to be augmented by more research on managing information flows in the supply chain. We develop a general (non-parametric) model of Information Systems (IS), and also develop methods to compare the information content or 'informativeness' of different IS.

Thus, this paper's contribution is both methodological and substantive. On the methodological front, we develop a model of a multiechelon supply chain maximizing its discounted expected profits over the infinite horizon under demand uncertainty, and derive the optimal dynamic production, sales and inventory policies as functions of the firm's IS structure. Sobel (1981) has derived sufficiency conditions for the optimality of *myopic* policies. We prove that the optimal policies for our model are myopic even though our constrained optimization problem does *not* satisfy Sobel (1981)'s conditions (a rare occurrence in such dynamic models), and derive them in closed form. Further, building on the seminal work of Marschak and Radner (1972), we are able to define IS non-parametrically, and study the effects of improved IS on the firm's production, sales and inventories.

A key, substantive result of our paper is that information and inventories are *complements*, in a fairly general setting, i.e., better information leads to higher levels of inventory in the supply chain. This result is driven by the *timing* of information flows in the supply chain, relative to production and distribution decisions. We establish that the timing of IS is a critical driver of supply chain behavior. Virtually all the previous research, which find that information and inventories are substitutes, make a set of assumptions (hidden or explicit) about the timing of information flows. Our results demonstrate the value of a supply-chain perspective in analytical modeling, incorporating both material and information flows.

1.1 Information Systems and Inventories

Understanding the role of inventory in a supply chain, and its relationship to IS, is important for several reasons. Controlling inventory-levels while meeting desired service objectives is critical for profitability. Joint inventory-service benchmarks enable rationalized service-targets, and are useful as performance metrics (*cf* Lee and Billington (1993)). Understanding the relationship between IS and inventory is also necessary for assessing the value added by investments in IS (*cf* Mukhopadhyay and Cooper (1993)).

To be useful as a performance metric, the drivers of inventories need to be understood. Alles et al (2000) argue that "Accountants are charged with the responsibility of determining the relevant costs and benefits of inventory. Managing accounting systems, though, often only consider the direct costs of inventory." In their model of a Just-In-Time (JIT) system, inventories inflict long-run costs by hampering process reliability, which are ignored in the conventional performance evaluations of managers. The solution proposed in Alles et al (2000) is to make the wage contract for workers depend on the inventory levels. Our analysis highlights other important caveats in using inventory-levels as a performance measure for managers. When the operation of the supply chain, including both material and information flows, is taken as a *whole*, we see that higher inventory-levels might be an artifact of greater supply chain responsiveness to end-market demand. This result is driven by IS-Inventory complementarity. Thus, the use of inventory *reduction* as a performance measure (*cf* Ishikawa (1985)) needs to be tempered with an analysis of the *drivers* of the inventories in the specific supply chain; otherwise, it could lead to perverse incentives for managers.

Conventional wisdom would argue that inventory and information are substitutes. The story is plausible enough: Consider a firm that faces demand uncertainty and has to produce its good(s) in advance because of the production and distribution lead-times. Under demand uncertainty, the firm's production, based on forecasts, will typically be greater than the expected demand. The surplus of production over expected demand is the 'safety stock'. It is intuitively obvious that the safety stock (and hence, inventories) increases with demand uncertainty. Furthermore, when used effectively, IS enable more accurate forecasts, reducing both demand uncertainty and the corresponding safety stock. Thus, by the above argument, inventory and information are substitutes: better IS reduces the need for inventory. Academic researchers (notably, Milgrom and Roberts (1988)) have developed models that capture this intuition. Section 2, and especially Section 2.2, reviews this literature.

Two key simplifying assumptions underlie the above argument, and need to be explicated. The first is that there is only one kind of inventory (*finished goods* in the above story); the second is that information has a single attribute– *precision* (the term favored in Bayesian parametric models) or, more generally, *information content* or *informativeness* (i.e., how well does the information enable demand prediction). In reality, there are many kinds of inventory, ranging from raw materials to various stages of work-in-process to finished goods, and inventory-measures must be normalized to take the actual cost of holding inventory at any stage of the process into account. Regarding the second assumption (which is the focus of this paper), the important but neglected attribute in models of the firm's IS is its *timing* (i.e., *when* the information is received, relative to material flows in the supply chain). The analysis in this paper explicitly addresses both timing and informativeness of IS, and their impact on firms' inventories. Under plausible conditions, information and inventories are shown to be complements: having more informative IS could actually *increase* inventories.

2 Literature Review

Since the classic Clark and Scarf (1960), research in multiechelon inventory systems (later broadened to 'supply chains') has exploded. Much of this literature focussed on material flows across echelons, neglecting the role of information flows. Our model incorporates both material and information flows. A fundamental thesis of this paper is that material and information flows (in other words, a firm's operations and its IS) interact in complex yet important ways, and hence need to be jointly optimized. Two streams of research are most relevant to our paper: *(i)* the literature that studies the value of information in a supply chain, and *(ii)* the literature that spotlights the relationship between information and inventories. These are reviewed below.

2.1 Value of Information in Supply Chains

Anand and Mendelson (1997), in their model of a two-echelon supply chain, parameterize the accuracy of IS, and distinguish between *data* (that IS can capture and transmit) and *local knowledge* (that cannot be captured/transmitted by IS). They study the value of Information Systems as well as the value of coordination (i.e., information sharing), and demonstrate that a firm's organizational structure and IS need to be *codetermined* for optimal performance. In a similar vein, Gavirneni (2002) and Anand and Goyal (2008) demonstrate the importance of *jointly* optimizing the firm's IS and its operations. Gavirneni (2002) argues that, in treating information-related strategies (such as information-sharing) as *auxiliary* to a firm's (fixed) operating policies, the supply chain literature does not go far enough: The optimal use of information flows may necessitate modifications in the firm's operations, to maximize profits. Gavirneni (2002) assumes that incentives are fully aligned within the supply chain, and so information is always shared truthfully. Anand and Goyal (2008) model both vertical relationships and horizontal competition, and incorporate incentive conflicts among the different entities in the supply chain. Anand and Goyal (2008)'s model stretches the concept of 'managing information flows' to the limit: In their model, demand information may be acquired, shared, fudged, inferred and even leaked by the different firms. They find that, in equilibrium, information flows are a critical determinant of the supply chain's material flows. For a comprehensive review of the supply chain literature pertaining to different facets of information flows (such as information *sharing* or *acquisition*), the reader is referred to Anand and Goyal (2008). Iyer *et al* (2007) and Anand and Mendelson (2008) are among the few inventory models that parameterize the precision of IS; their demand models arise as special cases of Anand and Mendelson (1997), with the '*local knowledge*' modeled in the latter being moot.

Huang and Iravani (2005) model a manufacturer supplying to two retailers under demand uncertainty, and study the value to the manufacturer of implementing *selective* information sharing (through a relationship with just one retailer) relative to complete or no information sharing. They find that *scale effects,* similar to those that lead to inventory pooling, lead to 'information pooling'. One implication is that supply chain mergers can create value due to both inventory and information pooling. Cachon and Fisher (2000) also model a two-echelon supply chain, with one supplier and multiple retailers, and periodic review of inventory by each firm. They derive firms' inventory policies under 'traditional information sharing' (wherein the supplier observes just the retailers' orders) and 'full information sharing' (wherein the supplier observes, additionally, retailers' inventory data): their comparison yields the *value of information sharing.* Zheng and Zipkin (1990) develop a queuing model of a production-inventory system for two products, in which the inventory position of each product *is* the information. They compare the performance of the system with and without this information. Mukhopadhyay and Cooper (1993) analyze the value of alternative IS configurations to a firm, with an application to inventory control.

Another related area is the value of *leadtime information* to a downstream retailer. Song and Zipkin (1996) model a two-echelon supply chain wherein *supply leadtimes* are determined by a Markov process, and characterize the optimal policies when this information is shared. In their setting, a longer leadtime does not necessarily lead to higher inventories.

Production capacity plays an analogous role to inventory in many settings. Gavirneni *et al* (1999) point out that the *value* of demand information depends on the supply chain's ability to act upon it; hence, capacity constraints (which limit responsiveness) dampen the value of information. Chen (2003) surveys the literature on both upstream and downstream information sharing within a supply chain, including the effects of mis-aligned incentives.

2.2 The relationship between information and inventory

A stream of literature argues that, under demand uncertainty, risk-pooling leads to lower inventories (*cf* Anand and Mendelson (2008)). Anand and Mendelson (2008) demonstrate that the ability to pool risks is intimately tied to the efficacy of a firm's IS. Alles *et al* (2000) study the relationship between Information and Inventory in the specific context of JIT production. In their model, lowering buffer inventories enhances process transparency, thus improving the context-specific information available to workers. Nagar et al (2007) model a two-stage assembly line in an agency framework. The principal uses an inventory control system for WIP to elicit agents' (workers') *private information* about their productivity.

A large body of research finds that information and inventory are substitutes in a variety of contexts. These include Cachon and Fisher (1997), Dudley and Lasserre (1989), Gallego and Ozer (2001), Gaur *et al* (2005), Iyer *et al* (2007), Jain and Moinzadeh (2005), Lee *et al* (2000), Milgrom and Roberts (1988), Ozer (2003), Ozer and Wei (2004) and Raghunathan (2001), and are discussed below.

Campbell Soup's Continuous Replenishment Program (CRP) is widely cited as an example of the benefits of information sharing between retailers and a manufacturer. Cachon and Fisher (1997) analyze demand data from Campbell, and find that inventories at the retailers' DCs fall by 66% on average, while maintaining or even improving fill rates. Lee *et al* (2000), Raghunathan (2001) and Gaur *et al* (2005) model a two-echelon supply chain, and study the value of demand information for the manufacturer under non-stationary demand. O. Ozer and coauthors analyze a supplier's optimal inventory policies when serving customers with varying demand lead times [Gallego and Ozer (2001), Ozer (2003), Ozer and Wei (2004)]. In these settings, the supplier gains demand information when some customers place advance orders. Jain and Moinzadeh (2005) model the sharing of supply-availability information between a manufacturer and a retailer. The downstream retailer's ordering policy is a state-dependent base-stock policy, where the "state" reflects whether the manufacturer has product available or is out-of-stock.

In an influential paper, Milgrom and Roberts (1988) study the factors that affect a firm's choice between "make-to-stock" (inventories) and "make-to-order" (collecting information on customers' demand) production structures, as well as hybrid combinations of the two. They demonstrate in their model that the firm will rely exclusively on one or the other structure. Dudley and Lasserre (1989) provide empirical support for the substitution relationship in specific industries.

Iyer *et al* (2007) (also discussed above) model buy-back contracts between a manufacturer and a retailer. They show that *(i)* information and inventory are substitutes in their model, and *(ii)* under perfect IS, no inventories are held in the channel. In sharp contrast, our analysis finds that *(i)* inventories and information are complements under the firm's optimal production and distribution policies, and hence, *(ii)* inventories are *highest* under perfect IS. As we demonstrate, these results are driven by the *timing* of demand information, which in turn affects material flows. In fact, Iyer *et al* (2007)'s insights (as well as those of Milgrom and Roberts (1988), and others) are consistent with information flows for a "black box" model of the supply chain, as demonstrated in our analysis of the "early-information" model of Section 6.

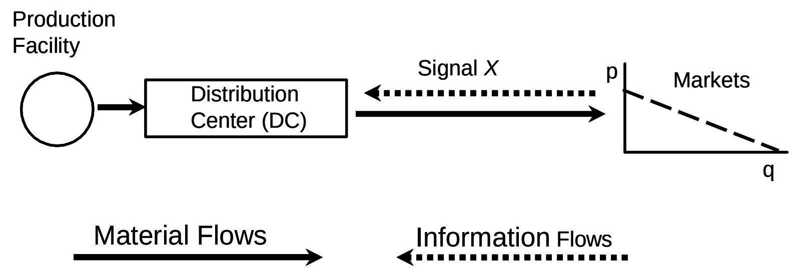

Figure 1. Model Setup

3 The Model

Consider a firm manufacturing and selling a single good, that faces uncertain but stationary demand. Its objective is to maximize its discounted expected profits over the long-term (infinite) horizon. The firm's supply chain consists of a production facility, a central warehouse / Distribution Center (DC), and one or more retail outlets that sell in the respective output markets (See Figure 1.). For concreteness and ease of exposition, the discussion initially focusses on a single outlet.[1] The firm's entire production is shipped to the DC, which then ships to the outlet. While the DC may choose to carry inventory, it is assumed, for simplicity, that the outlet cannot (equivalently, its holding cost is very high). Thus the DC's entire shipment quantity is sold by the retail outlet each period (with appropriate price adjustments). The production cost and DC inventory holding costs are both linear, at k and h per unit respectively. The one-period discount factor is β. Randomness in the revenues arises because of demand uncertainty and imperfect information. The model of IS builds on the seminal work of Marschak and Radner (1972). The comparisons of IS made in this paper use the criterion of *garbling*, rather than a (less general) parametric comparison.

Let $\mathcal{S}$ denote the state space of market-demands in each period, with a probability measure defined over it. The firm's IS provides information that partly counteracts the demand uncertainty. For each possible state $s \in \mathcal{S}$, the firm's Information System $\mathcal{X}$ generates some signal, a vector X from the support set $Su(\mathcal{X})$. Thus, the IS $\mathcal{X}$ provides a mapping from the set of states $\mathcal{S}$ to a set of signals $Su(\mathcal{X})$. In general, the same signal X may be generated for a group of states in $\mathcal{S}$, reflecting the imperfection (or noise) of the IS (Marschak and Radner (1972)). To minimize technicalities in the proofs, we will assume that the set $Su(\mathcal{X})$ is finite (although potentially very large); however, no such restriction is placed on the state space $\mathcal{S}$. At the start of each period, the firm receives the

[1]Section 5 demonstrates that the generalization of this model to multiple, possibly correlated markets is straight-forward.

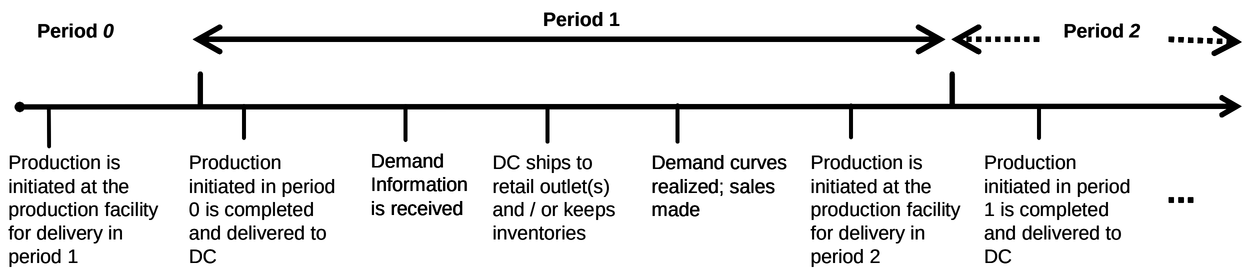

Figure 2. Model Timeline

information vector X on the states of the markets. The expected revenues from shipment quantity q are a function of the signal X, and denoted by $R(q|X)$.

Clearly, different assumptions are plausible with respect to IS timing. Initially, we assume that production leadtimes are long, so that the firm's IS gathers and provides information after the production decision but before shipment from the DC to the retail outlets (Section 6 discusses the effect of alternative assumptions.).

Figure 2 depicts the sequence of events. Production occurs with a one-period lag. For example, production commitments made in period 0 are delivered to the DC in the beginning of period 1. At the start of each period, the firm also receives the information signal X on that period's retail demand. Clearly, the signal X is received by the firm too late to influence production delivered in that period. However, the DC can use this information in its shipment and inventory strategies (i.e., in deciding how much to ship to the retail outlet, and how much inventory to retain). After the shipment is made to the outlet (which is a function of the total quantity available at the DC and the demand information received), the demand curve and the corresponding sales and revenues are realized. The firm then makes the production decisions for the next period. This cycle is repeated each period. The firm has to solve for the optimal production and shipment/inventory policies over the horizon.

The firm's decision problem can now be formulated mathematically. Let Q^t denote the production quantity delivered to the DC in period t, and let q^t denote the quantity shipped by the DC to the retail outlet in period t. Also let ξ^t denote the inventories carried over from period t to the next period ($\xi^0 = 0$ by assumption). Since the market state-distributions are stationary, inventories completely specify the firm's state at the start of each period. Let X_t denote the market information received in period t. $R(\cdot|X_t)$ is the corresponding expected revenue function from period t sales. For the sake of consistency, we adopt the following discounting convention in our dynamic formulation: All revenues and costs from a transaction are realized in the period in which that transaction is *initiated*. For example, the production costs of Q^t are incurred in period $(t-1)$, which is when the production is initiated, whereas the inventory holding costs from carrying ξ^t

at the DC are incurred in period t.[2] The firm's problem is to maximize its expected discounted profits, namely

$$\max_{\{Q^t,q^t,\xi^t\}_{t=1}^{\infty}} \left\{ -k \cdot Q^1 + E\left[\sum_{t=1}^{\infty} \beta^t \cdot \left(R(q^t|X_t) - \left(h \cdot \xi^t + k \cdot Q^{t+1} \right) \right) \right] \right\} \tag{1}$$

subject to the constraints

$$\{Q^t, q^t, \xi^t\}_{t=1}^{\infty} \geq 0, \text{ and} \tag{2}$$

$$\xi^t = \xi^{t-1} + Q^t - q^t, \tag{3}$$

where $\xi^0 = 0$. Conditions (2) are the non-negativity constraints on all the decision variables, and condition (3) is the inventory transition law: the inventory in each period is equal to the sum of the inventory from the previous period and the production for that period less the shipment quantity in the current period.

It is assumed that (i) $R'(0) = E_X\left[R'(0|X)\right] > \frac{k}{\beta}$, and (ii) the production cost k is greater than the DC's inventory holding cost h. The first assumption ensures that the firm will produce and ship positive quantities of the good– otherwise, the firm's trivial optimal policy is to cease operations and not produce the good at all. Similarly, the second assumption corresponds to the interesting case where the firm may carry inventory at its DC under its optimal policy. Otherwise, the firm will never hold inventory at the DC, and its optimal policy is trivial. Further, to minimize clutter, we make the technical assumption that for each possible realization of X, the expected revenue function $R(q|X)$ is bounded, concave and twice-differentiable.[3] The boundedness assumption ensures finite production and shipment under the optimal policy. The concavity assumption guarantees that the incremental revenues generated by additional shipments of the good (i.e., the marginal revenues) are non-increasing.

Theorem 1 derives the firm's optimal production, inventory and shipment policies. We show that the optimal policy is *myopic*, i.e., the decisions taken in each period can disregard the consequences for future periods (Heyman and Sobel (1984)). Sobel (1981) derives sufficiency conditions for the optimality of myopic policies. Myopic optimal policies when Sobel (1981)'s conditions are *not* met are a rare occurrence in such dynamic models (A notable exception in a multiechelon production-

[2] All our structural results and insights hold under alternative approaches to discounting, which would necessitate trivial changes to the problem formulation and solution.

[3] The "twice-differentiability" assumption is for ease of exposition. It is easily verified that all the analytical results hold even when $R(q|X)$ is twice-differentiable "almost everywhere", with the pointwise marginal revenue functions appropriately defined.

inventory context is Amihud and Mendelson (1983).). We prove that the optimal policies for our model are myopic even though our problem (objective function (1) and the constraints (2) and (3)) does not satisfy Sobel (1981)'s conditions, and derive them in closed form.

Theorem 1 (Optimal Policies) *The myopic production, shipment and inventory policies specified below are optimal:*

(A) ***Shipping/Inventory:*** *Conditional on the information signal X, the DC ships its entire on-hand quantity upto the threshold $q(X)$ given by*

$$q(X) = \max\{q : R'(q|X) = k - h, 0\}\cdot \tag{4}$$

Any quantity over and beyond this threshold is held as inventory. We call $q(X)$ the 'ship-up-to' level.

(B) ***Production:*** *The optimal production policy is a myopic threshold policy. Let I_A denote the indicator function of an event A that is 1 if the event A occurs and 0 otherwise. Each period, the firm produces to bring the quantity available at the DC next period to Q^*, where Q^* is the solution of the equation*

$$E_X\left[\left(R'(Q^*|X) - (k-h)\right) \cdot I_{\{Q^* < q(X)\}}\right] = \frac{k}{\beta} - k + h, \tag{5}$$

and $q(X)$ is the ship-up-to threshold from the Shipping/Inventory policy. We refer to Q^* as the 'build-up-to' level.

Proof: The proof of this Theorem is instructive, and provided in an Appendix at the end of this document. The space of feasible policies is expanded in a specific way, by allowing the firm access to a "spot market" for the intermediate good in each period. The optimal policy and its value function are derived for this "augmented" problem. Clearly, this value function is an upper bound on the value obtainable in the original problem. Next, the value function is derived for the policy proposed in Theorem 1, and shown to be identical to the value function of the optimal policy in the augmented problem. This completes the proof of optimality, since the augmented problem is a relaxation of the original problem.[4] ♦

From Theorem 1, we see that the optimal production, shipment and inventory policies are completely specified by the set of parameters $\{q(X)\}_{X\in\mathcal{X}}$ and $Q^*(\mathcal{X})$ (Recall that $\mathcal{X}$ denotes the Information System under consideration.). The myopic form and relatively simple structure of the optimal policies make the solution to the dynamic problem amenable to an intuitive interpretation. Observe that, when the DC ships a quantity q, the expected marginal revenue from shipping

[4] The proofs of all other results are provided separately in the document titled "Technical Appendix".

additional quantities (for any observed X) is $R'(q|X)$. Moreover, the marginal cost of replenishing the DC for the next period is k per unit, and the savings from holding a unit less at the DC is h. Thus, given a choice between shipping additional quantities and holding inventory, the DC would prefer to ship so long as the expected marginal revenue from the additional shipment is greater than the additional cost incurred; i.e., as long as $R'(q|X) \geq k - h$. Thus, $q(X)$ is the maximum quantity that the DC would ever ship to the branches when the information signal is X; above this threshold, the firm is better off keeping inventory at the DC. When $R'(0|X) \leq k - h$, the information X received is so unfavorable that keeping the entire quantity available in the DC as inventory (for possible sales next period) is optimal, and so the shipment is 0. These tradeoffs are reflected in the optimal ship-up-to level, given by expression (4) .

The firm's optimal production policy is a stationary threshold policy independent of demand information. For an arbitrary production 'build-up-to' level Q, the marginal cost-benefit analysis may be broken down into two parts. When X is sufficiently favorable, the shipment threshold $q(X) > Q$, and the marginal value of additional build-up of inventory, in net-present-value terms, is $R'(Q|X) - \frac{k}{\beta}$ (Since production occurs with a one-period lag, the marginal cost of producing a unit more in the previous period to sell in the current period is $\frac{k}{\beta}$ in current dollar terms.). However, if the realized X is such that the shipment threshold $q(X) \leq Q$, the additional build-up would have cost $\frac{k}{\beta} + h$ for producing in the previous period and storing as inventory in the current period, with a production saving of k for the current period. This implies that the incremental benefit from producing the *last* unit, at the build-up-to level Q, for a specific information signal X, is $\left(R'(Q|X) - \frac{k}{\beta}\right) \cdot I_{\{Q<q(X)\}} - \left(\frac{k}{\beta} - k + h\right) \cdot I_{\{Q \geq q(X)\}}$. Of course, the production decision has to be made before observing the information vector X, and so the *expected* incremental benefit from producing the *last* unit, at the build-up-to level Q, is

$$E_{X \in \mathcal{X}} \left[\left(R'(Q|X) - \frac{k}{\beta} \right) \cdot I_{\{Q<q(X)\}} - \left(\frac{k}{\beta} - k + h \right) \cdot I_{\{Q \geq q(X)\}} \right]. \tag{6}$$

The optimal build-up-to level Q^* is that value of Q at which expression (6) is driven to 0. Equating expression (6) to 0 and rearranging the terms gives Q^* as the solution of equation (5).

4 The Impact of Information Systems on the Optimal Policy

This Section studies the impact of IS informativeness on the firm's optimal policy, and in particular, its effect on the build-up-to level and inventories. ('Informativeness' will be formally defined below.) We will then develop an illustrative example.

Recall that $\mathcal{S}$ was the state space of market demands. For each $s \in \mathcal{S}$, the information system $\mathcal{X}$ generates some signal vector X. Under a *perfect* information system, the information signals generated would provide complete information on the state. In general, since the mapping from the state-space to the set of signals is not injective (i.e., one-to-one), the state cannot be precisely known from the signal, and hence IS are imperfect. The first objective of this Section is to define an appropriate metric to compare the informativeness of different (imperfect) IS. This will be used to study the effect of IS informativeness on the firm's production and distribution.

It is not always possible to make general comparisons of the value (or informativeness) of two different IS, in the absence of a specific payoff function that maps the firm's actions (driven by information) and states to the profits for the firm, i.e., the set of all IS is not an ordered set. Typically, one IS might more accurately capture certain states, while another might be more informative when a different set of states were to occur. Hence one IS might outperform another for one payoff function, while the second might do better than the first for a different payoff function. The important question is, what, if any, are the *sufficient* conditions for one IS to be more informative (and hence, 'valuable') than another, for *all* payoff functions?[5]

The notion of *garbling* is one answer to the above question. Let $\mathcal{X}$ and $\mathcal{Y}$ denote two different IS. For every $s \in \mathcal{S}$, let $\pi(X|s)$ and $\pi(Y|s)$ denote the respective probabilities of getting the signals X (under IS $\mathcal{X}$) or Y (under IS $\mathcal{Y}$) respectively. The IS $\mathcal{Y}$ is said to be a garbling of the IS $\mathcal{X}$ if the conditional probability $\pi(Y|s \cap X)$ is independent of s, for all $s \in \mathcal{S}$, $X \in Su(\mathcal{X})$ and $Y \in Su(\mathcal{Y})$. An equivalent characterization of the statement that "$\mathcal{Y}$ is a garbling of the $\mathcal{X}$" is that

$$(i)\ \ \pi(Y|s) = \sum_{X \in \mathcal{X}} \beta_{XY}\pi(X|s),\ \forall\ s \in \mathcal{S},\ Y \in Su(\mathcal{Y}); \text{ and}$$

$$(ii)\ \ \sum_{Y \in \mathcal{Y}} \beta_{XY} = 1,\ \forall\ X \in Su(\mathcal{X}),$$

where β_{XY} is the conditional probability $\pi(Y|X)$ of receiving the signal Y under IS $\mathcal{Y}$, when the signal X is received under the IS $\mathcal{X}$. Then the IS $\mathcal{X}$ outperforms the IS $\mathcal{Y}$ for all possible payoff functions (Marschak and Radner (1972), pp 64-67).

To understand why the IS $\mathcal{Y}$ that is a garbling of the IS $\mathcal{X}$ does no better than $\mathcal{X}$ for all possible payoff functions, imagine that the decision maker, upon receiving the signal $X \in \mathcal{X}$, uses a randomizing device that chooses the signal $Y \in \mathcal{Y}$ with probability β_{XY}, thus garbling $\mathcal{X}$, and

[5] The reader is referred to Marschak and Radner (1972), specifically Chapter 2, pp 53-70, for an outstanding elaboration of these issues.

bases her decisions on the results of this randomization. By this randomization, the decision maker who has the IS $\mathcal{X}$ can mimic any signal-action mapping possible under the IS $\mathcal{Y}$, and do at least as well as under the IS $\mathcal{Y}$, for *all* possible payoff functions. Further, the additional randomization under $\mathcal{Y}$ can only lead to a loss. Hence $\mathcal{X}$ may be described as a *more informative* IS than $\mathcal{Y}$—garbling is thus a sufficient condition to rank IS on their informativeness.[6]

We want to analyze the effects of improved informativeness on the firm's optimal policy, production, inventories and sales. However, working with arbitrary IS $\mathcal{X}$ and $\mathcal{Y}$, wherein $\mathcal{X}$ is more informative than $\mathcal{Y}$, and comparing the outcomes under each, is analytically intractable. To facilitate the analysis, we define the notion of "minimally more informativeness" and derive some of its useful properties in the following Section.

4.1 Minimally more informativeness

In what follows, let $X_1, X_2, ..., X_m$ be the m possible signals generated under the IS $\mathcal{X}$, and $Y_1, Y_2, ..., Y_n$ be the n signals generated under IS $\mathcal{Y}$; thus the respective supports of the two IS are $Su(\mathcal{X}) = \{X_1, X_2, ..., X_m\}$ and $Su(\mathcal{Y}) = \{Y_1, Y_2, ..., Y_n\}$.

Definition 1 *We say that the information system $\mathcal{X}$ is* ***minimally more informative*** *than the information system $\mathcal{Y}$, for a given state space $\mathcal{S}$, if and only if there are indices i and j such that*

$$\pi(Y_k|s) = \pi(X_k|s),\ \forall\ s \in \mathcal{S}, \forall\ k \neq i \ \textit{or}\ j;$$

$$\pi(Y_i|s) = \beta_i \pi(X_i|s),\ \forall\ s \in \mathcal{S};\ \textit{and}$$

$$\pi(Y_j|s) = \pi(X_j|s) + (1 - \beta_i)\pi(X_i|s),\ \forall\ s \in \mathcal{S},$$

where $\beta_i \in [0, 1]$ is the conditional probability $\pi(Y_i|X_i)$.

The case of "minimally more informativeness" is a special case of the general requirements for greater informativeness. Imagine that the decision maker, upon receiving signal $X \in \mathcal{X}$, chooses a unique signal $Y \in \mathcal{Y}$ for all signals except one. (For example, the original signal might be allowed to pass through without any obstruction, in which case $Y_k = X_k$ for $k \neq i$.) In exactly one case

[6] In Bayesian parametric models, the information content of an IS with respect to the attribute being estimated is measured by its '*precision*'– usually defined as the reduction in the variance of the attribute's *posterior* distribution (conditional on the information signal) vis-a-vis the variance of its prior (unconditional) distribution. This presupposes the availability of appropriate metrics to quantify and compare the information precisions of different IS. Our model of information, building on Marschak and Radner (1972), is a more general, non-parametric formulation that nevertheless preserves a partial ordering of IS based on their information content. Hence we prefer the term '*informativeness*' to the term 'precision', for the general (non-parametric) model. When we specialize the model to specific parameters in a metric space, we will use measures of information precision to capture the informativeness of IS.

(for the signal X_i), the decision-maker uses the randomizing device. Further, the randomization follows a special structure: only one of two possible signals (Y_i and Y_j) are chosen, with probabilities that add up to one. This "minimal randomization" leads to minimally more informativeness. An alternative viewpoint is that the IS $\mathcal{Y}$ is generated by a "minimal garbling" of the IS $\mathcal{X}$. Observe that both "informativeness" and "minimally more informativeness" are reflexive: any IS is both more informative and minimally more informative than itself, by definition.

Before performing the comparative statics on the effect of IS informativeness, we need two further results. The first establishes the transitivity of informativeness, and second breaks down informativeness into a finite sequence in which each predecessor is minimally more informative than its successor.

Lemma 1 *If the IS $\mathcal{X}$ is at least as informative as the IS $\mathcal{Y}$, and the IS $\mathcal{Y}$ is at least as informative as the IS $\mathcal{Z}$, then $\mathcal{X}$ is at least as informative as $\mathcal{Z}$.*

Lemma 2 *For any two IS $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X}$ is at least as informative as $\mathcal{Y}$, there exists a finite sequence of IS $\mathcal{X} = \mathcal{Y}_0, \mathcal{Y}_1, ..., \mathcal{Y}_{T-1}, \mathcal{Y}_T = \mathcal{Y}$ (for some $T \geq 0$) such that $\mathcal{Y}_t$ is minimally more informative than $\mathcal{Y}_{t+1}$ for $t = 0, 1, ..., T-1$.*

The properties established in Lemmas 1 and 2 imply that, rather than working with informativeness of IS (which is analytically intractable in the general case), we can compare two IS $\mathcal{X}$ and $\mathcal{Y}$ wherein one of them (say, $\mathcal{X}$) is *minimally more informative* than the other. Results derived for "minimally more informativeness" then extend to informativeness in general by applying transitivity (Lemma 1) repeatedly (We are guaranteed that we can do so in a finite number of steps by Lemma 2.).

4.2 Production, Inventories, Sales and Information

In this Section, we study the behavior of production, inventories and sales with respect to IS informativeness. We initially focus on the general (non-parametric) framework developed above. To gain additional insights, we then apply these results to a specific, parametric demand model in Section **??**.

4.2.1 Impact of informativeness on production:

As established previously, the optimal production policy under an IS $\mathcal{X}$ is a threshold production policy with a per-period build-up-to level of $Q^*(\mathcal{X})$, given by equation (5). Theorem 2 shows the effect of changes in IS informativeness on production.

Theorem 2 *The optimal build-up-to level Q^* is increasing (non-decreasing) in the informativeness of the Information Systems used.*

Theorem 2 demonstrates that the firm's IS affects its production policy– even though the information signal on the demand in each period is received too late to affect that period's production. Since the information is received in time to influence shipment and inventory decisions at the DC, a more informative IS is helpful to finetune the firm's shipments to reflect actual demand at the retail outlet and optimize profits. In other words, the expected marginal revenue from each unit shipped to the DC is *increasing* in IS informativeness.[7] Since the optimal build-up-to level $Q^*(\mathcal{X})$ for *any* IS $\mathcal{X}$ equalizes the expected marginal revenues and the marginal costs, Q^* is increasing in IS informativeness, as Theorem 2 asserts.

4.2.2 Impact of informativeness on inventories:

Clearly, inventories depend on both the production policy and the shipment/inventory policy. Specifically, under the optimal policy of Theorem 1, the DC inventory in any period is determined by both the build-up-to and the ship-up-to thresholds– the latter in turn depends on the information signal received in that period. Mathematically, the average DC inventories under the IS $\mathcal{X}$ are $E_{X\in\mathcal{X}}\left\{[Q^*(\mathcal{X}) - q(X)]^+\right\} = E_{X\in\mathcal{X}}\left\{\max\{Q^*(\mathcal{X}) - q(X), 0\}\right\}$, where $q(X)$, given by equation (4), depends on the specific information signal X received under the IS $\mathcal{X}$. Observe that $q(\cdot)$ is a mapping from the set of "information signals" comprising the IS $\mathcal{X}$ to the positive real line. To compare inventories under different IS, we need to impose additional structure on the function $q(\cdot)$, for which we introduce the notion of *informational concavity.*

Definition 2 (*Informational Concavity*) *: Suppose that X_1, X_2 and Y are information signals in the set Ω, and $\exists\ p \in [0,1]$ such that $\Pr[X_1|Y] = p$ and $\Pr[X_2|Y] = (1-p)$. A function $f : \otimes \to \mathcal{R}$ is* informationally concave *in the set of information signals Ω iff $f(Y) \geq p \cdot f(X_1) + (1-p) \cdot f(X_2)$ for all such X_1, X_2, $Y \in \otimes$, where $\mathcal{R}$ is the Real line.*

While the analogy between informational concavity and regular concavity is readily apparent, an important difference is that concavity is defined over metric spaces closed with respect to addition and multiplication operators. Hence, the notion of 'weighted averages' over the domain of concavity is well-defined. In our development of IS, we haven't contrained information signals to metric spaces. The definition of informativeness earlier in this Section got around this lack of metrics in the signal

[7] The expected marginal value of each unit carried at the DC could be alternatively described as the *option value* or the *shadow price* of that unit, in the vocabulary of the finance and mathematical programming literatures.

space by working with probability measures. Informational concavity also works with probabilities over signals, with the advantage of not constraining the signal space to any specific metric, but necessitating additional structure on the function $f(\cdot)$ defined over the signal space.

To understand the intuition behind informational concavity, suppose that the signal X_1 is generated with probability p_1 and that the signal X_2 is generated with probability p_2, under an IS $\mathcal{X}$. Now consider an IS $\mathcal{Y}$ that garbles the signals X_1 and X_2 into an indistinguishable signal Y, and is otherwise identical to the IS $\mathcal{X}$. The signal Y will be generated under $\mathcal{Y}$ with a probability $(p_1 + p_2)$. Applying Definition 1, it is easy to see that the IS $\mathcal{X}$ is minimally more informative than the IS $\mathcal{Y}$, with β_i of Definition 1 set to 0. Note further that $\Pr[X_1|Y] = \frac{p_1}{p_1+p_2}$ and $\Pr[X_2|Y] = \frac{p_2}{p_1+p_2}$; thus a *necessary* condition for informational concavity of a function $f(\cdot)$ is that $f(Y) \geq p \cdot f(X_1) + (1-p) \cdot f(X_2)$, where $p = \frac{p_1}{p_1+p_2}$.

Now consider two IS $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X}$ is at least as informative as $\mathcal{Y}$. By Lemma 2, we know that there exists a finite sequence of IS $\mathcal{X} = \mathcal{Y}_0, \mathcal{Y}_1, ..., \mathcal{Y}_{T-1}, \mathcal{Y}_T = \mathcal{Y}$ (for some $T \geq 0$) such that $\mathcal{Y}_t$ is minimally more informative than $\mathcal{Y}_{t+1}$ for $t = 0, 1, ..., T-1$. Informational concavity of a function $f(\cdot)$ guarantees by transitivity that evaluations of $f(\cdot)$ can be rank-ordered based solely on the relative informativeness of the IS $\mathcal{X}$ and $\mathcal{Y}$.

The following Theorem derives *sufficient* conditions for information and inventories to be complements.

Theorem 3 *If the ship-up-to level $q(\cdot)$ is informationally concave, information and inventories are complements.*

To understand the drivers of the result of Theorem 3, note that inventories are increasing in the threshold production (build-up-to) level. Further, the production build-up-to level always increases with IS informativeness, as shown by Theorem 2. When $q(\cdot)$ is informationally concave, the firm's optimal distribution strategy works in tandem with its production strategy, so that inventories increase with IS informativeness. Thus the informational concavity of $q(\cdot)$ is a *sufficient* condition for complementarity. (In fact, the proof shows that even for a fixed production strategy, IS and inventories are complementary when $q(\cdot)$ is informationally concave.) When the concavity condition is not satisfied, the distribution strategy works against the complementarity relationship for certain parameter values, while the production strategy continues to work for the complementarity relationship. In this case, the complementarity/substitution relationship between IS and inventories depends on which factor dominates.

4.2.3 Impact of informativeness on sales:

The expected sales quantity $S(\mathcal{X})$ for an IS $\mathcal{X}$, under the optimal policies of Theorem 1, is $Q^*(\mathcal{X}) - I(\mathcal{X})$, where $I(\mathcal{X}) = E_{X \in \mathcal{X}} \{\max\{Q^*(\mathcal{X}) - q(X), 0\}\}$ is the average per-period inventory. This simplifies to $S(\mathcal{X}) = E_{X \in \mathcal{X}} \{\min\{Q^*(\mathcal{X}), q(X)\}\}$ .With an increase in informativeness, both $Q^*(\mathcal{X})$ and $I(\mathcal{X})$ increase under informational concavity of $q(\cdot)$. In this case, the expected sales quantity could increase or decrease with IS informativeness.

To make our insights concrete and understand how to operationalize Theorems 1, 2 and 3, we develop an illustrative application in the next Section, with parameterized demand curves and information systems.

5 Illustrative Applications

We demonstrate the power and generality of the modeling framework through illustrative applications. Once an application is established as a special instance of the model of Section 3, all the results derived for the model (including the solution structure, the optimal production and distribution strategies, and the relationship between the firm's IS and its production, inventories and sales) will also apply to the specific application. We begin in Section 5.1 by analyzing the simplest possible setting of a *linear* supply chain, with a single output market with linear demand curves, where the demand intercept is uncertain. We then analyze arborescent supply chains, starting in Section 5.2 with a supply chain operating in two output markets with correlated demand, and then extending the analysis to the general setting of n (≥ 2) horizontal markets in Section 5.3. Due to the generality of the demand model in Section 3, our analysis and results, developed in the context of a linear supply chain, can be applied to arborescent market structures. The crux of this approach, as summarized in Section 5.4, is the *separability* (which we prove) of the *optimal allocation problem* across the n horizontal markets from the *upstream production/inventory decision.* All these results are derived in the context of a discounted dynamic program with inventories.

5.1 Linear Supply Chain with Linear demand curves

Suppose that the firm considered previously sells through a retail outlet in a single output market, characterized by the demand curve $P(q) = \widetilde{a} - b \cdot q$ in each period, where the slope $b > 0$ and $\widetilde{a}$ is the uncertain, binary-valued intercept. The value of $\widetilde{a}$ depends on the market state s, which is binary: $s = 1$, corresponding to the "high" demand state wherein $\widetilde{a} = a_H$, occurs with probability p $(0 \leq p \leq 1)$, while $s = 0$, corresponding to "low" demand wherein $\widetilde{a} = a_L$, occurs with probability

$(1-p)$, where $a_H > a_L$. The production and holding costs are k and h per unit, as in the general model. In addition, let the transportation cost from the DC to the outlet be θ per unit. As before, β is the one-period discount factor.

The timing of information generated by the model is the same as before (See Figure 2). At the start of each period, the DC receives the binary signal X (taking the value 0 or 1) from the market (The binary information signal is an artifact of the binary state variable.). Assume that $X = s$ with probability $(1-\alpha)$ and $(1-s)$ with probability α. Without loss of generality, it is assumed that $0 \leq \alpha \leq \frac{1}{2}$. Thus, α is a measure of the *imprecision* of the information signal. While $\alpha = 0$ corresponds to perfect IS (which predicts the demand state perfectly), the IS X is pure noise when $\alpha = \frac{1}{2}$.

While our results hold more generally, we make the following assumptions to minimize technical distractions and clutter: *(i)* $a_L > \frac{k}{\beta} + \theta$: this eliminates relatively uninteresting corner solutions by ensuring that *some positive* production is profitable even when "low" demand is certain, and *(ii)* $p = \frac{1}{2}$.

The high and low demand revenue functions for each market, inclusive of the transportation costs, are given by $R(q|\ s=1) = (a_H - b \cdot q) \cdot q - \theta \cdot q$ and $R(q|\ s=0) = (a_L - b \cdot q) \cdot q - \theta \cdot q$. The expected revenue functions conditional on X are thus given by

$$\begin{aligned}
R(q|X=1) &= (1-\alpha) \cdot [(a_H - b \cdot q) \cdot q - \theta \cdot q] + \alpha \cdot [(a_L - b \cdot q) \cdot q - \theta \cdot q] \\
&= (\alpha \cdot a_L + (1-\alpha) \cdot a_H) \cdot q - b \cdot q^2 - \theta \cdot q; \text{ and} \\
R(q|X=0) &= ((1-\alpha) \cdot a_L + \alpha \cdot a_H) \cdot q - b \cdot q^2 - \theta \cdot q.
\end{aligned}$$

Observe that the conditional expected revenue functions for all realizations of X are bounded from above, twice-differentiable and concave in q. Thus, all the conditions for applying Theorem 1 are satisfied in this specification, and hence the myopic optimal policy of Theorem 1 applies here.

5.1.1 Production, Inventories, Sales and Informativeness

Firm's optimal policies: To illustrate the effect of increased informativeness on production, inventories and sales, we first apply Theorem 1 to derive the optimal ship-up-to and build-up-to levels. Using expression (4) of Theorem 1, the optimal ship-up-to levels can be shown to be:

$$q(X) = \begin{cases} \frac{(1-\alpha) \cdot a_H + \alpha \cdot a_L - (k+\theta-h)}{2 \cdot b}, & \text{when } X = 1; \text{ and} \\ \frac{(1-\alpha) \cdot a_L + \alpha \cdot a_H - (k+\theta-h)}{2 \cdot b}, & \text{when } X = 0. \end{cases} \tag{7}$$

Similiarly, the build-up-to level, using expression (5) of Theorem 1, is

$$Q^*=\begin{cases} \frac{(1-\alpha)\cdot a_H+\alpha\cdot a_L-(2\frac{k}{\beta}-k+\theta+h)}{2\cdot b}, & \text{if } (1-2\cdot\alpha)\cdot(a_H-a_L)>2\left[\frac{k}{\beta}-k+h\right]; \\ \frac{\frac{a_H+a_L}{2}-(\frac{k}{\beta}+\theta)}{2\cdot b}, & \text{otherwise.} \end{cases}$$

which simplifies to

$$Q^*=\max\left\{\frac{(1-\alpha)\cdot a_H+\alpha\cdot a_L-(2\frac{k}{\beta}-k+\theta+h)}{2\cdot b},\frac{\frac{a_H+a_L}{2}-(\frac{k}{\beta}+\theta)}{2\cdot b}\right\}. \tag{8}$$

Measuring informativeness: Before analyzing the behavior of production, inventories and sales with respect to changes in IS informativeness, we need to define informativeness in our specific context. Consider two IS X_1 and X_2 with respective precision parameters α_1 and α_2, such that $\alpha_1<\alpha_2$. It is clear that X_1 is a more precise IS than X_2. It can also be shown that the IS X_1 is *more informative* than the IS X_2. The proof is straightforward– by demonstrating that the IS X_2 can be generated as a *garbling* of the IS X_1 (Section 4). Thus, the 'precision' ranking of IS in the current demand model is *equivalent to* ranking them on their informativeness.

Production and Information: From (8) , Q^* is decreasing in α, i.e., increasing in informativeness, consistent with the prediction of Theorem 2.

Inventory and Information: Observe from expression (7) that the optimal ship-up-to levels $q(X)$ for $X\in\{0,1\}$ are linear (and hence, weakly concave) in α. Hence $q(\cdot)$ is *informationally concave* (Recall discussion in Section 4.2.2.). By Theorem 3, we know that inventory and information must be complements. In fact, this can be directly verified. Average inventory is $\frac{1}{2}\cdot\left[\{Q^*-q(1)\}^++\{Q^*-q(0)\}^+\right]$, which, using (7) and (8) , simplifies to:

$$I=\frac{\left[(1-2\cdot\alpha)\cdot(a_H-a_L)-2\left(\frac{k}{\beta}-k+h\right)\right]^+}{4b}. \tag{9}$$

Expression (9) for inventory highlights the relevance of the distinction between *ex ante demand uncertainty* (which reflects market conditions, including macroeconomic and other environmental factors) and *ex post uncertainty* (which is mediated in large measure by the efficacy of the firm's IS). The intercept spread (a_H-a_L) (while keeping the mean $\frac{a_H+a_L}{2}$ constant) measures the level of *ex ante* demand uncertainty. Expression (9) shows that, in accord with 'conventional wisdom', inventories are decreasing in the *ex ante* demand uncertainty. However, we also find that inventories *fall* as α increases; equivalently, inventories are *increasing* in IS informativeness. Thus, inventories and information are complements, consistent with the prediction of Theorem 3.

Sales and Information: Finally, the sales quantity $S(X)$ for any signal $X \in \mathcal{X}$ is $\min\{Q^*(\mathcal{X}), q(X)\}$ under the optimal policies, and derived using (7) and (8). When the demand spread is low, given by the condition $(1 - 2 \cdot \alpha) \cdot (a_H - a_L) \leq 2 \left[\frac{k}{\beta} - k + h\right]$, $Q^* \leq q(0) \leq q(1)$.Then $S(X) = Q^* = \frac{\frac{a_H + a_L}{2} - (\frac{k}{\beta} + \theta)}{2 \cdot b}, \forall\, X$, i.e., the optimal sales policy is to sell the quantity Q^* every period, and keep zero inventories. The more interesting case is that of high demand spread, given by the condition $(1 - 2 \cdot \alpha) \cdot (a_H - a_L) > 2 \left[\frac{k}{\beta} - k + h\right]$. In this case, $q(0) < Q^* < q(1)$, and

$$S(X) = \begin{cases} \frac{(1-\alpha)\cdot a_H + \alpha \cdot a_L - (2\frac{k}{\beta} - k + \theta + h)}{2 \cdot b}, & \text{when } X = 1; \text{ and} \\ \frac{(1-\alpha)\cdot a_L + \alpha \cdot a_H - (k + \theta - h)}{2 \cdot b}, & \text{otherwise.} \end{cases} \tag{10}$$

The *expected* sales quantity is $E_{X \in \mathcal{X}}[S(X)] = \frac{\frac{a_H + a_L}{2} - (\frac{k}{\beta} + \theta)}{2 \cdot b}$, which is the same as under low demand spread. Thus for *all* parameter values, $E_{X \in \mathcal{X}}[S(X)]$ is a *constant*, and hence *invariant* with respect to informativeness. This is probably an artifact of the linear demand model. Even so, we have the apparent conundrum that expected sales are constant but inventories increase with informativeness. To explain this, observe from (10) that the *variance* of sales quantities is increasing in IS informativeness (i.e., decreasing in α), because market shipments are better tailored to actual demand conditions (evidenced by improved sales revenues and profits, under better IS). Higher sales variance, in turn, leads to higher inventories.

5.2 Arborescent Supply Chains: Two Correlated Markets

Arborescent supply chains are supply chains that exhibit downstream "fan-out"– for example, a single upstream distributor might ship product to several (sometimes even several hundred) retail outlets. Clearly, arborescence is commonly observed in practical supply chains. Yet, in comparison to linear supply chains, they have not been modeled much in the academic literature, probably because of analytical difficulties. Two notable exceptions are Eppen and Schrage (1981) and Erkip et al (1990). Eppen and Schrage (1981) derive *approximately optimal*, cost-minimizing policies for an arborescent structure, under independent, normally distributed demands; Erkip et al (1990) generalize the analysis to correlated demands. Neither model IS, which would contribute an additional layer of complexity.

We extend the analysis of Section 5.1 to an arborescent supply chain: a firm, with a supply chain as described in Section 3, that produces and sells a product in *two* horizontal markets (Equivalently, think of a supply chain selling two related products.). The demand models for each market are the same as in Section 5.1. Additionally, the demand in the two markets may be correlated, with correlation coefficient ρ. Let $i(=1,2)$ be the market index. Thus, $X=(X_1,X_2)$ is the market information available to the DC, where $X_i = S_i$ with probability $(1-\alpha)$. θ is the per-unit transportation cost from the DC to either market; other parameters are identical to those of the model of Section 5.1. The firm's objective is to maximize its discounted expected profits over the long-term (infinite) horizon.

The joint distribution of the vector of demand states (S_1,S_2) is given by $P[S_1=S_2=1] = P[S_1=S_2=0]=\frac{1+\rho}{4}$, and $P[S_1=1,\ S_2=0]=P[S_1=0,\ S_2=1]=\frac{1-\rho}{4}$. Let $P(x_1,x_2)$ be the probability that $X_1=x_1$ *and* $X_2=x_2$, and $P_{x_1x_2}=Pr[S_1=1 \mid X_1=x_1,\ X_2=x_2]$ be the probability that a market is in the "high-demand" state, when x_1 is the signal from that same market and x_2 is the signal from the *other* market. These are derivable as functions of ρ by the repeated application of Bayes' rule. The revenue functions $R_H(.)$and $R_L(.)$ corresponding to high and low demand curves are $R_H(q)=(a_H-bq-\theta)q$ and $R_L(q)=(a_L-bq-\theta)q$ (inclusive of customization costs).

In order to derive the firm's optimal policies, we need to consider the allocation across the two markets, in addition to production and inventory. Suppose that the firm ships the total quantity q (≥ 0) to the two markets, when the market-signals it obtains from its IS are $X_1=x_1$ and $X_2=x_2$. The expected revenues from shipping quantity q_1 to market 1 and quantity q_2 to market 2 are

$$\begin{aligned} R(q|X_1=x_1,\,X_2=x_2) &= P_{x_1x_2}\cdot R_H(q_1)+(1-P_{x_1x_2})\cdot R_L(q_1) \\ &+ P_{x_2x_1}\cdot R_H(q_2)+(1-P_{x_2x_1})\cdot R_L(q_2), \end{aligned}$$

which simplifies to

$$\begin{aligned} R(q|X_1=x_1,\,X_2=x_2) = \quad & \left(P_{x_1x_2}\cdot a_H+(1-P_{x_1x_2})\cdot a_L-bq_1-\theta\right)q_1 \\ & +\left(P_{x_2x_1}\cdot a_H+(1-P_{x_2x_1})\cdot a_L-bq_2-\theta\right)q_2. \end{aligned} \tag{11}$$

The optimal allocation policy maximizes the expression (11) subject to the constraint that $q_1+q_2=q$. For convenience, define $f(P_{x_1x_2})=P_{x_1x_2}\cdot a_H+(1-P_{x_1x_2})\cdot a_L$ and $d(x_1,x_2)=\frac{f(P_{x_1x_2})-f(P_{x_2x_1})}{2\cdot b}$.

The optimal allocation of the intermediate good to market 1, is

$$q_1 = \begin{cases} q, & \text{if } q < d(x_1, x_2); \\ \frac{q+d(x_1,x_2)}{2}, & \text{if } q \geq |d(x_1, x_2)|\,; \\ 0, & \text{if } q < -d(x_1, x_2). \end{cases}$$

obtained by solving the constrained optimization problem (11) subject to $q_1 + q_2 = q$. The optimal allocation to market 2 is, of course, $q - q_1$. Plugging the solution back into the expression (11) , and simplifying, the conditional revenue function under optimal allocation is given by

$$R(q|X_1 = x_1, X_2 = x_2) = \begin{cases} -\theta \cdot q + q \cdot f(P_{x_1 x_2}) - bq^2, & \text{if } q < d(x_1, x_2); \\ -\theta \cdot q + q \cdot \left(\frac{f(P_{x_1 x_2}) + f(P_{x_2 x_1})}{2}\right) - \frac{b}{2} \cdot \left(q^2 - (d(x_1, x_2))^2\right), & \text{if } q \geq |d(x_1, x_2)|\,; \\ -\theta \cdot q + q \cdot f(P_{x_2 x_1}) - bq^2, & \text{if } q < -d(x_1, x_2). \end{cases}$$

Observe that this is concave, bounded from above, and differentiable almost everywhere (except at the point $|d(x_1, x_2)|$). Hence Theorem 1 can be applied, to derive the optimal production and distribution policies.

5.3 Arborescent Supply Chains: Multiple Horizontal Markets

We now extend the analysis to a supply chain operating in n independent, horizontal markets labelled $1, \cdots, n$, through retail outlets selling in each market. (The analysis extends straightforwardly to markets with correlated demand, by appropriately specifying the correlation parameters. Section 5.2 analyzed correlated demands for $n = 2$.) For each market, the demand and information structures are identical to those of the linear supply chain of Section 5.1. Thus, the IS generates the information vector $X = (X_1, \cdots, X_n)$, where i is the market index, and $X_i = s_i$ with probability $(1 - \alpha)$.

To demonstrate that Theorem 1 can be applied here, the optimal allocation rule is derived, for any shipment quantity q. Let $K_n(X)$ be the number of markets from which a "high" signal $(X_i = 1)$ is received. Thus K_n takes on one of the values $0, 1, \cdots, n$. The number of markets from which a "low" signal $(X_i = 0)$ is received is $(n - K_n)$. The marginal revenue function for each "high"-signal (i.e., favorable) market is thus $MR_H(q) = (1 - \alpha) \cdot a_H + \alpha \cdot a_L - \theta - 2 \cdot b \cdot q$, and that for each "low"-signal (i.e., unfavorable) market is $MR_L(q) = \alpha \cdot a_H + (1 - \alpha) \cdot a_L - \theta - 2 \cdot b \cdot q$. From the marginal revenue functions, it is clear that each favorable market (which has $X_i = 1$) should receive the same quantity under optimal allocation. Let q_H be the optimal allocation to each favorable market when the total shipment is q. Similiarly, each of the unfavorable markets should receive equal quantities; let this be q_L. Thus $K_n \cdot q_H + (n - K_n) \cdot q_L = q$. It is clear from the above arguments $K_n(X)$ is a *sufficient statistic* to compute the firm's expected revenues, in lieu of the vector of binary signals $X = (X_1, \cdots, X_n)$.

Under some conditions, it may be optimal to ship zero quantities to the unfavorable markets. Define

$$k^* = \begin{cases} \left\lceil \frac{2 \cdot b \cdot q}{(1-2\alpha)\cdot(a_H - a_L)} \right\rceil, & \text{when } \alpha < \frac{1}{2}; \\ +\infty, & \text{when } \alpha = \frac{1}{2}. \end{cases}$$

where q is the shipment quantity. k^* is the critical threshold such that $q_L = 0$ *if and only if* $K_n \geq k^*$. When $K_n \geq k^*$, $MR_H(\frac{q}{K_n}) = (1-\alpha) \cdot a_H + \alpha \cdot a_L - \theta - 2 \cdot b \cdot \frac{q}{K_n} \geq \alpha \cdot a_H + (1-\alpha) \cdot a_L - \theta = MR_L(0)$, and thus it is optimal for the firm to allocate exactly $\frac{q}{K_n}$ to each favorable market and 0 to the unfavorable markets. For $K_n < k^*$, the optimal allocation sets $MR_H(q_H) = MR_L(q_L)$, which implies that $q_H = q_L + \frac{(1-2\alpha)(a_H - a_L)}{2 \cdot b}$. Since $K_n \cdot q_H + (n - K_n) \cdot q_L = q$, $q_H(K_n) = \frac{q}{n} + \left(\frac{n-K_n}{n}\right) \cdot \left(\frac{(1-2\alpha)(a_H-a_L)}{2 \cdot b}\right)$, and $q_L(K_n) = \frac{q}{n} - \left(\frac{K_n}{n}\right) \cdot \left(\frac{(1-2\alpha)(a_H-a_L)}{2 \cdot b}\right)$. The expected revenues under *optimal* allocation, conditional on the information K_n, when the *total* shipment quantity to the markets is q, is given by:

$$R(q|K_n) = \begin{cases} K_n \left[(1-\alpha) \cdot a_H + \alpha \cdot a_L - b\frac{q}{K_n}\right] \cdot \frac{q}{K_n} - \theta \cdot q, & \text{when } K_n \geq k^*; \text{ and} \\ K_n \left[(1-\alpha) \cdot a_H + \alpha \cdot a_L - b \cdot q_H(K_n)\right] \cdot q_H(K_n) & \\ + (n - K_n)\left[\alpha \cdot a_H + (1-\alpha) \cdot a_L - b \cdot q_L(K_n)\right] \cdot q_L(K_n) - \theta \cdot q, & \text{otherwise.} \end{cases}$$

The corresponding expected marginal revenues are

$$R'(q|K_n) = \begin{cases} (1-\alpha) \cdot a_H + \alpha \cdot a_L - 2b\frac{q}{K_n} - \theta, & \text{when } K_n \geq k^*; \text{ and} \\ \frac{K_n((1-\alpha)\cdot a_H + \alpha \cdot a_L) + (n-K_n)(\alpha \cdot a_H + (1-\alpha)\cdot a_L)}{n} - \frac{2b}{n}q - \theta, & \text{otherwise.} \end{cases}$$

$R'(q|K_n)$ is linear and decreasing in q. Hence the expected revenue function conditional on the information vector X, $R(q|X) \equiv R(q|K_n(X))$ is bounded from above, twice-differentiable and concave for each X, under optimal allocation of q. Thus the necessary conditions to apply Theorem 1 are met. Hence the optimal production, shipment and inventory policies for this supply chain selling in multiple horizontal markets are myopic, and as specified by Theorem 1; additionally, the optimal allocation policy for any shipment quantity was derived above. The other insights from the analysis of the main model will also apply here.

5.4 Additional Remarks: Applications to Arborescent Supply Chains

Although the general framework presented in Section 3 was focussed on a linear supply chain, the above applications show that the solution structure can be adapted for arborescent market-structures, incorporating both material and information flows.

This generality was a result of the following modeling features– *(i)* an abstract (and very general) IS structure, *(ii)* minimal assumptions on the output markets– only the *aggregate* revenue function conditional on market information (which, for arborescent structures, is imputed from the optimal allocation of shipments across markets) is assumed to be concave, and *(iii)* the use of a non-parametric approach– garbling– to compare IS. Specifically, the solution approach enables the separation of the optimal *allocation* problem (among multiple markets) from that of determining the optimal (upstream) production and shipment/inventory policies.

Thus the optimal dynamic production and distribution policies for an arborescent supply chain may be derived using the following three step procedure: *(i)* derive the optimal allocation policy across the markets in a *static* (single-period) setting, conditional on each possible realization of the information vector; *(ii)* show that the conditional revenue functions induced by the optimal allocation policy are concave, bounded from above and twice-differentiable almost everywhere; and *(iii)* 'fold' this back into Theorem 1 to get the optimal production and distribution policies, which were shown to be myopic. The applicability of Theorem 1 for arborescent supply chains hinges critically on step *(ii)* above; fortunately, we would expect concavity and boundedness of revenue functions to hold in most (non-pathological) situations.

Additionally, the results derived in Section 4 (relating IS informativeness to the firm's production, inventories and sales) will also apply to these applications. Future research can build on these results.

6 The Role of Information Timing

In the preceding analysis, the modeling of information *timing*, relative to material flows, played a crucial role. Recall the model timeline of Figure 2: Demand information was received in each period *after* production commitments had been made, but *before* distribution from the DC. (However, as we proved, the production strategy was a function of the *quality* of the firm's IS, although not of the specific signal of that period.). In addition to the baseline model, we analyze two other plausible models that differ *only* in the timing of IS: *(i) Model of "late-information":* Demand information is received *after* both production and distribution decisions for that period, and *(ii) Model of "early-information":* Demand information is received in each period *before* both production and distribution decisions. Comparing the "late-information" and "early-information" models to the baseline "intermediate-information" model sheds light on the role of information timing (relative to material flows) in determining the firm's optimal production and distribution policies.

6.1 Model of "late-information"

The firm's optimal policies under "late-information", which is the case when the firm does not have an appropriate IS for timely capture of demand information (or alternatively, has no IS at all), can be analyzed as a special case of the previous model. Intuitively, the "late-information" case is equivalent to the baseline "intermediate-information" model where the IS signals are pure noise; i.e., there is no informational value in any signal received under such an IS. We formally define a "no-information" IS below, and establish that every IS is at least as informative as a no-information IS (or equivalently, the no-information IS can be generated from any other IS by appropriate garbling).

Definition 3 *An IS $\mathcal{Y}$ conveys no information when $\pi(s|Y) = \pi(s), \forall s \in S, \forall Y \in Su(\mathcal{Y})$.*

Lemma 3 *Every IS is at least as informative as a no-information IS.*

Thus, the structure of the optimal policies for the no-information model is the same as that derived previously in Theorem 1. Further, the preceding Lemma establishes that a "no-information" IS is equivalent to a "least-informative" IS. In this case, the posterior $R'(\cdot|X)$ is the same as the prior $R'(\cdot)$. Thus the ship-up-to level is the same for all X, and given by

$$q(X) = q = {R'}^{-1}(k - h),$$

which is obtained by simplifying equation (4) of Theorem 1. Simplifying equation (5) of Theorem 1 yields

$$Q^* = {R'}^{-1}\left(\frac{k}{\beta}\right),$$

which sets production at a level such that the costs of an additional unit of production, k, equals the expected marginal revenues $\beta \cdot R'(Q^*)$. Since $R(\cdot)$ is concave, $R'(\cdot)$ is a decreasing function, and so is ${R'}^{-1}(\cdot)$. Thus $Q^* < q$, and no inventories are ever held in the "no-information" case. Intuitively, there is no point in building up inventories at the DC, since there are no informational gains in the second (shipment) stage. Thus, the firm can decide on shipment quantities to the market in the production stage itself. The expected sales each period is equal to the actual sales, which is Q^*. Thus, the optimal solution for the no-information model is consistent with the earlier analysis: the *lower bound* for the optimal build-up-to level under any IS, ${R'}^{-1}\left(\frac{k}{\beta}\right)$, is attained here, since, as established by Theorem 2, the build-up-to level is an increasing function of informativeness. Further, the lower bound on *inventories* (zero) is also attained here, consistent with the complementary relationship between inventory and IS informativeness.

6.2 Model of "early-information"

In the early information case, the information signal $\mathcal{X}$ is received in time to affect the production decision. The ship-up-to threshold under the optimal inventory policy for the early information model is the same as before and given by

$$q(X) = \max\{q : R'(q|X) = k - h, 0\}.$$

The optimal production policy is to build up the DC inventory level to $Q^*(X)$ each period, where

$$Q^*(X) = \max\{Q : R'(Q|X) = \frac{k}{\beta}, 0\}.$$

(The proof is along the lines of that for Theorem 1.) By the concavity of $R(\cdot|X)$, $q(X)$ is at least as large as the build-up-to level $Q^*(X)$, and no inventory is ever held under the optimal policy. Intuitively, this "no-inventory" result is driven by the same reasons as in the no-information model: since no *additional* useful information is received between the time of the production decision and that of shipment, it is optimal to produce only for shipment. Here, the build-up-to level $Q^*(X)$ may be interpreted as setting expected marginal revenues equal to marginal costs for each period, subject to the non-negativity constraint. (Once again, the myopia of the optimal policies facilitate such an intuitive interpretation.)

6.3 Comparisons: The role of Information Timing

Clearly, the expected profits would be higher under early information than under no information. However, the per-period expected production (equivalently, sales) under the early-information model could be greater or less than that under no information, depending on the the specific demand curves and parameters. For linear demand, the expected sales are *identical* under the two models, provided a minor condition– positive sales should occur with probability one under early information– holds (See Section 4.2.3 for the analysis of sales under the main model, and Section 5.1.1 for the specialization of this model to linear demand.). The *variance* of sales is identically zero only in the no-information model: here, the production quantity in each period is the same because no demand information is ever available. In the "early information" model, the information received each period changes the expected demand distribution, and production is adjusted accordingly.

Unlike in the main model, the DC never holds any inventory in both the early-information and the no-information models, under their respective optimal policies. *The value of warehousing (to* *pool inventories and delay shipment to the output markets) is thus driven strongly by the timing and informativeness of IS.* (There could be other reasons for warehousing, such as transportation scale economies, that are not captured in our analysis.)

7 Concluding Remarks

We modeled both material and information flows in a multiechelon supply chain seeking to maximize its expected discounted profits over the infinite horizon, under demand uncertainty. We proved that the optimal production, inventory and sales policies for the supply chain are myopic, and derived them in closed form, *as a function of the firm's IS.* The 'informativeness' of Information Systems was characterized non-parametrically, using the notion of *garbling.* This provided a crucial element for a general framework: a *payoff-independent* partial ordering of all possible IS. The production build-up-to level was increasing in informativeness. Further, under reasonable conditions, information and inventories could be complements, not substitutes. Thus, average inventories could increase with IS informativeness. Average sales, the difference between the build-up-to level and average inventories, could thus increase or decrease with IS informativeness. For example, under the widely used linear demand curve, we proved that inventories and information are *strict* complements, even though the expected sales (in units) is invariant to IS informativeness (see Section 5.1).

There are two important reasons for the complementarity of information and inventory – a result that runs counter to conventional wisdom as well as much of the existing literature. The first is that changes in the informativeness of IS have an "accordion effect" all along the supply chain, which have to be taken into account. In the general model, for example, changes in IS informativeness affect production strategies in addition to distribution strategies. Knowing that more informative IS is available, the firm raises its build-up-to levels. Secondly, when a firm is modelled as a single point (without making its internal operations explicit), the whole issue of the *timing* of information is trivialized. In contrast, the firm under the general framework takes certain actions (production) before the receipt of demand information and other actions (distribution) after the information is known, which obviously affects the relationship between IS and inventory. This was illustrated using models differing only in the timing of information: the complementarity relationship (and indeed, the need to hold inventories at the DC) vanished under both the "early" and "late" information models. Thus the paper's results underscore *(a)* the pivotal role of IS timing in a supply chain, which affects all other decisions, including production, inventories and sales; and consequently, *(b)* the importance of information flow analysis, in conjunction with material flows, to optimize supply chain performance.

The combination of the following modeling features– *(i)* an abstract (and very general) IS structure, *(ii)* minimal assumptions on the output market(s)– only the aggregate revenue function is assumed to be concave, and *(iii)* the use of garbling to compare IS, enable the model to be specialized to study different arborescent supply chains. Section 5 developed a solution technique for arborescent supply chains, hinging on the separation of the optimal *allocation* problem (among the multiple end-markets) from the problem of determining the optimal (upstream) production and shipment/inventory policies. Section 5.4 laid out a three-step procedure to derive the optimal dynamic production and distribution policies for an arborescent supply chain. As illustrated through the applications in Section 5, the model can be specialized to derive the optimal policies for a more precise (parametric) characterization of market, IS and supply chain structures – thus integrating marketing, IS and operational perspectives.

An interesting issue not studied in this paper, that could build on this framework, is the optimal trade-off between IS informativeness and timing. A firm could structure its production processes (and the concomitant IS) to exploit rough (less informative) information early in the production process, or exploit more accurate information downstream with some delay costs (Cost and feasibility considerations might preclude its doing both.).

An assumption underlying this analysis was that the entire supply chain (including production, distribution and sales) is controlled by a single decision-maker, or equivalently, functions as a *team* with the shared objective of maximizing total supply chain profits. Often, different elements of the supply chain (such as different firms, or divisions within a firm) have diverging objectives. Future research could study the effects of such incentive-conflicts on supply chain strategies and profits. Also of interest is the role of supply chain contracts in aligning the incentives of the different players constituting the supply chain (*cf* Anand *et al* (2008)).

While game-theoretic analyses of competition typically are rich in their modeling of markets (and information), they tend to simplify the firms' internal structures. On the other hand, the extant supply chain models focus more on modeling the firm's internal activities (especially material flows), with very simple modeling of both information and markets. The difficulty with trying to do both is, of course, analytical intractability. A logical extension of the present model would be to competing supply chains, retaining the richness in the modeling of the firms' internal material and information flows. How competition affects, and is affected by, each firm's supply chain activities promises to be a fertile area for future research.

8 Appendix: Proof of Theorem 1

The proof of the optimality of Theorem 1 proceeds in three steps:

(i) The space of feasible policies is expanded, by allowing the firm to also sell its product in a "spot market" for the intermediate good (as described below). A specific policy is considered for this "augmented" problem, and its value function is derived.

(ii) The policy proposed in *(i)* is proved to be optimal in the "augmented" problem. Thus, its value function is an upper bound on the value obtainable in the original problem.

(iii) The value function is derived for the policy proposed in Theorem 1, and shown to be identical to the value function of the optimal policy in the augmented problem. This completes the proof since the augmented problem is a relaxation of the original problem.

Step (i): Let Ψ denote the space of all feasible policies for the firm. Now suppose there is a "spot market" available every period, in which the firm can sell any amount of its available intermediate good at the price $k-h$. (This may be interpreted as the salvage value.) Denote the space of feasible policies with the additional spot-price option as $\widetilde{\Psi}$. Clearly, $\widetilde{\Psi} \supseteq \Psi$.

Now consider the following policy in the "augmented" problem:

Augmented policy:

(A) Production: Build up the DC inventory of the product each period to Q^*. If the quantity available at the DC exceeds Q^*, produce nothing.

(B) Shipment / Inventories: Conditional on the available quantity Q, and information X, ship up to the quantity $q(X)$ to the market. If there is any quantity left over, keep up to $[Q^* - q(X)]^+ = Q^* - \min\{Q^*, q(X)\}$ in inventory. Sell the remaining quantity (if any; given by $[Q - \max\{Q^*, q(X)\}]^+$) in the spot market.

Now the infinite horizon discounted value function $\widetilde{v}(Q)$ is derived for the augmented policy, where the initial "state" (DC inventory) is Q. Under the augmented policy, the production quantity for the next period is $Q^* - Q$ when $Q \leq Q^*$, and 0 otherwise. The shipment quantity, when Q units are available and the information signal is X, is $\min\{Q, q(X)\}$, yielding the expected revenues $R(\min\{Q, q(X)\}|X)$. The inventory carried forward is the minimum of the remaining quantity and the inventory build-up-to level, which is $\min\{Q - \min\{Q, q(X)\}, Q^* - \min\{Q^*, q(X)\}\}$. Finally, the quantity sold in the spot market is the leftover after setting aside the shipment and inventory quantities, which simplifies to $[Q - \max\{Q^*, q(X)\}]^+$.

The value function is specified by the following recursive equations.

When $Q \leq Q^*$,

$$\begin{aligned}\widetilde{v}(Q) &= -k \cdot (Q^* - Q) + \beta \cdot E_X \left[R\left(\min\{Q^*, q(X)\}|X\right) - h \cdot \left(Q^* - \min\{Q^*, q(X)\}\right)\right] \\ &\quad -\beta \cdot E_X \left[k \cdot \min\{Q^*, q(X)\}\right] + \beta \cdot \widetilde{v}(Q^*). \qquad (12)\end{aligned}$$

When $Q > Q^*$,

$$\begin{aligned}\widetilde{v}(Q) &= \beta \cdot E_X \left[R\left(\min\{Q, q(X)\}|X\right) - h \cdot \min\left\{\left(Q - \min\{Q, q(X)\}\right), \left(Q^* - \min\{Q^*, q(X)\}\right)\right\}\right] \\ &\quad +\beta \cdot (k-h) \cdot E_X \left[\left(Q - \max\left\{Q^*, q(X)\right\}\right)^+\right] \qquad (13) \\ &\quad -\beta \cdot k \cdot E_X \left[Q^* - \min\left\{\left(Q - \min\{Q, q(X)\}\right), \left(Q^* - \min\{Q^*, q(X)\}\right)\right\}\right] + \beta \cdot \widetilde{v}(Q^*).\end{aligned}$$

Observe that the quantity available at the DC never exceeds Q^* after the first period; thus, under the augmented policy, the spot market is never used after the first period. Also, when $Q^* < Q$, $Q^* - \min\{Q^*, q(X)\} < Q - \min\{Q, q(X)\}$. Simplifying and solving equations (12) and (13),

$$\widetilde{v}(Q) = \begin{cases} k \cdot Q + \widetilde{v}(0), & \text{if } Q \leq Q^*; \\ \beta \cdot E_X \left[\Phi(Q; X)\right] + k \cdot Q^* + \widetilde{v}(0), & \text{if } Q > Q^*; \end{cases} \qquad (14)$$

where

$$\widetilde{v}(0) = -k \cdot Q^* + \frac{\beta}{1-\beta} \cdot E_X \left[R\left(\min\{Q^*, q(X)\}|X\right) - h \cdot \left(Q^* - \min\{Q^*, q(X)\}\right) - k \cdot \min\{Q^*, q(X)\}\right], \qquad (15)$$

and

$$\Phi(Q; X) = R\left(\min\{Q, q(X)\}|X\right) - R\left(\min\{Q^*, q(X)\}|X\right) + (k-h) \cdot \left(Q - \max\left\{Q^*, q(X)\right\}\right)^+. \qquad (16)$$

This completes the step (i) of the proof.

Step (ii) Straightforward differentiation of equation (16) shows that, for $Q \neq q(X)$ and $Q > Q^*$,

$$\Phi'(Q; X) = (k-h) + \big(R'(Q|X) - (k-h)\big) \cdot I_{\{Q < q(X)\}},$$

where $I_{\{A\}}$ is the indicator function that takes the value 1 if event A is true and 0 otherwise. Since $R'(\cdot|X)$ is decreasing and $R'(q(X)|X) = k - h$, $\Phi'(\cdot; X)$ is also decreasing in Q. Since in addition $\Phi(\cdot; X)$ is continuous, it must also be concave. Thus, by equation (14), $\widetilde{v}(Q)$ is also concave in Q in the range (Q^*, ∞). Further, $\widetilde{v}'(Q)$ is well-defined almost everywhere, and decreasing. For $Q < Q^*$, $\widetilde{v}'(Q) = k$. For $Q > Q^*$,

$$\begin{aligned}\widetilde{v}'(Q) &= \beta \cdot E_X \left[\Phi'(Q; X)\right] < \beta \cdot E_X \left[\Phi'(Q^*; X)\right] \\ &= \beta \cdot (k-h) + \beta \cdot E_X \left[\left(R'(Q^*|X) - (k-h)\right) \cdot I_{\{Q^* < q(X)\}}\right] \\ &= \beta \cdot (k-h) + \beta \cdot \left(\frac{k}{\beta} - k + h\right) = k \text{ (by equation (5))}.\end{aligned}$$

It follows that under the augmented policy,

$$\widetilde{v}'(Q) = k, \text{ for } Q < Q^*, \text{ and} \tag{17}$$

$$\widetilde{v}'(Q) < k, \text{ for } Q > Q^*. \tag{18}$$

The proof of the optimality of the augmented policy is by backward induction. Assume that the augmented policy is followed for the production decision at the end of period 1 (after sales in period 1) and for production, shipment and inventory decisions from period 2 onwards for the rest of the horizon. Then the augmented policy will be proved optimal for the production policy in period 0, and the shipment and inventory policies in period 1, as well, for any initial state (DC inventory). Thus, the augmented policy will be shown to be *unimprovable* and hence optimal.

First, the optimal distribution policy in period 1 is derived, for any information signal X. Let $\overline{Q}$ be the quantity on hand at the beginning of period 1. $\overline{Q}$ is the sum of the on-hand quantities at the end of period 0 (the start of the horizon) and the production in period 0 for period 1. Let Q be the shipment quantity to the output market, and Q_I the inventory carried forward to the next period. Then, the quantity sold in the spot market is $\overline{Q} - (Q + Q_I)$. The optimal distribution/inventory policy is given by the solution to the concave maximization problem

$$\max_{Q,Q_I} R(Q|X) + (k-h) \cdot \left(\overline{Q} - (Q + Q_I) \right) - h \cdot Q_I + \widetilde{v}\,(Q_I)$$

subject to the constraint $\overline{Q} \geq (Q + Q_I)$. The solution is simple to derive. The firm has three options: shipping to the output market, selling in the spot market and carrying inventory. The expected revenue from carrying inventory forward is $\widetilde{v}\,(Q_I) - h \cdot Q_I$; the expected marginal revenue of this inventory (from equations (17) and (18)) is $(k-h)$ for $Q_I < Q^*$, and less than $(k-h)$ otherwise. The marginal revenue (= price) from the spot market is $(k-h)$. Since $R'(Q|X) > (k-h)$ for $Q < q(X)$, shipping to the output market dominates the other options until the shipment quantity reaches $q(X)$. Since $R'(Q|X) < (k-h)$ for $Q > q(X)$, the spot market dominates shipping additional quantities above $q(X)$. So, the optimal quantity shipped never exceeds $q(X)$. The firm is *indifferent* between holding inventories and selling in the spot market as long as the inventories don't exceed Q^*; above the inventory level of Q^*, the spot market option dominates. Thus, if the available quantity $\overline{Q}$ is above $q(X)$, it is optimal to ship $q(X)$ to the output market; with regard to the remaining quantity, one optimal course of action is to carry inventory forward up to the level $[Q^* - q(X)]^+$ (which is $< Q^*$), and sell the rest in the spot market. Thus, the distribution/inventory policy specified by the augmented policy, is also optimal for period 1, for any available quantity $\overline{Q}$.

Given this period-1 shipment/inventory policy, we now derive the optimal production quantity in period 0. Let $\widetilde{I}(Q)$ denote the infinite horizon discounted returns starting in period 1, under the augmented policy, where $Q \geq 0$ is the quantity on hand (before the market information is received). Clearly, under the augmented policy,

$$\widetilde{I}(Q) = \begin{cases} \begin{array}{c} E_X\left[R\left(\min\{Q, q(X)\}|X\right) - h \cdot (Q - \min\{Q, q(X)\})\right] \\ -E_X\left[k \cdot (Q^* - Q + \min\{Q, q(X)\})\right] + \widetilde{v}(Q^*), \end{array} & \text{when } Q \leq Q^*; \\ \begin{array}{c} E_X\left[R\left(\min\{Q, q(X)\}|X\right) - h \cdot (Q^* - \min\{Q^*, q(X)\})\right] + \\ E_X\left[(k-h) \cdot (Q - \max\{Q^*, q(X)\})^+ - k \cdot \min\{Q^*, q(X)\}\right] + \widetilde{v}(Q^*) \end{array} & \text{when } Q > Q^*. \end{cases} \tag{19}$$

Differentiating equation (19) with respect to Q, we get (for $Q \neq Q^*$)

$$\begin{aligned} \widetilde{I}'(Q) &= E_X\left[R'\left(\min\{Q, q(X)\}|X\right) + (k-h) \cdot I_{\{Q > q(X)\}}\right] \\ &= E_X\left[R'\left(\min\{Q, q(X)\}|X\right) - (k-h) \cdot I_{\{Q < q(X)\}}\right] + (k-h) \end{aligned} \tag{20}$$

Observe that, since $R'\left(q(X)\}|X\right) = k - h$ (by definition $q(X)$), $\widetilde{I}'(Q)$ is a decreasing function of Q, for all $Q \neq Q^*$. Further, by equation (5) of Theorem 1, $\widetilde{I}'(Q^*) = \frac{k}{\beta}$. Thus, by continuity of $\widetilde{I}(Q)$, $\widetilde{I}'(Q) > \frac{k}{\beta}$ for $Q < Q^*$, and $\widetilde{I}'(Q) < \frac{k}{\beta}$ for $Q > Q^*$.

Now suppose, in period 0 (prior to the production decision), the on-hand inventory is (arbitrarily) Q. The optimal period 0 production quantity $\widetilde{Q}$ is the solution to the concave maximization problem

$$\max_{\widetilde{Q}} \quad \left\{-k \cdot \widetilde{Q} + \beta \cdot \widetilde{I}(\widetilde{Q} + Q)\right\},$$

where $\widetilde{Q} \geq 0$. Since $\widetilde{I}'(\widetilde{Q} + Q) > \frac{k}{\beta}$ for $\left(\widetilde{Q} + Q\right) < Q^*$, and $\widetilde{I}'(\widetilde{Q} + Q) < \frac{k}{\beta}$ for $\left(\widetilde{Q} + Q\right) > Q^*$ (as shown above), it is optimal to build up the DC inventory in period 1 to Q^* when $Q < Q^*$; i.e., to produce the quantity $\widetilde{Q} = Q^* - Q$, when $Q^* > Q$. When $Q \geq Q^*$, it is optimal to produce nothing, since the marginal cost k outweighs the expected marginal benefit. Thus, the optimal production quantity is $\widetilde{Q} = (Q^* - Q)^+$, which is identical to the production under the augmented policy.

Thus, it has been shown that if the augmented policy is followed in future periods, it is optimal in the current period as well, for all initial states (DC's starting inventories). This shows that the augmented policy is unimprovable and hence optimal in the space $\widetilde{\Psi}$. This completes step *(ii)* of the proof.

Step (iii) Now consider the production and distribution/inventory policies proposed in the Theorem. When the starting inventory is $Q \leq Q^*$, and the space of feasible policies is Ψ (i.e., there is no spot market available), the infinite horizon returns under the proposed policy are

$$\begin{aligned} v(Q) &= -k\cdot(Q^*-Q)+\beta\cdot E_X\left[R\left(\min\{Q^*,q(X)\}|X\right)-h\cdot(Q^*-\min\{Q^*,q(X)\})\right] \\ &\quad -\beta\cdot E_X\left[k\cdot\min\{Q^*,q(X)\}\right]+\beta\cdot\widetilde{v}(Q^*). \end{aligned}$$

It is clear that $v(Q)=k\cdot Q+v(0)$ for all $Q\leq Q^*$. Simplifying,

$$v(0)=-k\cdot Q^*+\frac{\beta}{1-\beta}\cdot E_X\left[R\left(\min\{Q^*,q(X)\}|X\right)-h\cdot(Q^*-\min\{Q^*,q(X)\})-k\cdot\min\{Q^*,q(X)\}\right],$$

which is identical to $\widetilde{v}(0)$ under the augmented policy (compare with equations (14) and (15)). Since $\widetilde{\Psi}\supseteq\Psi$ and the augmented policy is optimal in $\widetilde{\Psi}$, the value function $\widetilde{v}(\cdot)$ from the augmented policy is an upper bound on the value from any policy in Ψ. Since $v(\cdot)=\widetilde{v}(\cdot)$, the policy proposed in Theorem 1 attains this upper bound, and is hence optimal for Ψ. This completes the proof of Theorem 1. ⧫

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Technical Appendix: Proofs for the Paper "Can Information and Inventories be Complements?"

Proof of Theorem 1: The proof is provided in the Appendix in the main document. ⧫

Proof of Lemma 1: Given that $\mathcal{X}$ is at least as informative as $\mathcal{Y}$, and $\mathcal{Y}$ is at least as informative as $\mathcal{Z}$, we must have non-negative sets of numbers β_{XY} and β_{YZ} such that $\pi(Y|s) = \sum_{X\in\mathcal{X}} \beta_{XY} \cdot \pi(X|s)$, $\forall$ $s \in \mathcal{S}$, $Y \in \mathcal{Y}$, where $\sum_{Y\in\mathcal{Y}} \beta_{XY} = 1$, $\forall$ $X \in \mathcal{X}$, and $\pi(Z|s) = \sum_{Y\in\mathcal{Y}} \beta_{YZ} \cdot \pi(Y|s)$, $\forall$ $s \in \mathcal{S}$, $Z \in \mathcal{Z}$, where $\sum_{Z\in\mathcal{Z}} \beta_{YZ} = 1$, $\forall$ $Y \in \mathcal{Y}$. Thus, for an arbitrary Z and s,

$$\begin{aligned} \pi(Z|s) &= \sum_{Y\in\mathcal{Y}} \beta_{YZ} \cdot \pi(Y|s) \\ &= \sum_{Y\in\mathcal{Y}} \beta_{YZ} \left(\sum_{X\in\mathcal{X}} \beta_{XY} \cdot \pi(X|s) \right) \\ &= \sum_{X\in\mathcal{X}} \left(\sum_{Y\in\mathcal{Y}} \beta_{YZ} \cdot \beta_{XY} \right) \pi(X|s) \\ &= \sum_{X\in\mathcal{X}} \beta_{XZ} \cdot \pi(X|s), \end{aligned}$$

where $\beta_{XZ} = \sum_{Y\in\mathcal{Y}} \beta_{YZ} \cdot \beta_{XY}$ is non-negative for all X and Z. Further, observe that

$$\begin{aligned} \sum_{Z\in\mathcal{Z}} \beta_{XZ} &= \sum_{Z\in\mathcal{Z}} \left(\sum_{Y\in\mathcal{Y}} \beta_{YZ} \cdot \beta_{XY} \right) \\ &= \sum_{Y\in\mathcal{Y}} \left(\beta_{XY} \cdot \sum_{Z\in\mathcal{Z}} \beta_{YZ} \right) \\ &= \sum_{Y\in\mathcal{Y}} (\beta_{XY} \cdot 1) \\ &= 1\cdot \end{aligned}$$

This proves that $\mathcal{Z}$ is a garbling of $\mathcal{X}$. Thus, $\mathcal{X}$ is at least as informative as $\mathcal{Z}$.⧫

Proof of Lemma 2: If $\mathcal{X} = \mathcal{Y}$, set $T = 0$ and we are done. We now consider the case where $\mathcal{X} \neq \mathcal{Y}$. We offer a constructive proof. By the definition of garbling, there are non-negative numbers β_{XY} such that

$$\pi(Y|s) = \sum_{X\in\mathcal{X}} \beta_{XY} \cdot \pi(X|s),\ \forall\ s \in \mathcal{S},\ Y \in \mathcal{Y};\text{ and}$$

$$\sum_{Y\in\mathcal{Y}} \beta_{XY} = 1,\ \forall\ X \in \mathcal{X}.$$

Recall that $Su(\mathcal{X}) = \{X_1, X_2, \cdots, X_m\}$ and $Su(\mathcal{Y}) = \{Y_1, Y_2, \cdots, Y_n\}$. Consider a specific signal $X_1 \in \mathcal{X}$. There are non-negative numbers $\beta_1, \beta_2, \cdots \beta_n$ such that upon observing the signal X_1 under IS $\mathcal{X}$, the decision-maker's randomizing device chooses the signal Y_i under the IS $\mathcal{Y}$ with probability β_i ($i = 1, 2, \cdots, n$), with $\sum_{i=1}^{n} \beta_i = 1$. Now construct the following IS (call it $\mathcal{Y}_1$) from the IS $\mathcal{X}$. The randomizing device (under $\mathcal{Y}_1$) allows all signals received under the IS $\mathcal{X}$ to pass through as is, *except for the signal* X_1. When the signal generated under $\mathcal{X}$ is X_1, the randomizing device generates one of the following two signals: Y_1 with probability β_1 and X_1 with probability $(1 - \beta_1)$. By construction, $\mathcal{X}$ is minimally more informative than

$\mathcal{Y}_1$.

Now construct the IS $\mathcal{Y}_2$ from $\mathcal{Y}_1$ as follows. The randomizing device (under $\mathcal{Y}_2$) transmits all signals received under $\mathcal{Y}_1$ unaltered, except for the signal X_1. When the signal generated under $\mathcal{Y}_1$ is X_1, the randomizing device operates as follows: *(i)* when $\beta_2 \neq 0$, the device generates one of the following two signals: Y_2 with probability $\frac{\beta_2}{1-\beta_1}$ (which is well-defined when $\beta_2 \neq 0$) or X_1 with probability $(1- \frac{\beta_2}{1-\beta_1})$; and *(ii)* when $\beta_2 = 0$, the device doesn't randomize at all (i.e., generates Y_2 with probability 0). The net result is that when the signal X_1 is generated under the IS $\mathcal{X}$, the IS $\mathcal{Y}_2$ generates Y_1 with probability β_1, Y_2 with probability β_2 and X_1 with probability $(1- \beta_1 - \beta_2)$. By construction, $\mathcal{Y}_1$ is minimally more informative than $\mathcal{Y}_2$.

Repeat this process n times. Under IS $\mathcal{Y}_k$, all signals received under $\mathcal{Y}_{k-1}$ are transmitted unaltered, except for the signal X_1. When X_1 is generated under the IS $\mathcal{Y}_{k-1}$ (an event with probability $(1-\sum_{j=1}^{k-1}\beta_j)$), the randomizing device for $\mathcal{Y}_k$ operates as follows: *(i)* when $\beta_k \neq 0$, the device generates one of the following two signals: Y_k with probability $\frac{\beta_k}{1-\sum_{j=1}^{k-1}\beta_j}$ (which is well-defined when $\beta_k \neq 0$) or X_1 with probability $\left(1 - \frac{\beta_k}{1-\sum_{j=1}^{k-1}\beta_j}\right)$; and *(ii)* when $\beta_k = 0$, the device doesn't randomize at all (i.e., generates Y_k with probability 0). By construction, $\mathcal{Y}_{k-1}$ is minimally more informative than $\mathcal{Y}_k$. Furthermore, under the IS $\mathcal{Y}_k$, the signals generated under the original IS $\mathcal{X}$ are preserved, except when $\mathcal{X}$ generates the signal X_1. In the latter case, $\mathcal{Y}_k$ generates the signal Y_1 with probability β_1, Y_2 with probability β_2,..., Y_k with probability β_k, and the signal X_1 with probability $(1-\sum_{j=1}^{k}\beta_j)$. Finally, under the IS $\mathcal{Y}_n$, when $\mathcal{X}$ generates the signal X_1, the randomizing device generates one of the signals $\{Y_j\}_{j=1}^{n}$, with the respective probabilities $\{\beta_j\}_{j=1}^{n}$, and the signal X_1 with probability $(1-\sum_{j=1}^{n}\beta_j) = 0$. Thus *under the IS $\mathcal{Y}_n$, all signals generated under the original IS $\mathcal{X}$ other than the signal X_1 pass through as is; the randomizing behavior for the signal X_1 alone mimicks that of the IS $\mathcal{Y}$.* Further, $\mathcal{Y}_t$ is minimally more informative than $\mathcal{Y}_{t+1}$, for $t = 0, 1, \cdots, n-1$.

Starting with the IS $\mathcal{Y}_n$, a similiar process of increasing randomization can be implemented for the signal X_2, so that, after n more iterations, the resulting IS mimicks the randomization under the IS $\mathcal{Y}$ for both the signals X_1 and X_2 (the signals $X_3, \cdots, X_m$ generated under $\mathcal{X}$ pass through without any randomization). Repeating this process for each of the signals $X_3, \cdots, X_m$, we can construct the IS $\mathcal{Y}$ from the IS $\mathcal{X}$. Further, each member of the resulting sequence of IS is minimally more informative than its successor. ♦

Proof of Theorem 2: Consider two arbitrary IS $\mathcal{X}$ and $\mathcal{Y}$, where $\mathcal{X}$ is minimally more informative than $\mathcal{Y}$. Let $X_1, X_2, \cdots, X_m$ be the m possible signals generated under the IS $\mathcal{X}$, and $Y_1, Y_2, \cdots, Y_n$ be the n signals generated under IS $\mathcal{Y}$. By the definition of "minimally more informativeness", a randomizing device generates signals in $\mathcal{Y}$ from the signals in $\mathcal{X}$ as follows. Upon receiving a particular signal X_i under the IS $\mathcal{X}$, the device randomizes between two signals Y_i and Y_j: Y_i is chosen with probability β_i and Y_j is chosen with probability $(1-\beta_i)$. Upon receiving any other signal X_k $(k \neq i)$ under the IS $\mathcal{X}$, the device generates a unique signal $Y_k \in \mathcal{Y}$. Abstracting away from the state space, suppose that, under the IS $\mathcal{X}$, the signals $\{X_k\}_{k=1}^m$ occur with probabilities $\{p_k\}_{k=1}^m$, where $\sum_{k=1}^m p_k = 1$. Then, under the IS $\mathcal{Y}$, the randomizing device generates the signal Y_k $(k \neq i$ or $j)$ with probability p_k, Y_i with probability $p_i \cdot \beta_i$ and Y_j with probability $p_j + p_i \cdot (1-\beta_i)$. Furthermore, there is loss of information under $\mathcal{Y}$ only when the signal Y_j is generated (since the original signal could have been either X_i with probability $\frac{p_i \cdot (1-\beta_i)}{p_i \cdot (1-\beta_i)+p_j}$ or X_j with probability $\frac{p_j}{p_i \cdot (1-\beta_i)+p_j}$). For all other signals Y_k, the corresponding signal X_k under $\mathcal{X}$ can be deduced with probability 1.

Since the optimal ship-up-to level for a signal X is given $q(X) = \max\{q : R'(q|X) = k-h, 0\}$, it is clear from the preceding discussion that $q(Y_k) = q(X_k)$ for all $k \neq j$, and $q(Y_j) \geq q(X_j)$ if and only if the signal X_i is more favorable than the signal X_j .

Recall that under the IS $\mathcal{X}$, the optimal build-up-to level is given by the solution $Q^*(\mathcal{X})$ of

$$E_X\left[\big(R'(Q^*|X) - (k-h)\big) \cdot I_{\{Q^* < q(X)\}}\right] = \frac{k}{\beta}.$$

$Q^*(\mathcal{Y})$ is computed similiarly. Observe that the left hand side of the preceding equation is decreasing in Q^*; the right side is a constant independent of Q^*. Thus, to show that $Q^*(\mathcal{X}) \geq Q^*(\mathcal{Y})$, it is enough to show that

$$\begin{aligned} E_X\left[(R'(Q^*(\mathcal{Y})|X) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y}) < q(X)\}}\right] &\geq E_X\left[(R'(Q^*(\mathcal{X})|X) - (k-h)) \cdot I_{\{Q^*(\mathcal{X}) < q(X)\}}\right] \\ &= E_Y\left[(R'(Q^*(\mathcal{Y})|Y) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y}) < q(Y)\}}\right], \end{aligned}$$

where the last equality arises because both expressions are equal to $\frac{k}{\beta}$. Now,

$$E_X\left[(R'(Q^*(\mathcal{Y})|X) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y}) < q(X)\}}\right] \geq E_Y\left[(R'(Q^*(\mathcal{Y})|Y) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y}) < q(Y)\}}\right].$$

Equivalently,

$$\begin{aligned} &\sum_{k=1}^{m} p_k \left(R'(Q^*(\mathcal{Y})|X_k) - (k-h)\right) I_{\{Q^*(\mathcal{Y}) < q(X_k)\}} \\ \geq\ &\sum_{k=1}^{n} \Pr(Y_k) \left(R'(Q^*(\mathcal{Y})|Y_k) - (k-h)\right) I_{\{Q^*(\mathcal{Y}) < q(Y_k)\}} \end{aligned} \tag{21}$$

By the nature of the randomizing device that generates the signals $\{Y_k\}_{k=1}^n$ from $\{X_k\}_{k=1}^m$ (discussed above),

$$\begin{aligned}
& \sum_{k=1}^{n} \Pr(Y_k) \cdot (R'(Q^*(\mathcal{Y})|Y_k) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y})<q(Y_k)\}} \\
= & \sum_{\substack{k=1 \\ k\neq i,j}}^{n} p_k \cdot (R'(Q^*(\mathcal{Y})|X_k) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y})<q(X_k)\}} \\
& +(p_i \cdot \beta_i) \cdot (R'(Q^*(\mathcal{Y})|X_i) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y})<q(X_i)\}} + (p_j + p_i \cdot (1-\beta_i)) \times \\
& \left\{ \frac{p_i \cdot (1-\beta_i)}{(p_j + p_i \cdot (1-\beta_i))} R'(Q^*(\mathcal{Y})|X_i) + \frac{p_j}{(p_j + p_i \cdot (1-\beta_i))} R'(Q^*(\mathcal{Y})|X_j) - (k-h) \right\} \\
& \cdot I_{\{Q^*(\mathcal{Y})<q(Y_j)\}}.
\end{aligned}$$

Simplifying the expressions in the inequality 21 and rearanging terms, we see that 21 holds if and only if

$$\begin{aligned}
& \frac{p_i \cdot (1-\beta_i)}{(p_j + p_i \cdot (1-\beta_i))} (R'(Q^*(\mathcal{Y})|X_i) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y})<q(X_i)\}} \\
& + \frac{p_j}{(p_j + p_i \cdot (1-\beta_i))} (R'(Q^*(\mathcal{Y})|X_j) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y})<q(X_j)\}} \\
\geq & \left\{ \frac{p_i \cdot (1-\beta_i)}{(p_j + p_i \cdot (1-\beta_i))} R'(Q^*(\mathcal{Y})|X_i) + \frac{p_j}{(p_j + p_i \cdot (1-\beta_i))} R'(Q^*(\mathcal{Y})|X_j) - (k-h) \right\} \\
& \cdot I_{\{Q^*(\mathcal{Y})<q(Y_j)\}}. \qquad (22)
\end{aligned}$$

Without loss of generality, we assume that $q(X_i) \leq q(X_j)$. Since $q(Y_j) = \max\{q : R'(q|Y_j) = k - h, 0\}$ $=\max\{q : \frac{p_i \cdot (1-\beta_i)}{(p_j+p_i\cdot(1-\beta_i))} R'(q|X_i) + \frac{p_j}{(p_j+p_i\cdot(1-\beta_i))} R'(q|X_j) = k-h, 0\}$, it is clear that $q(X_i) \leq q(Y_j) \leq q(X_j)$. We will now show that the inequality 22 always holds. Observe that both the LHS and the RHS of the inequality are non-negative. Now first consider the case that $Q^*(\mathcal{Y}) \geq q(Y_j)$. Since the RHS is equal to 0, the inequality holds. Now suppose $Q^*(\mathcal{Y}) < q(Y_j)$. This leads to two possibilities. The first is that $Q^*(\mathcal{Y}) < q(X_i)$. In this case, the LHS is equal to the RHS identically. The remaining possibility is that $q(Y_j) > Q^*(\mathcal{Y}) \geq q(X_i)$. In this case, $R'(Q^*(\mathcal{Y})|X_i) - (k-h) \leq 0$, and hence the LHS is

$$\begin{aligned}
& \frac{p_i \cdot (1-\beta_i)}{(p_j + p_i \cdot (1-\beta_i))} (R'(Q^*(\mathcal{Y})|X_i) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y})<q(X_i)\}} \\
& + \frac{p_j}{(p_j + p_i \cdot (1-\beta_i))} (R'(Q^*(\mathcal{Y})|X_j) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y})<q(X_j)\}} \\
= & \frac{p_j}{(p_j + p_i \cdot (1-\beta_i))} (R'(Q^*(\mathcal{Y})|X_j) - (k-h)) \\
\geq & \frac{p_i \cdot (1-\beta_i)}{(p_j + p_i \cdot (1-\beta_i))} (R'(Q^*(\mathcal{Y})|X_i) - (k-h)) + \frac{p_j}{(p_j + p_i \cdot (1-\beta_i))} (R'(Q^*(\mathcal{Y})|X_j) - (k-h)) \\
= & \frac{p_i \cdot (1-\beta_i)}{(p_j + p_i \cdot (1-\beta_i))} R'(Q^*(\mathcal{Y})|X_i) + \frac{p_j}{(p_j + p_i \cdot (1-\beta_i))} R'(Q^*(\mathcal{Y})|X_j) - (k-h) \\
= & \text{RHS}.
\end{aligned}$$

Thus in all cases, LHS $\geq$ RHS, and hence the inequalities 22 and **??** hold. This implies that

$$\begin{aligned}
& E_X\left[(R'(Q^*(\mathcal{Y})|X) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y})<q(X)\}}\right] \\
\geq & E_Y\left[(R'(Q^*(\mathcal{Y})|Y) - (k-h)) \cdot I_{\{Q^*(\mathcal{Y})<q(Y)\}}\right] = \frac{k}{\beta},
\end{aligned}$$

and so, $Q^*(\mathcal{X}) \geq Q^*(\mathcal{Y})$. But $\mathcal{X}$ and $\mathcal{Y}$ were two arbitrary IS, with $\mathcal{X}$ minimally more informative than $\mathcal{Y}$.

Further, by Lemma 2, when there are two IS such that one is at least as informative as the other, the more informative IS can be constructed from the less informative one by a sequence of minimally more informative IS. Thus by transitivity, the build-up-to levels are increasing in informativeness, which proves Theorem 2. ♦

Proof of Theorem 3: As in the proof of Theorem 2, consider two arbitrary IS $\mathcal{X}$ and $\mathcal{Y}$, with $\mathcal{X}$ minimally more informative than $\mathcal{Y}$. The expected inventories are $E_X\left[(Q^*(\mathcal{X})-q(X))^+\right]$ under IS $\mathcal{X}$, and $E_Y\left[(Q^*(\mathcal{Y})-q(Y))^+\right]$ under IS $\mathcal{Y}$. We will show that $E_X\left[(Q^*(\mathcal{X})-q(X))^+\right] \geq E_Y\left[(Q^*(\mathcal{Y})-q(Y))^+\right]$, i.e, that as we move to a minimally more informative IS, the expected inventories increase. Further, applying Lemmas 2 and 1 repeatedly, this would imply that inventories are increasing in informativeness, i.e., information and inventory are complements in our model setting.

Clearly, inventories under any IS $\mathcal{X}$ are increasing in the build-up-to level $Q^*(\mathcal{X})$. Since $Q^*(\mathcal{X}) \geq Q^*(\mathcal{Y})$ (by Theorem 2), it is enough to show that $E_X\left[(Q^*(\mathcal{Y})-q(X))^+\right] \geq E_Y\left[(Q^*(\mathcal{Y})-q(Y))^+\right]$.

Recall that, given the signal Y_j under $\mathcal{Y}$, it may be deduced that the signal generated under $\mathcal{X}$ (prior to the randomization that generated Y_j) was either X_i with probability $p_i \cdot (1-\beta_i)$ or X_j with probability p_j . For all other signals Y_k $(k \neq j)$, the corresponding signal X_k under $\mathcal{X}$ can be deduced with probability 1 (and hence, there is no loss of information). Further, given that under $\mathcal{X}$, the signals $\{X_k\}_{k=1}^m$ occur with probabilities $\{p_k\}_{k=1}^m$, the probabilities of the various signals under $\mathcal{Y}$ may be deduced as follows: the signal Y_k $(k=1,\cdots,m;\ k \neq i$ or $j)$ occurs with probability p_k, Y_i occurs with probability $p_i \cdot \beta_i$ and Y_j with probability $p_j + p_i \cdot (1-\beta_i)$ (All other signals occur with probability $0\cdot$). Thus,

$$E_X\left[(Q^*(\mathcal{Y})-q(X))^+\right] = \sum_{k=1}^{m} p_k \cdot (Q^*(\mathcal{Y})-q(X_k))^+ ,$$

and

$$\begin{aligned} E_Y\left[(Q^*(\mathcal{Y})-q(Y))^+\right] &= \sum_{k=1}^{n} \Pr(Y_k)\cdot (Q^*(\mathcal{Y})-q(Y_k))^+ \\ &= \sum_{\substack{k=1\\k\neq i,j}}^{n} p_k \cdot (Q^*(\mathcal{Y})-q(X_k))^+ + (p_i\cdot\beta_i)\cdot(Q^*(\mathcal{Y})-q(X_i))^+ \\ &\quad + (p_j + p_i\cdot(1-\beta_i))\cdot(Q^*(\mathcal{Y})-q(Y_j))^+ \cdot \end{aligned}$$

From the previous two equations,

$$\begin{aligned} E_X\left[(Q^*(\mathcal{Y})-q(X))^+\right] &\geq E_Y\left[(Q^*(\mathcal{Y})-q(Y))^+\right] \\ \Leftrightarrow p_i\cdot(Q^*(\mathcal{Y})-q(X_i))^+ + p_j\cdot(Q^*(\mathcal{Y})-q(X_j))^+ &\geq (p_i\cdot\beta_i)\cdot(Q^*(\mathcal{Y})-q(X_i))^+ \\ &\quad + (p_j+p_i\cdot(1-\beta_i))\cdot(Q^*(\mathcal{Y})-q(Y_j))^+ , \end{aligned}$$

or equivalently,

$$\frac{p_i\cdot(1-\beta_i)}{p_j+p_i\cdot(1-\beta_i)}\cdot(Q^*(\mathcal{Y})-q(X_i))^+ + \frac{p_j}{p_j+p_i\cdot(1-\beta_i)}\cdot(Q^*(\mathcal{Y})-q(X_j))^+ \geq (Q^*(\mathcal{Y})-q(Y_j))^+ \cdot$$

Both the LHS and the RHS are non-negative. We now consider two cases:

Case *(i)* $Q^*(\mathcal{Y}) \leq q(Y_j)$. In this case, the RHS is 0, and hence LHS $\geq$ RHS.

Case *(ii)* $Q^*(\mathcal{Y}) > q(Y_j)$. In this case,

$$\begin{aligned}
& \frac{p_i \cdot (1-\beta_i)}{p_j + p_i \cdot (1-\beta_i)} \cdot (Q^*(\mathcal{Y}) - q(X_i))^+ + \frac{p_j}{p_j + p_i \cdot (1-\beta_i)} \cdot (Q^*(\mathcal{Y}) - q(X_j))^+ \\
\geq \; & \frac{p_i \cdot (1-\beta_i)}{p_j + p_i \cdot (1-\beta_i)} \cdot (Q^*(\mathcal{Y}) - q(X_i)) + \frac{p_j}{p_j + p_i \cdot (1-\beta_i)} \cdot (Q^*(\mathcal{Y}) - q(X_j)) \\
= \; & Q^*(\mathcal{Y}) - \left(\frac{p_i \cdot (1-\beta_i)}{p_j + p_i \cdot (1-\beta_i)} \cdot q(X_i) + \frac{p_j}{p_j + p_i \cdot (1-\beta_i)} \cdot q(X_j) \right) \\
\geq \; & Q^*(\mathcal{Y}) - q(Y_j) \text{ (By the concavity of } q(\cdot)) \\
= \; & (Q^*(\mathcal{Y}) - q(Y_j))^+ . \text{(under Case } (ii))
\end{aligned}$$

Thus, $E_X\left[(Q^*(\mathcal{Y}) - q(X))^+\right] \geq E_Y\left[(Q^*(\mathcal{Y}) - q(Y))^+\right]$, and so the inventory under $\mathcal{X}$ is at least as great as that under $\mathcal{Y}$. But $\mathcal{X}$ and $\mathcal{Y}$ were arbitrarily chosen. Further, by applying applying Lemmas 2 and 1 repeatedly, the result on complementarity of information and inventories holds for any two arbitrary IS where one IS is at least as informative as the other. This completes the proof of Theorem 3. ♦

Proof of Lemma 3: Consider a no-information IS $\mathcal{Y}$ with support $Su(\mathcal{Y}) = \{Y_1, Y_2, \cdots, Y_n\}$. By definition, $\pi(s|Y_j) = \pi(s), \forall s \in S$, for $j = 1, \cdots, n$. Now consider an arbitrary IS $\mathcal{X}$ with support $Su(\mathcal{X}) = \{X_1, X_2, \cdots, X_m\}$. Starting with the IS $\mathcal{X}$, construct an IS $\mathcal{Z}$ using a randomizing device that operates as follows: For each $X \in Su(\mathcal{X})$, the device generates one of the signals from the set $Su(\mathcal{Z}) = \{Y_1, Y_2, \cdots, Y_n\}$, each with probability $\frac{1}{n}$. Defining $\beta_{XZ} = \frac{1}{n}$, for all $X \in Su(\mathcal{X}), Z \in Su(\mathcal{Z})$, we can check that (i) $\pi(Z|s) = \sum_{X \in \mathcal{X}} \beta_{XZ} \pi(X|s)$, $\forall\ s \in \mathcal{S}$, $Z \in Su(\mathcal{Z})$, and (ii) $\sum_{Z \in \mathcal{Z}} \beta_{XZ} = 1$, $\forall\ X \in Su(\mathcal{X})$. Thus by definition, $\mathcal{X}$ is at least as informative as $\mathcal{Z}$. If we now establish that the IS $\mathcal{Z}$ is equivalent to the IS $\mathcal{Y}$, our proof that the arbitrarily chosen IS $\mathcal{X}$ is at least as informative as the no-information IS $\mathcal{Y}$ is complete. To prove the equivalence of $\mathcal{Y}$ and $\mathcal{Z}$, we need to show that under the IS $\mathcal{Z}$, $\pi(s|Y_j) = \pi(s), \forall s \in S$, for all $Y_j \in Su(\mathcal{Z})$. Now, under the IS $\mathcal{Z}$,

$$\begin{aligned}
\pi(Y_j|s) &= \sum_{i=1}^{m} \pi(Y_j, X_i|s) \\
&= \sum_{i=1}^{m} \pi(Y_j|X_i, s) \cdot \pi(X_i|s) \\
&= \frac{1}{n} \sum_{i=1}^{m} \pi(X_i|s) \text{ (because } \pi(Y_j|X_i, s) = \pi(Y_j|X_i) = \frac{1}{n}, \text{ by construction)} \\
&= \frac{1}{n} \cdot
\end{aligned}$$

Further, for each $s \in \mathcal{S}$, and for $j = 1, \cdots, n$,

$$\begin{aligned}
\pi(s|Y_j) &= \frac{\pi(Y_j|s) \cdot \pi(s)}{\pi(Y_j)} \\
&= \frac{\pi(Y_j|s) \cdot \pi(s)}{\sum_{s \in \mathcal{S}} \pi(Y_j|s) \cdot \pi(s)} \\
&= \frac{\frac{1}{n} \cdot \pi(s)}{\sum_{s \in \mathcal{S}} \frac{1}{n} \cdot \pi(s)} \\
&= \pi(s) \cdot
\end{aligned}$$

Hence the result. ♦